LECTURES

ON THE

THEORY OF ELLIPTIC FUNCTIONS

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VOLUME I

ANALYSIS

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GENERAL PREFACE

In the publication of these lectures, it is proposed to present the Theory of Elliptic Functions in three volumes, which are to include in general the following three phases of the subject:

I. *Analysis;*

II. *Applications to Problems in Geometry and Mechanics;*

III. *General Arithmetic and Higher Algebra.*

In Volume I an attempt is made to give the essential principles of the theory. The elliptic functions considered as the inverse of the elliptic integrals have their origin in the immortal works of Abel and Jacobi. I have wished to treat from a philosophic, as well as from a formal standpoint, the existence, and as far as possible, the ultimate meaning of the functions introduced by these mathematicians, to discuss the theories which originated with them, to follow their development, and to extend as far as possible the principles which they established. In this development great assistance has been rendered by the works of Hermite, who contributed so much not only to the theory of elliptic functions but also to almost every form of mathematical thought. The theory of Weierstrass is studied side by side with the older theory, and the beautiful formulas which we owe to him are contrasted with the corresponding formulas of the earlier writers. Riemann introduced certain surfaces upon which he represented algebraic integrals, and by thus expressing his conceptions of analytic functions he revealed a clearer insight into their meaning. Instead of generalizing either the theory of Jacobi or that of Weierstrass so as to embrace the whole subject, it is thought better to make these theories specializations of a more general theory. This general theory is treated by means of the Riemann surface, which at the same time shows the intimate relation between the two theories just mentioned.

In Volume II a treatment of elliptic integrals is given. Here much attention is paid to the work of Legendre, whom we may rightly regard as the founder of the elliptic functions, for upon his investigations were established the theories of Abel and Jacobi, and indeed, in the very form given by Legendre. Abel in a published letter to Legendre wrote: "Si je suis assez heureux pour faire quelques découvertes, je les attribuerai à vous plutôt qu' à moi"; and Jacobi wrote as follows to the genial Legendre: "Quelle satisfaction pour moi que l'homme que j'admirais tant en

dévorant ses écrits a bien voulu accueillir mes travaux avec une bonté si rare et si précieuse! Tout en manquant de paroles qui soient de dignes interprètes de mes sentiments, je n'y saurai réprondre qu'en redoublant mes efforts a pousser plus loin les belles théories dont vous êtes le créateur."

True Fagnano, Euler, Landen, Lagrange, and possibly others had discovered certain theorems which proved fundamental in the future development of the elliptic functions; but by the patient devotion of a long life to these functions, Legendre systematized an independent theory in that he reduced all integrals which contain no other irrationality than the square root of an expression of degree not higher than the fourth into three canonical forms of essentially different character. Thus he was enabled to discover many of their most important properties and to overcome great difficulties, which with the means then at hand appear almost insurmountable. Methods were devised which furnished immediate results and which, extended by subsequent investigations, enriched the science of mathematics and the fields of knowledge. In this direction the great English mathematician Cayley has done much work, and to him a considerable portion of this volume is due. The admirable work of Greenhill has also been of great assistance. Much space is given in Volume II to the applications of the theory. These applications are usually in the form of integrals and the results required are real quantities, and for the most part the variables must be taken real. Thus the complex variable of Volume I must be limited to some extent in the second volume. The problems selected serve to illustrate the different phases treated in the previous theory; sometimes preference, as the occasion warrants, is given to Legendre's formulas, sometimes to those of Weierstrass. While the most of these problems are taken from geometry, physics, and mechanics, there are some which have to do with algebra and the theory of numbers.

All true students of applied mathematics, engineers, and physicists should have some knowledge of elliptic functions; at the same time it must be recognized that one cannot do all things, and it is not expected that such students should be as well versed in the theoretical side of this subject as are pure mathematicians. For this reason Volume II has been so prepared that without dwelling too long upon the intrinsic meaning of the subject, one may obtain a practical idea of the formulas. Much of the theory of Volume I is therefore not presupposed, and many of the results that have hitherto been derived are again deduced in Volume II by other methods, which, without emphasizing the theoretical significance, are often more direct. This is especially true of the addition-theorems. A table of elliptic integrals of the first and second kinds will be found at the end of this volume, which may consequently, for the reasons stated, be regarded as an advanced calculus.

Volume III will be of interest especially to the lovers of pure mathe-

matics. In this volume the theory becomes more abstract. Many problems of higher algebra occur which lie within the realms of general arithmetic. This includes the theories of complex multiplication; of the division and transformation of the elliptic functions; a study of the modular equations and the solution of the algebraic equation of the fifth degree, etc.

The discoveries of Kronecker in the theory of the complex multiplication not only prove the theorems left in fragmentary form by Abel and give a clear insight into them, but they show the close relationship of this theory with algebra and the theory of numbers. The problem of division resolves itself into the solution of algebraic equations, and the introduction of the roots of these equations into the ordinary realm of rationality forms a "realm of algebraic numbers"; the same is true of the modular equations. Kronecker, Dedekind, Hermite, Weber, Joubert, Brioschi, and other mathematicians have developed these lines of thought into an independent branch of mathematics which in its further growth is susceptible of extension in many directions, notably to the treatment of the Abelian transcendents on the one hand and of the modular systems on the other.

Jacobi in a letter to Crelle wrote: "You see the theory [of elliptic functions] is a vast subject of research, which in the course of its development embraces almost all algebra, the theory of definite integrals, and the science of numbers." It is also true that when a discovery is made in any one of these fields the domains of the others are also thereby extended.

INTRODUCTION TO VOLUME I

Every one-valued analytic function which has an algebraic addition-theorem is an elliptic function or a limiting case of one. The existence, formation, and treatment of the elliptic functions as thus defined are given in Chapters I–VII of the present volume.

An algebraic equation connecting the function and its derivative, which we have called the *eliminant equation*, is emphasized. This differential equation due to Méray is first used as a latent test to ascertain whether or not a function in reality has an algebraic addition-theorem, and, secondly, as shown by Hermite, its integrals when restricted to one-valued functions are one or the other of the three classes of functions: rational functions, simply periodic functions, or doubly periodic functions. We regard the first two types as limiting cases of the third, the three types forming the general subject of elliptic functions. All three types of functions are shown to have algebraic addition-theorems, and consequently the existence of the eliminant equation is found to be coëxtensive with that of the elliptic functions.

In Chapter I some preliminary notions are given. In particular it is found that the rational and the trigonometric, and later, in Chapter V, that the doubly periodic functions may be expressed in terms of simple elements, and it is seen that all three forms of expression are the same; a treatment is given of infinite products and also of the *primary factors* of an integral transcendental function; analytic functions are defined.

The properties of functions which have algebraic addition-theorems are considered in Chapter II, and it is shown that these properties exist for the *whole* region in which the *function* has a meaning.

After establishing the existence of the simply and doubly periodic functions in Chapters III and IV and after studying the nature of the periods, we proceed in Chapter V to the actual formation of the doubly periodic functions. It is shown that the doubly periodic functions may be represented as the quotients of two Hermitean "intermediary functions," of which the Jacobi Theta-functions are special cases. The derivation of such functions with their characteristic properties is then treated. Further, by a method also due to Hermite, it is shown that the most general elliptic functions may be expressed in terms of a simple functional element, which is in fact the simplest intermediary function.

After proving the theorem that the most general elliptic function may be expressed algebraically through an elliptic function of the second order (the simplest kind of an elliptic function), a form of eliminant equation is derived in which the derivative appears only to the second power. The functions connected with this equation are treated by means of the Riemann surface, which is given at length in Chapter VI, where also the "one-valued functions of position" are introduced.

The integrals defining the circular functions contain radicals under which the variable appears to the second degree; while the variable appears to the third or fourth degree under the radicals in the elliptic integrals. It is therefore natural to consider the elliptic functions as the generalization of the circular functions, just as the latter functions may be regarded as limiting cases of the former. The methods followed by Legendre, Abel and Jacobi seem the natural and inevitable methods of presenting these functions. History also gives them precedence. Weierstrass built his theory on the foundation already established by these earlier mathematicians, and it is impossible to realize the real significance of Weierstrass's functions without a prior knowledge of the older theory. Riemann's theory forms an important extension of the purely analytic treatment of Legendre and Jacobi as well as of the Weierstrassian theory. The characteristics of Riemann's theory lie on the one hand in the simple application of geometrical representations such as the two-leaved surface and its conformal representation upon the period parallelogram, and on the other hand it shows how the formulas are founded synthetically on the basis of the fundamental properties of the functions and integrals; and thus a deeper and a clearer insight into their true nature is gained.

Mr. Poincaré has said, "By the instrument of Riemann we see at a glance the general aspects of things — like a traveler who is examining from the peak of a mountain the topography of the plain which he is going to visit and is finding his bearings. By the instrument of Weierstrass analysis will in due course throw light into every corner and make absolute clearness shine forth."

The universal laws of Riemann are particularized in the one direction of the Legendre-Jacobi theory and in the other direction of the Weierstrassian theory, the two theories being interconnected. Accordingly in the present volume the Legendre-Jacobi functions are first developed and often side by side with them the corresponding Weierstrassian functions.

Owing to a theorem due to Liouville, we are able to show the real significance of the one-valued functions of position on the Riemann surface, viz., they are the general elliptic functions. These one-valued functions form a "class of algebraic functions" or "a closed realm of rationality," since the sum, difference, product, or quotient of any two such functions

is a function of the realm. This realm of rationality is of the first order, corresponding to the connectivity of the associated Riemann surface, the realm of the ordinary rational functions being of the zero order. The former realm is derived from the latter by adjoining an algebraic quantity, which quantity defines the Riemann surface. This latter realm, which we call the "elliptic realm," includes as special cases the natural realm of all rational functions, and also the realm of the simply periodic functions. It therefore follows that all one-valued analytic functions which have algebraic addition-theorems form a closed realm; for every element (function) that belongs to this elliptic realm has an algebraic addition-theorem. Thus simultaneously with the development of the elliptic functions, the realm in which they enter is shown to be a closed one, and the reader gradually finds himself studying these functions in their own realm.

The elliptic or doubly periodic realm degenerates into a simply periodic realm when any two branch-points coincide, and it degenerates into the realm of rational functions when any two pairs of branch-points are equal. Thus again it is seen that the elliptic realm includes the three types of functions: rational functions, simply periodic functions, and doubly periodic functions. In Chapter VII the eliminant equation is further simplified and it is finally shown what form this equation must have that the upper limit of the resulting integral be a one-valued function of the integral. The *problem of inversion* is thereby solved in a remarkably simple manner. Thus by means of the Riemann surface, as it is possible in no other way, we may study the integral as a one-valued function of its upper limit and *vice versa*.

In Chapter VIII the most general integral involving the square root of an expression of the third or fourth degree in the variable is made to depend upon three types of integrals. The normal forms of integrals are derived, and in particular Weierstrass's normal form, in a manner which illustrates the meaning of the invariants. The realms of rationality in which the normal forms of Legendre and of Weierstrass are defined are shown to be equivalent.

The further contents of this volume are indicated through the headings of the different chapters. To be noted in particular is Chapter XIV, in which it is shown how the Weierstrassian functions are derived directly from those of Jacobi; in Chapter XX are given several different methods of representing any doubly periodic function; while in Chapter XXI we find a method of determining all analytic functions which have algebraic addition-theorems. A table of the most important formulas is found at the end of this volume.

Professor Fuchs made the Riemann surfaces fundamental in his treatment of the Theory of Functions and the Differential Equations. It was

my privilege to hear him lecture on these subjects, and the present work, so far as it has to do with the Riemann surfaces, is founded upon the theory of that great mathematician. Although Professor Weierstrass lectured twenty-six times (from 1866 to 1885) in the University of Berlin on the theory of elliptic functions including courses of lectures on the application of these functions, no authoritative account of his work has been published, a quarter of a century having in the meanwhile elapsed. It is therefore difficult to say in that part of the theory which bears his name what is due to him, what to other mathematicians. I have derived considerable help in this respect from the lectures of Professor H. A. Schwarz, the results of which are published in his *Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen.*

While it has not been my purpose to make the book encyclopedic, I have tried to give the principal authorities which have been of service in its preparation. The pedagogical side is insisted upon, as the work in the form of lectures is intended to be introductory to the theory in question.

To Messrs. John Wiley and Sons, Scientific Publishers, and to the Stanhope Press, I am under great obligation for the courteous co-operation which has minimized my labor during the progress of printing.

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CHAPTER I

PRELIMINARY NOTIONS

ARTICLE 1. *One-valued function.* — A function of the complex variable $u = x + iy$ is said to be *one-valued* when it has only one value for each value of u; for example, $\frac{1}{u}$, $\sin u$, $\tan u$ are one-valued functions.

If we represent the variable $u = x + iy$ by a point on the plane with coördinates x and y, we also speak of the function as being one-valued in the whole plane, or in any part of the plane for which the function is defined.

Regular function. — A one-valued function is *regular* * at a point a when we may develop this function by Taylor's Theorem within a circle with a as center in a convergent series of the form

$$f(u) = f(a) + \frac{u-a}{1!}f'(a) + \frac{(u-a)^2}{2!}f''(a) + \cdots + \frac{(u-a)^n}{n!}f^{(n)}(a) + \cdots,$$

the exponents $1, 2, \ldots, n, \ldots$ being positive integers.

The power series on the right is denoted by $P(u - a)$. Any such point a is called an *ordinary* or *regular point* of the function, and the function is said to *behave regularly* † in the *neighborhood* of such a point. At these points the function has the *character* of *an integral* function.

Zeros. — If the function $f(u)$ is regular for all points in the neighborhood of a, and if $f(a) = 0$, the point a is a *zero* of the function $f(u)$; if $f'(a) \neq 0$, the point a is a simple zero, or a zero of the *first* order. If the derivatives $f'(a), f''(a), \ldots, f^{(n-1)}(a)$ are all zero, while $f^{(n)}(a) \neq 0$, the zero $u = a$ is of the nth *order*. In the latter case the function $f(u)$ may be written

$$f(u) = (u-a)^n g(u),$$

* Weierstrass, *Zur Theorie der eindeutigen analytischen Functionen*, Werke, Bd. 2, p. 77; Berl. Abh. 1876, p. 11; *Abhandlungen aus der Funktionenlehre*, Werke, Bd. 2, p. 135; Zur Funktionentheorie, Ber. Ber. 1880, p. 719; Werke, 2, p. 201.

Mittag-Leffler, *Sur la représentation analytique des fonctions monogènes uniformes*, Acta Math., Bd. IV, p. 3.

† "*Ich sage von einer eindeutigen definirten Function einer Veränderlichen u, dass sie sich in der Nähe eines bestimmten Werthes u_0 der letzteren regulär verhalte, wenn sie sich für alle einer gewissen Umgebung der Stelle u_0 angehörigen Werthe von u in der Form einer gewöhnlichen Potenzreihe von $u - u_0$ darstellen lässt.*" Weierstrass, Werke, 2, p. 295, 1883.

where $g(u)$ is a regular function that is *not* zero for $u = a$. The function $g(u)$ may consequently be developed in a convergent series of the form

$$g(u) = g(a) + \frac{u-a}{1!}g'(a) + \frac{(u-a)^2}{2!}g''(a) + \cdots .$$

ART. 2. *Singular points.* — If the one-valued function $f(u)$ is not regular at a definite point a, we say that this point is a *singular point* or a *singularity* of the function. It is an *isolated* singular point when we may draw around a as center a circle with radius as small as we wish, within which there is no other singularity of the function.

Pole or infinity. — A singular point a is a *pole* or *infinity* when it is isolated and when the function regular in the vicinity of this point becomes at the point infinite in the same way as, say, the function

$$f(u) = \frac{\phi(u)}{(u-a)^n},$$

where n is a positive integer and where $\phi(u)$ is a regular function at the point a and $\phi(a) \neq 0$. The function $\phi(u)$ may be expanded in a convergent power series of the form

$$\phi(u) = \phi(a) + \frac{(u-a)}{1!}\phi'(a) + \frac{(u-a)^2}{2!}\phi''(a) + \cdots,$$

so that $f(u)$, when expanded in the neighborhood of $u = a$, is

$$f(u) = \frac{A_n}{(u-a)^n} + \frac{A_{n-1}}{(u-a)^{n-1}} + \cdots + \frac{A_1}{u-a} + F(u),$$

where $F(u)$ is a regular function in the neighborbood of $u = a$. The constants $A_n, A_{n-1}, \ldots, A_1$ are determinate, $A_n = \phi(a)$, etc.

The integer n is the *order* or *degree* of the pole.

The coefficient A_1 of $\dfrac{1}{u-a}$ is the *residue* relative to the pole a and

$$\frac{A_n}{(u-a)^n} + \frac{A_{n-1}}{(u-a)^{n-1}} + \cdots + \frac{A_1}{u-a}$$

is called the *principal part* of the function relative to the pole $u = a$.

ART. 3. *Essential singular points.* — In the neighborhood of such a point, the function is completely *indeterminate*. Consider,* for example, the function

$$\frac{1}{e^{u-a}} = 1 + \frac{1}{1!}\frac{1}{u-a} + \frac{1}{2!}\frac{1}{(u-a)^2} + \frac{1}{3!}\frac{1}{(u-a)^3} + \cdots$$

in the neighborhood of the point $u = a$.

* Cf. Hermite, *Cours rédigé par* M. Andoyer (Quatrième édition, 1891), p. 97.

If $\alpha + i\beta$ be any arbitrary point whatever, then it is always possible to give to $u - a$ a value $\xi + i\eta$ as small as we wish, such that

$$e^{\frac{1}{\xi + i\eta}} = \alpha + i\beta.$$

For writing $\alpha + i\beta = e^{p+iq}$, the preceding equation becomes

$$\frac{1}{\xi + i\eta} = p + iq = \frac{p^2 + q^2}{p - iq}.$$

It follows at once that

$$\xi = \frac{p}{p^2 + q^2} \quad \text{and} \quad \eta = -\frac{q}{p^2 + q^2}.$$

From this it is seen that ξ and η are completely determined. On the other hand the proposed equation is satisfied if for q we write $q + 2k\pi$, where k is an arbitrary integer, since $2i\pi$ is the period of the exponential function. Thus since q may be increased beyond every limit, the quantities ξ and η are susceptible of becoming as small as we wish.

The origin is an essential singularity of the function $e^{\frac{1}{u}}$. A characteristic distinction between the poles and the essential singularities is: If we take the inverse of the proposed function, the poles are transformed into zeros; while an essential point remains an essential point, the reciprocal of the function in the neighborhood of such a point being as the function itself completely indeterminate.*

In the present theory we have to treat such functions which have poles as the only singular points in the finite portion of the plane.

ART. 4. *Remark concerning the zeros and the poles.*—If the point a is a zero of order n of the function $f(u)$, it is a simple zero with residue n in the logarithmic derivative $\dfrac{f'(u)}{f(u)}$.

For in the neighborhood of $u = a$ we have

$$f(u) = (u - a)^n g(u),$$

where $g(a) \neq 0$.

It follows that

$$\frac{f'(u)}{f(u)} = \frac{n}{u - a} + \frac{g'(u)}{g(u)},$$

$\dfrac{g'(u)}{g(u)}$ being a regular function at the point $u = a$.

Similarly it is seen that if $u = a$ is a pole of order m of the function $f(u)$, it is a simple pole of residue $-m$ for $\dfrac{f'(u)}{f(u)}$.

* Briot and Bouquet (*Fonctions Elliptiques*, p. 94) employ what seems a more appropriate name, "*point d'indétermination.*"

For writing

$$f(u) = \frac{G(u)}{(u-a)^m}, \text{ where } G(a) \neq 0,$$

we have

$$\frac{f'(u)}{f(u)} = \frac{-m}{u-a} + \frac{G'(u)}{G(u)},$$

where $\frac{G'(u)}{G(u)}$ is a regular function at the point $u = a$.

ART. 5. *The point at infinity.* — If we write $u = \frac{1}{v}$, a definite point in the u-plane corresponds to a definite point in the v-plane, and *vice versa.* The infinite point in the u-plane corresponds to the origin in the v-plane.

Hence if the function $f(u)$ is regular at the point $u = \infty$, the function $f\left(\frac{1}{v}\right)$ must be regular at the point $v = 0$. It must consequently for small values of v in the vicinity of $v = 0$ take the form

$$f\left(\frac{1}{v}\right) = a_0 + a_1 v + a_2 v^2 + \cdots = P(v), \text{ say,}$$

where the a's are constants. It follows that for large values of u we must have

$$f(u) = a_0 + \frac{a_1}{u} + \frac{a_2}{u^2} + \cdots + \frac{a_{n-1}}{u^{n-1}} + \frac{a_n}{u^n} + \cdots .$$

If the function is regular in the neighborhood of the point ∞, the infinite point is a zero of the nth order, when $a_0 = 0 = a_1 = \cdots = a_{n-1}$; $a_n \neq 0$. This function then vanishes at infinity as $\frac{1}{u^n}$ (where $u = \infty$).

The point at infinity is a pole or an essential singularity of the function $f(u)$, when $v = 0$ is a pole or essential singularity of $f\left(\frac{1}{v}\right)$. If $u = \infty$ is a pole, we must have for small values of v

$$f\left(\frac{1}{v}\right) = \frac{A_1}{v} + \frac{A_2}{v^2} + \cdots + \frac{A_n}{v^n} + c_0 + c_1 v + c_2 v^2 + \cdots ,$$

where the A's and c's are constants; or, for large values of u,

$$f(u) = A_1 u + A_2 u^2 + \cdots + A_n u^n + c_0 + \frac{c_1}{u} + \frac{c_2}{u^2} + \cdots .$$

The part $A_1 u + A_2 u^2 + \cdots + A_n u^n$, which becomes infinite at the pole $u = \infty$, is *the principal part relative to this pole* and n is the *order* of the pole.

ART. 6. *Convergence of series.* — We have spoken above of the convergence of the series which represents the function $f(u)$ in the neighborhood of a point a. We said that the function $f(u)$, one-valued in a defined region, is regular at a point a of this region, when it is developed by Taylor's Theorem in a circle with a as the center.

This series is convergent within the circle having a for center and a radius which extends to the nearest singular point of the function* $f(u)$. We shall presuppose the fundamental tests for *absolute* convergence. The criterion for *uniform* convergence as stated by Weierstrass is as follows: The infinite series $u_1(z)+u_2(z)+u_3(z)+\cdots$, the individual terms of which are functions of z defined for a fixed interval, converges *uniformly* within this interval, provided there exists an *absolutely* convergent series,

$$M_1 + M_2 + \cdots,$$

where the M's are quantities independent of z and are such that within the fixed interval the following inequality is true:

$$|u_n(z)| \leqq M_n, \quad \text{where } n \geqq \mu,$$

μ being a fixed integer. (See Osgood, *Lehrbuch der Funktionentheorie*, p. 75.)

ART. 7. *A one-valued function that is regular at all points of the plane (finite and infinite) is a constant.*

For the function supposed regular at $u = 0$ is developable in the series

$$f(u) = a_0 + a_1 u + a_2 u^2 + \cdots = P(u), \quad \text{say},$$

which is convergent within a circle which may extend to infinity, since by hypothesis there are no singular points in the plane.

Writing $u = \dfrac{1}{v}$, the expansion in the neighborhood of infinity is

$$f\left(\frac{1}{v}\right) = a_0 + \frac{a_1}{v} + \frac{a_2}{v^2} + \cdots.$$

This function being by hypothesis regular in the neighborhood of infinity, can contain no negative powers.

It follows that $a_1 = 0 = a_2 = a_3 = \ldots$, and consequently

$$f(u) = f\left(\frac{1}{v}\right) = a_0.$$

Another statement of this theorem is the following: *A one-valued function that is finite at all points of the plane (including the infinite point) is a constant.*

For at each one of its poles a one-valued function becomes infinite. It may also be shown that if the variable u tends towards an essential singularity in a manner which has been suitably chosen, the modulus of the function increases beyond limit. If then a one-valued function is every-

* See Cauchy, *Cours d'Analyse de l'École Royale Polytechnique*, 1ère Partie. *Analyse Algébrique*, Chapitre 9, § 2, Théorème I, p. 286. Paris. 1821. Unless stated otherwise, by "convergent" is meant *absolutely* convergent. (See Osgood, *Lehrbuch der Funktionentheorie*, pp. 75 *et seq.*; pp. 285 *et seq.*); and when the variable enters, *uniformly* convergent. In the latter case by "within the circle of convergence" we understand "within any interval that lies wholly within this circle."

where finite, it cannot have singular points; it is regular throughout the whole plane and reduces to a constant.

ART. 8. *The zeros and the poles of a one-valued function, which has no other singularities than poles in the finite portion of the plane, are necessarily isolated the one from the other.*

By this we mean to say that there cannot exist a point a of the plane in whose immediate neighborhood there are an infinite number of poles or an infinite number of zeros. In other words, wherever the point a is situated, one may always draw around a as center a circle with radius sufficiently small that within the circle there are (1) neither zero nor pole; or (2) a zero but no pole; or (3) a pole but no zero.

This follows immediately from the preceding developments. For if a point a is taken in the plane, three cases are possible: (1) the function $f(u)$ may be *regular* at a without vanishing at this point; or (2) the point a is a *zero* of $f(u)$; or (3) the point a is a *pole* of $f(u)$. In the first case we may draw about a as center a circle with radius sufficiently small that within the circle there is neither zero nor pole; in the second case we may draw a circle sufficiently small that it does not contain a pole and contains the only zero $u = a$, and similarly in the third case.

It follows that if for a one-valued function there exists a point a such that within an area as small as we choose inclosing this point there exists an infinity of poles or an infinity of zeros, this point is an essential singularity. The function is *not* regular at this point. As examples of what has been said are the rational functions and the trigonometric functions, which shall be first studied as introductory to the general theory of elliptic functions.

RATIONAL FUNCTIONS.

ART. 9. Methods are given here, (1) of decomposing a rational fraction into its simple (or partial) fractions; (2) of representing such a fraction as a quotient of two products of linear factors. The same methods will be adopted later in the general theory of elliptic functions, there existing analogous relations for these functions.

Consider first as a particular case * the function

$$f(u) = \frac{u}{(u-1)(u-2)},$$

which is regular at all finite points of the plane except the points $u = 1$ and $u = 2$. These points are poles of the first order. The principal part of $f(u)$ relative to the pole $u = 1$ is

$$-\frac{1}{u-1} = \phi_1(u), \quad \text{say},$$

* See Appell et Lacour, *Fonctions Elliptiques*, p. 7.

as is seen by noting that the difference

$$f(u) - \phi_1(u)$$

is regular at the point $u = 1$. The residue relative to the pole $u = 1$ is -1.

Similarly the principal part relative to the pole $u = 2$ is

$$\phi_2(u) = \frac{2}{u - 2},$$

with the residue 2.

At the point $u = \infty$ the function is regular, for

$$f\left(\frac{1}{v}\right) = \frac{v}{(1 - v)(1 - 2v)}$$

is a regular function at the point $v = 0$.

It is further seen that $v = 0$ or $u = \infty$ is a simple zero. The function $f(u)$ has then two simple poles $u = 1$, $u = 2$ and two simple zeros $u = 0$, $u = \infty$. The function is said to be of *order* or *degree* 2.

It may also be observed that the equation

$$f(u) = C$$

has two roots, whatever be the constant C. Further, since the functions $\phi_1(u)$ and $\phi_2(u)$ are everywhere regular except at the poles $u = 1$, $u = 2$, the difference

$$f(u) - \phi_1(u) - \phi_2(u)$$

is a function that is everywhere regular. It is therefore a constant, and since $f(u)$, $\phi_1(u)$, $\phi_2(u)$ all vanish for $u = \infty$, this constant is zero.

We therefore have

$$f(u) = \phi_1(u) + \phi_2(u),$$

a formula, which gives immediately the decomposition of the rational function $f(u)$ into its simple fractions.

ART. 10. *The general case.* — A rational function

$$f(u) = \frac{a_0u^m + a_1u^{m-1} + \cdots + a_m}{b_0u^n + b_1u^{n-1} + \cdots + b_n} = \frac{Q_1(u)}{Q(u)},$$

where Q_1 and Q are integral functions (polynomials) of degree m and n, is a function which has no other singularities than poles in the finite portion of the plane or at infinity. At a finite distance it has as poles the roots of $Q(u) = 0$. The number of these poles at a finite distance, where each is counted with its order of multiplicity, is n.

1°. If $m > n$, the point at ∞ is a pole of order $m - n$. Hence the total number of poles at finite and infinite distances is $n + m - n = m$.

There are also m zeros, viz., the roots of $Q_1(u) = 0$. It is thus seen that the function $f(u)$ has m zeros and m poles. We say that it is of

order or *degree* m. The equation $f(u) = C$ has m roots, whatever the value of the constant C.

2°. If $n > m$, the point ∞ is a zero of order $n - m$. The function has n poles and an equal number of zeros. For there are m zeros at finite distances, viz., the roots of $Q_1(u) = 0$ and $n - m$ zeros at infinity. The function is of order n and the equation $f(u) = C$ has n roots.

3°. If $m = n$, the point at infinity is neither a pole nor a zero. There are also here as many zeros as infinities, and the function is of order $m = n$.

It follows that a rational function $f(u)$ has always in the whole plane, including infinity, as many zeros as poles. The number of zeros or poles is the order of the function, and the equation $f(u) = C$, where C is an arbitrary constant, has a number of roots equal to the order of the function $f(u)$. *In particular we note that the rational functions have only polar singularities.*

Principal Analytical Forms of Rational Functions.

ART. 11. *First form: where the poles and the corresponding principal parts are brought into evidence. Decomposition into simple fractions.*

Let $a_1, a_2, \ldots, a_\nu$ be poles of order $n_1, n_2, \ldots, n_\nu$ of the function $f(u)$ and let the *principal parts with respect to these poles* be

$$\phi_1(u) = \frac{A_{11}}{u - a_1} + \frac{A_{21}}{(u - a_1)^2} + \cdots + \frac{A_{n_1 1}}{(u - a_1)^{n_1}},$$

$$\phi_2(u) = \frac{A_{12}}{u - a_2} + \frac{A_{22}}{(u - a_2)^2} + \cdots + \frac{A_{n_2 2}}{(u - a_2)^{n_2}},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\phi_\nu(u) = \frac{A_{1\nu}}{u - a_\nu} + \frac{A_{2\nu}}{(u - a_\nu)^2} + \cdots + \frac{A_{n_\nu \nu}}{(u - a_\nu)^{n_\nu}}.$$

Further for the most general case, suppose that the point ∞ is also a pole, which is the case in the previous Article when $m > n$; and let the principal part relative to this pole be

$$\phi(u) = A_{10}u + A_{20}u^2 + \cdots + A_{s0}u^s,$$

where $s = m - n$ is the order of the pole.

Since each of the principal parts is everywhere regular except at the associated pole, the difference

$$f(u) - \phi_1(u) - \phi_2(u) - \cdots - \phi(u)$$

is regular everywhere including infinity and consequently is a constant, $= A$, say.

It follows that

$$f(u) = A + A_{10}u + A_{20}u^2 + \cdots + A_{s0}u^s$$
$$+ \sum_{i=1}^{i=\nu}\left(\frac{A_{1i}}{u - a_i} + \frac{A_{2i}}{(u - a_i)^2} + \cdots + \frac{A_{n_i i}}{(u - a_i)^{n_i}}\right),$$

where the index i refers to the indices of the poles $a_1, a_2, \ldots, a_\nu$. This formula may be written in a somewhat simpler form if we symbolize $\frac{1}{u}$ by $u - a_0$, where $a_0 = \infty$, and let $n_0 = s$. We then have

$$f(u) = A + \sum_i\left(\frac{A_{1i}}{u - a_i} + \frac{A_{2i}}{(u - a_i)^2} + \cdots + \frac{A_{n_i i}}{(u - a_i)^{n_i}}\right),$$

where the summation index i refers to the indices of the poles $a_1, a_2, \ldots, a^\nu, a_0$.

If we put $\frac{1}{u - a_i} = v_i$, we have finally

$$f(u) = A + \sum_i\left(A_{1i}v_i - \frac{A_{2i}}{1!}v_i' + \frac{A_{3i}}{2!}v_i'' - \cdots \pm \frac{A_{n_i i}}{(n_i - 1)!}v_i^{(n_i-1)}\right).$$

The formula is convenient especially for the integration of a rational function.*

ART. 12. *Second form: where the zeros and the infinities are brought into evidence.* It is sufficient here to decompose the polynomials $Q_1(u)$ and $Q(u)$ of the preceding article into their linear factors, so that

$$f(u) = C\frac{(u - c_1)(u - c_2)\ldots(u - c_m)}{(u - b_1)(u - b_2)\ldots(u - b_n)},$$

where C is a constant. Of course, some of the factors may be equal. We may derive the second form from the first by noting (Art. 4) that

$$\frac{f'(u)}{f(u)} = \frac{1}{u - c_1} + \frac{1}{u - c_2} + \cdots + \frac{1}{u - c_m}$$
$$- \frac{1}{u - b_1} - \frac{1}{u - b_2} - \cdots - \frac{1}{u - b_n}.$$

Integrating and passing from logarithms to numbers, we have the form required.

In the next Chapter it will be shown that any rational function $f(u)$ has an algebraic addition-theorem; that is, if u and v are two independent variables, $f(u + v)$ may be expressed *algebraically* in terms of $f(u)$ and $f(v)$.

* Cf. Appell et Lacour, *loc. cit.*, p. 9.

TRIGONOMETRIC FUNCTIONS.

ART. 13. In the presentation of some of the fundamental properties of the trigonometric functions we shall apply methods which are later used in a similar manner in the theory of the elliptic functions.

The polynomial $a_0 + a_1 u + a_2 u^2 + \cdots + a_n u^n = F(u)$ is a one-valued function with a *finite* number of terms each having a positive integral exponent. This integral function is of the nth degree.

Another class of one-valued functions are those where n has an infinite value. Such functions, when convergent for all finite values of the variable, are known as integral *transcendental functions.*

For example,

$$\sin u = u - \frac{u^3}{3!} + \frac{u^5}{5!} - \cdots$$

is a series which is convergent for all finite values of u and is a regular function at all points at a finite distance from the origin. It becomes zero for the values

$$u = 0, \pm \pi, \pm 2\pi, \pm 3\pi, \cdots .$$

We know that the decomposition of a polynomial into a product of linear factors is the fundamental problem of algebra. It is natural to seek whether the integral transcendents may not also be decomposed into their prime factors. Euler gave the celebrated formula

$$\frac{\sin \pi u}{\pi u} = \left(1 - \frac{u^2}{1}\right)\left(1 - \frac{u^2}{4}\right)\left(1 - \frac{u^2}{9}\right)\cdots\left(1 - \frac{u^2}{n^2}\right)\cdots,$$

a formula which is true for every finite value of u. Cauchy was the first to treat the subject in general. Although he did not complete the theory, he recognized that if a is a root of the integral transcendent $f(u)$, it is necessary in many cases to join to the product of the infinite number of factors such as $1 - \frac{u}{a}$ a certain exponential factor $e^{P(u)}$, where $P(u)$ is a power series in positive powers of u. Weierstrass gave a complete treatment of this subject.

ART. 14. We may establish first the results derived by Cauchy. With Hermite (*loc. cit.*, p. 84) suppose that $a_1, a_2, a_3, \ldots$ are the roots of the integral transcendental function $f(u)$ which are arranged in the order of increasing moduli. Further suppose that they are all different and none of them is zero.

Suppose *first* that the series

$$\sum_{i=1}^{i=\infty} \frac{1}{\text{mod } a_i},$$

formed by the inverse of the moduli of the roots, is convergent. The same will (as shown below) also be true of the series

$$\sum_{i=1}^{i=\infty} \frac{1}{\text{mod } (a_i - u)},$$

whatever the value of u, excepting the values $u = a_1, a_2, \ldots$, which make the series infinite. It follows then that $\sum_{i=1}^{i=\infty} \frac{1}{u - a_i}$ will represent an analytic function in the whole plane.*

To prove the above statement consider the two infinite series Σu_n and Σv_n, of which the first is convergent.

The second series will also be convergent if we have

$$v_n < k u_n \quad (k \text{ constant})$$

for all values of n starting with a certain limit. If we write $u_n = \frac{1}{\text{mod } a_n}$ and $v_n = \frac{1}{\text{mod } (a_n - u)}$, the condition just written is

$$\frac{\text{mod } a_n}{\text{mod } (a_n - u)} < k.$$

From the inequality

$$\text{mod } a_n < \text{mod } (a_n - u) + \text{mod } u,$$

we have

$$\frac{\text{mod } a_n}{\text{mod } (a_n - u)} < 1 + \frac{\text{mod } u}{\text{mod } (a_n - u)},$$

which demonstrates the theorem since $\frac{\text{mod } u}{\text{mod } (a_n - u)}$ decreases indefinitely when n increases.

It is seen at once that

$$\frac{f'(u)}{f(u)} - \sum \frac{1}{u - a_n}$$

is a regular function for all finite points of the plane. This difference we may represent by $G'(u) = \frac{dG(u)}{du}$.

We thus have

$$\frac{f'(u)}{f(u)} - \sum \frac{1}{u - a_n} = G'(u).$$

Multiplying by du and integrating, taking zero as the lower limit, we have

$$\log \frac{f(u)}{f(o)} - \sum \log \left(1 - \frac{u}{a_n}\right) = G(u) - G(o) = G_1(u), \text{ say;}$$

or

$$\frac{f(u)}{f(o)} = e^{G_1(u)} \prod \left(1 - \frac{u}{a_n}\right),$$

* See Osgood, *Lehrbuch der Funktionentheorie*, p. 75 and p. 259.

where the product is to be taken over the finite or infinite number of factors $1-\frac{u}{a_1}$, $1-\frac{u}{a_2}$, ...

This result is due to Cauchy, *Exercises de Mathématiques*, IV.

ART. 15. We may *next* consider the general case and, following the methods of Mr. Mittag-Leffler,* establish the important results of Weierstrass † who extended to these integral transcendents the fundamental theorem of algebra. When the series of the preceding article $\sum\frac{1}{\operatorname{mod} a_n}$ is *not* convergent, the sum $\sum\frac{1}{u-a_n}$ no longer represents an analytic function; but by subtracting from each term a part of its development arranged according to decreasing powers of n, Mr. Mittag-Leffler has shown that it is possible to form with these differences an absolutely convergent series.

Let
$$P_\omega(u)=\frac{1}{a_n}+\frac{u}{a_n{}^2}+\cdots+\frac{u^{\omega-1}}{a_n{}^\omega},$$

so that
$$\frac{1}{u-a_n}+P_\omega(u)=\frac{u^\omega}{a_n{}^\omega(u-a_n)}.$$

We may next show that by a suitable choice of ω we may render the series
$$\sum\left(\frac{1}{u-a_n}+P_\omega(u)\right),\quad\text{or}\quad\sum\frac{u^\omega}{a_n{}^\omega(u-a_n)}$$

convergent.

In the first place it may happen that $\sum\frac{1}{\operatorname{mod} a_n}$ being divergent, the series formed by raising each term of the divergent series to a certain power is convergent. For example, in the case of the divergent harmonic series $\sum\frac{1}{n}$, we know that $\sum\frac{1}{n^\mu}$, where $\mu>1$, is convergent.

Hence we may fix a number ω such that the series $\sum\frac{1}{\operatorname{mod} a_n{}^{\omega+1}}$ is convergent.

We may then conclude from this series the convergence of
$$\sum\frac{1}{\operatorname{mod} a_n{}^\omega(u-a_n)},\quad\text{and consequently of}\quad\sum\frac{u^\omega}{a_n{}^\omega(u-a_n)}.$$

* See Mittag-Leffler, *loc. cit.*, p. 38; and *Comptes rendus*, t. 94, pp. 414, 511, 713, 781, 938, 1040, 1105, 1163; t. 95, p. 335.

† Weierstrass, Werke, Bd. III, p. 100. See also Casorati, *Aggiunte a recenti lavori dei Sigi Weierstrass e Mittag-Leffler; Annali di Matematica*, serie ii, t. X; Harkness and Morley, *Theory of Functions*, p. 188; Forsyth, *Theory of Functions*, p. 335.

For, if we put

$$u_n = \frac{1}{\operatorname{mod} a_n^{\omega+1}}, \qquad v_n = \frac{1}{\operatorname{mod} a_n^{\omega}(a_n - u)},$$

we have for the ratio $\frac{v_n}{u_n}$ the same value as before,

$$\frac{v_n}{u_n} = \frac{\operatorname{mod} a_n}{\operatorname{mod} (a_n - u)}.$$

We must, however, always know that we are passing to a convergent series when we raise each term of the divergent series to a certain power.

For example,* consider the divergent series $\sum \frac{1}{\log n}$. It is seen that the series $\sum \frac{1}{(\log n)^{\omega}}$ is also divergent, however great ω be taken.

For writing

$$S_n = \frac{1}{(\log 2)^{\omega}} + \frac{1}{(\log 3)^{\omega}} + \cdots + \frac{1}{(\log n)^{\omega}},$$

it is seen that

$$S_n > \frac{n-1}{(\log n)^{\omega}}.$$

Note that

$$\frac{n-1}{(\log n)^{\omega}} = \frac{n}{(\log n)^{\omega}} - \frac{1}{(\log n)^{\omega}},$$

and that the first term on the right increases with increasing n, while the second term tends towards zero. The series is therefore divergent.

ART. 16. In such cases as the above Weierstrass took for ω a value which changes with n. With Weierstrass write $\omega = n - 1$. The given series may be written

$$\sum \frac{u^{n-1}}{a_n^{n-1}(u - a_n)} = \frac{1}{u} \sum \frac{-u^n}{a_n^n\left(1 - \frac{u}{a_n}\right)}.$$

This series is convergent; for writing

$$U_n = \operatorname{mod} \frac{u^n}{a_n^n\left(1 - \frac{u}{a_n}\right)},$$

it is seen that $\sqrt[n]{U_n}$ tends towards zero for $n = \infty$. We know (cf. Art. 86) that it is sufficient for this limit to be less than unity for a convergent series.

It follows as before that the expression

$$\frac{f'(u)}{f(u)} - \sum \left[\frac{1}{u - a_n} + P_{\omega}(u)\right]$$

* This example is due to Mr. Stern and cited by Hermite, *loc. cit.*, p. 86.

is a function that remains regular for all finite values of u. It must therefore be expressible in a convergent power series in ascending powers of u.

Write this series $= \dfrac{dG(u)}{du}$; and for brevity write

$$Q_\omega(u) = \frac{u}{1} + \frac{u^2}{2} + \cdots + \frac{u^\omega}{\omega},$$

so that

$$\int_0^u P_\omega(u)\,du = \frac{u}{a_n} + \frac{u^2}{2\,a_n^2} + \cdots + \frac{u^\omega}{\omega a_n^\omega} = Q_\omega\left(\frac{u}{a_n}\right).$$

We have at once

$$\frac{f(u)}{f(o)} = e^{G(u)} \prod_n \left\{\left(1 - \frac{u}{a_n}\right) e^{Q_\omega\left(\frac{u}{a_n}\right)}\right\},$$

which formula gives an analytic expression, in which the roots are set forth, of the integral transcendental function.

The quantities $\left(1 - \dfrac{u}{a_n}\right) e^{Q_\omega\left(\frac{u}{a_n}\right)}$ are called *primary functions** by Weierstrass.

Suppose next that $f(u)$ has equal roots, say, of the pth order of multiplicity. We see immediately that the formula does not undergo any analytic modification, it being sufficient to raise the corresponding primary factor to the pth power.

Finally if we admit the case of a function having a zero root of the qth order, we have only to proceed with the quotient $\dfrac{f(u)}{u^q}$, the result differing from the preceding only by the presence of the factor u^q. (See Hermite, *loc. cit.*)

INFINITE PRODUCTS.

ART. 17. It may be shown that the infinite product

$$(1 + a_1)(1 + a_2) \ldots (1 + a_n) \ldots$$

has a definite value, if

$$|a_1| + |a_2| + \cdots + |a_n| + \cdots$$

represents a convergent series.†

* See Osgood, *Ency. der math. Wiss.*, Band II², Heft 1, pp. 78 *et seq.*; Forsyth, *Theory of Functions*, pp. 92 *et seq.*; Weierstrass, Werke, II, p. 100; Harkness and Morley, *Theory of Functions*, p. 190.

† Cf. Mittag-Leffler, *Acta Math.*, Vol. IV, pp. 30 *et seq.*; Dini, *Ann. di mat.* (2), 2, 1870, p. 35; Harkness and Morley, *Theory of Functions*, p. 82; and especially Pringsheim, *Ueber die Convergenz unendlicher Producte*, *Math. Ann.*, Bd. 33; Weierstrass, Werke, I, p. 173.

For write

$$P_n = (1 + a_1)(1 + a_2) \ldots (1 + a_n).$$

Then evidently

$$P_n - P_{n-1} = a_n P_{n-1},$$

and

$$P_n = 1 + a_1 + a_2 P_1 + a_3 P_2 + \cdots + a_n P_{n-1}.$$

Hence when n becomes indefinitely large, the series P_n will tend towards a definite limit if the series

$$1 + a_1 + a_2 P_1 + a_3 P_2 + \cdots + a_n P_{n-1} + a_{n+1} P_n + \cdots \tag{1}$$

is convergent, the limit, if there is one, being the sum of this series.

Consider first the case where the quantities $a_1, a_2, \ldots$ are real and positive or zero. The quantities $P_1, P_2, \ldots$ are then at least equal to unity, and consequently, in order that the series (1) be convergent, it is necessary that the series

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots \tag{2}$$

be convergent.

Further, if (2) is convergent, it may be shown as follows that (1) is convergent.

The product

$$P_n = (1 + a_1)(1 + a_2) \ldots (1 + a_n),$$

when developed is

$$1 + a_1 + a_2 + \cdots + a_n + a_1 a_2 + \cdots + a_1 a_2 \cdots a_n.$$

Writing

$$A_n = a_1 + a_2 + \cdots + a_n \quad \text{and}$$
$$A = a_1 + a_2 + \cdots + a_n + a_{n+1} + \cdots,$$

it is seen that

$$P_n < 1 + \frac{A_n}{1!} + \frac{A_n^2}{2!} + \cdots + \frac{A_n^n}{n!},$$

or

$$P_n < e^{A_n} < e^A,$$

which proves the proposition.

Next let the quantities $a_1, a_2, \ldots, a_n, \ldots$, previously supposed to be real and positive, take any values.

Then the series

$$1 + a_1 + a_2 + \cdots + a_n + a_1 a_2 + \cdots + a_1 a_2 \cdots a_n$$

is evidently convergent if the corresponding series made by taking the absolute values of the a's is convergent.

Hence the condition for the convergence of the product $\prod_{n=1}^{n=\infty} (1 + a_n)$ is that the series $\sum_{n=1}^{n=\infty} | a_n |$ be convergent.

ART. 18. If further the series

$$|a_1| + |a_2| + |a_3| + \cdots + |a_n| + \cdots \tag{3}$$

is convergent, the product

$$(1 + a_1 u)(1 + a_2 u)(1 + a_3 u) \ldots (1 + a_n u) \ldots \tag{4}$$

is convergent for all values of the variable u, except infinity. For if r is the modulus of u, the series

$$|a_1| r + |a_2| r + \cdots + |a_n| r + \cdots$$

is convergent whatever be the modulus r.

ART. 19. We shall next show that *when the series* (3) *is convergent, the product* (4) *may be expressed as an integral power series* $P(u)$ *which is convergent** *for all finite values of* u.

Consider the product of n real factors

$$P_n(r) = (1 + a_1 r)(1 + a_2 r) \ldots (1 + a_n r),$$

$a_1, a_2, \ldots$ being taken as real quantities and positive. Let these n factors be multiplied together.

If $s_1^{(n)}$ denotes the sum of the n quantities $a_1, a_2, \ldots a_n$; $s_2^{(n)}$ the sum of the products of these n quantities taken two at a time; $s_3^{(n)}$ the sum of the products taken three at a time, etc., we will have

$$P_n(r) = 1 + s_1^{(n)} r + s_2^{(n)} r^2 + \cdots + s_m^{(n)} r^m + \cdots + s_n^{(n)} r^n.$$

Since any term $s_m^{(n)} r^m$ is less than $P_n(r)$ or its limit $P(r)$, where $n = \infty$, it follows that $s_m^{(n)} r^m$ tends toward a definite limit $s_m r^m$ when n increases indefinitely; thus the sum $s_m^{(n)}$ of the products taken m at a time of the n first terms of the convergent series

$$a_1 + a_2 + a_3 + \cdots$$

tends toward a definite limit s_m when n increases indefinitely.

But since

$$P_n(r) > 1 + s_1^{(n)} r + s_2^{(n)} r^2 + \ldots + s_m^{(n)} r^m,$$

if leaving m fixed we let n increase indefinitely, it is seen that

$$P(r) > 1 + s_1 r + s_2 r^2 + \ldots + s_m r^m.$$

Since the sum $S_m(r)$ of the m first terms (m indefinitely large) of the series

$$1 + s_1 r + s_2 r^2 + \cdots \tag{5}$$

is less than a finite quantity $P(r)$, we conclude that this sum tends toward a limit $S(r)$ which is less than or equal to $P(r)$.

* See Briot et Bouquet, *Fonctions Elliptiques*, pp. 301 *et seq.;* Osgood, *Lehrbuch der Funktionentheorie*, pp. 460 *et seq.;* Tannery et Molk, *Fonctions Elliptiques*, t. I, pp. 28 *et seq.;* Picard, *Traité d'Analyse*, I, 2, p. 136; Bromwich, *Theory of Infinite Series*, pp. 101 *et seq.*

On the other hand, each of the terms of the product

$$P_n(r) = 1 + s_1^{(n)}r + s_2^{(n)}r^2 + \cdots + s_n^{(n)}r^n$$

being less than the corresponding term of the sum

$$S_n(r) = 1 + s_1r + s_2r^2 + \cdots + s_nr^n,$$

the sum $S_n(r)$ is greater than $P_n(r)$ and consequently its limit $S(r)$ is greater than or equal to $P(r)$.

It follows that $S(r) = P(r)$.

Consider next the product of n imaginary factors

$$P_n(u) = (1 + a_1u)(1 + a_2u) \ldots (1 + a_nu),$$

where $|a_1| + |a_2| + |a_3| + \cdots$ is a convergent series.

It follows as above that

$$P_n(u) = 1 + \sigma_1^{(n)}u + \sigma_2^{(n)}u^2 + \cdots + \sigma_m^{(n)}u^m + \cdots + \sigma_n^{(n)}u^n.$$

Any coefficient $\sigma_m^{(n)}$ is a sum of imaginary terms whose moduli form quantities corresponding to $s_m^{(n)}$ above. Consequently when n increases indefinitely, since $s_m^{(n)}$ tends towards a limit s_m, the sum $\sigma_m^{(n)}$ tends towards a limit σ_m.

The series

$$1 + \sigma_1u + \sigma_2u^2 + \sigma_3u^3 + \cdots = P(u), \quad \text{say,} \tag{6}$$

is convergent, since the moduli of its terms are less than the corresponding terms of (5).

The sum $S_n(u)$ of the n first terms of this series contains all the terms of $P_n(u)$. Further, the terms of the difference $S_n(u) - P_n(u)$ have for their moduli the corresponding terms of the difference $S_n(r) - P_n(r)$ and consequently tend towards zero, when n increases indefinitely. We conclude that $S_n(u)$ tends towards a limit $P(u)$.

Thus the function defined by the product (4) is developable in a uniformly convergent series (6) arranged according to increasing powers of u.

ART. 20. *The sine-function.* — As an example of Art. 16, we note that the function $f(u) = \dfrac{\sin \pi u}{\pi u}$ has for its roots all the positive and negative integers $\pm 1, \pm 2, \pm 3, \cdots$.

The series $\sum \dfrac{1}{\text{mod } a_n}$ is here divergent, but the series $\sum \dfrac{1}{(\text{mod } a_n)^2}$ is convergent.

We may consequently put $\omega = 1$ in Weierstrass's formula. The primary factors are therefore

$$\left(1 - \frac{u}{n}\right)e^{\frac{u}{n}}.$$

Noting that $f(o) = 1$, and admitting* that $G(u) = 0$ (see Vivanti-Gutzmer, *Eindeutige analytische Funktionen*, p. 163), we have the formula

$$\frac{\sin \pi u}{\pi u} = \prod_n \left\{\left(1 - \frac{u}{n}\right)e^{\frac{u}{n}}\right\},$$

$$n = \pm 1, \pm 2, \pm 3, \cdots .$$

Uniting the integers that are equal and of opposite sign we have Euler's formula:

$$\frac{\sin \pi u}{\pi u} = \left(1 - \frac{u^2}{1}\right)\left(1 - \frac{u^2}{4}\right)\cdots\left(1 - \frac{u^2}{n^2}\right)\cdots .$$

The periodic property of the sine-function may be deduced from this definition. For write

$$F(u) = Au(u-1)(u-2)\ldots(u-n) \quad \textit{multiplied by}$$
$$\cdot\,(u+1)(u+2)\ldots(u+n),$$

where A is a constant.

Changing u into $u + 1$, we have

$$F(u+1) = A(u+1)\,u\,(u-1)\ldots(u-n+1) \quad \textit{multiplied by}$$
$$(u+2)\quad(u+3)\ldots(u+n+1).$$

It follows that

$$F(u+1) = F(u)\,\frac{u+n-1}{u-n};$$

or, when $n = \infty$,

$$F(u+1) = -F(u).$$

From this we may derive at once the relation

$$\sin(u + \pi) = -\sin u, \quad \text{or} \quad \sin(u + 2\pi) = \sin u.$$

ART. 21. We may write

$$\sin u = u \prod_m{}' \left\{\left(1 - \frac{u}{m\pi}\right)e^{\frac{u}{m\pi}}\right\},$$

where the product extends over all integers $m = \pm 1, \pm 2, \pm 3, \ldots,$ the accent over the product-sign denoting that m does *not* take the value zero.

Owing to the factor $e^{\frac{u}{m\pi}}$, the above product is convergent whatever be the order of the factors.

For any one of the factors $\left(1 - \dfrac{u}{m\pi}\right)e^{\frac{u}{m\pi}}$ may be written

$$e^{\frac{u}{m\pi} + \log\left(1 - \frac{u}{m\pi}\right)} = e^{-\left[\frac{1}{2}\left(\frac{u}{m\pi}\right)^2 + \frac{1}{3}\left(\frac{u}{m\pi}\right)^3 + \cdots\right]},$$

* If we expand the sine-function on the left by Maclaurin's Theorem, and equate like powers of u on either side of the equation, it follows that $e^{G(u)} = 1$.

and passing to the product of such terms we note that the series

$$\frac{1}{2}\sum_m\left(\frac{u}{m\pi}\right)^2,\quad \frac{1}{3}\sum_m\left(\frac{u}{m\pi}\right)^3,\ \cdots$$

are absolutely convergent.

Since m takes all integral values from $-\infty$ to $+\infty$ excepting zero, we may change the sign in the above product and have

$$\sin u = u\prod_m{}' \left\{\left(1+\frac{u}{m\pi}\right)e^{-\frac{u}{m\pi}}\right\}.$$

Next changing u to $-u$ and comparing the resulting product with the one previously derived, we see that

$$\sin(-u) = -\sin u.$$

The point $u = \infty$ is an essential singularity of $\sin u$. For if we put $u = \frac{1}{v}$ we see that within an area as small as we wish about $v = 0$, the function $\sin\frac{1}{v}$ admits an infinity of zeros $v = \frac{1}{m\pi}$, m being any indefinitely large integer. It follows from what we saw in Art. 3 that $v = 0$ or $u = \infty$ is an essential singularity.

ART. 22. *The function cot u.* — This function may be derived from the sine-function from the formula

$$\cot u = \frac{d}{du}\log\sin u.$$

It follows from the above formula * for $\sin u$ that

$$\cot u = \frac{1}{u}+\left(\frac{1}{u-\pi}+\frac{1}{\pi}\right)+\left(\frac{1}{u-2\pi}+\frac{1}{2\pi}\right)+\cdots$$
$$+\left(\frac{1}{u+\pi}-\frac{1}{\pi}\right)+\left(\frac{1}{u+2\pi}-\frac{1}{2\pi}\right)+\cdots.$$

From this expression we have at once

$$\cot(-u) = -\cot u.$$

We also note that the points $0, \pm\pi, \pm 2\pi, \ldots$ are simple poles and that the residue with respect to each of these poles is unity.

With respect to any of these poles, say $u = \pi$, the difference

$$\cot u - \frac{1}{u-\pi}$$

is a regular function in the neighborhood of $u = \pi$.

* Eisenstein (*Crelle's Journ.*, Bd. 35, p. 191) makes use of this formula for $\sin u$ together with the expression for $\cot u$ and establishes a complete theory for the trigonometric functions.

The point $u = \infty$ is an essential singularity.

In a more condensed form we may write

$$\cot u = \frac{1}{u} + \sum_m{}' \left(\frac{1}{u - m\pi} + \frac{1}{m\pi} \right),$$

where the summation extends over all integers from $-\infty$ to $+\infty$ excepting zero.

The function

$$\frac{1}{\sin^2 u} = -\frac{d}{du} \cot u = \frac{1}{u^2} + \sum_m{}' \frac{1}{(u - m\pi)^2}$$

is an *even* function which has as *double poles* the points $0, \pm\pi, \pm 2\pi, \cdots$. The principal part relative to the pole $u = m\pi$ is $\frac{1}{(u - m\pi)^2}$.

From the preceding formulas the periodicity of the circular functions is easily established.

The expression of $\frac{1}{\sin^2 u}$ is seen to remain unchanged when π is added to u.

For the cotangent consider the difference $\cot(u + \pi) - \cot u$. We find that the expression

$$\left(\frac{1}{u + \pi} - \frac{1}{u} \right) + \left(\frac{1}{u} - \frac{1}{u - \pi} \right) + \left(\frac{1}{u - \pi} - \frac{1}{u - 2\pi} \right) + \cdots$$
$$+ \left(\frac{1}{u + 2\pi} - \frac{1}{u + \pi} \right) + \left(\frac{1}{u + 3\pi} - \frac{1}{u + 2\pi} \right) + \cdots$$

is zero.

Further, from the relation

$$\cot(u + \pi) = \cot u$$

we may derive the periodicity of the sine-function. For multiplying both sides of this expression by du and integrating, we have

$$\log \sin(u + \pi) = \log \sin u + \log C, \quad \text{or}$$
$$\sin(u + \pi) = C \sin u.$$

In this formula put $u = -\frac{\pi}{2}$, and we have $C = -1$.

ART. 23. *Development in series.* — If we note that

$$\frac{1}{u - m\pi} = -\frac{1}{m\pi} - \frac{u}{m^2\pi^2} - \frac{u^2}{m^3\pi^3} - \cdots,$$

it is seen from the expansion of the cotangent that

$$\cot u = \frac{1}{u} - s_1 \frac{u}{\pi^2} - s_2 \frac{u^3}{\pi^4} - s_3 \frac{u^5}{\pi^6} - \cdots,$$

where $s_1 = \sum' \frac{1}{m^2}$, $s_2 = \sum' \frac{1}{m^4}$, $s_3 = \sum' \frac{1}{m^6}$, etc. The sums $\sum' \frac{1}{m}$, $\sum' \frac{1}{m^3}$, etc., are evidently zero, since the positive terms are destroyed by the corresponding negative terms.

To determine the values $s_1, s_2, \ldots$, multiply the above formula by du and integrate.

We thus have

$$\log \sin u = \log A + \log u - \frac{s_1}{2}\frac{u^2}{\pi^2} - \frac{s_2}{4}\frac{u^4}{\pi^4} - \frac{s_3}{6}\frac{u^6}{\pi^6} - \cdots$$

or
$$\sin u = Au\, e^{-\frac{s_1}{2}\frac{u^2}{\pi^2} - \frac{s_2}{4}\frac{u^4}{\pi^4} - \cdots}.$$

Since $\frac{\sin u}{u} = 1$, when u approaches zero, it follows that $A = 1$. Further, since

$$\sin u = u - \frac{u^3}{\underline{|3}} + \frac{u^5}{\underline{|5}} - \cdots,$$

we have by equating like powers of u, after the exponential function on the right has been developed in series,

$$s_1 = \frac{\pi^2}{3},\quad s_2 = \frac{\pi^4}{3^2 \cdot 5},\quad s_3 = \frac{2\pi^6}{3^3 \cdot 5 \cdot 7},\quad \cdots$$

(see Bertrand's *Calcul Différentiel*, p. 421).

Noting that

$$s_\nu = \sum' \frac{1}{m^{2\nu}} = 2\sum_{n=1}^{n=\infty} \frac{1}{n^{2\nu}},$$

we have *

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} = \frac{1}{6} \cdot \frac{2\pi^2}{2!},$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90} = \frac{1}{30} \cdot \frac{2^3\pi^4}{4!},$$

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \cdots = \frac{\pi^6}{945} = \frac{1}{42} \cdot \frac{2^5\pi^6}{6!},$$

$$1 + \frac{1}{2^8} + \frac{1}{3^8} + \cdots = \frac{\pi^8}{9450} = \frac{1}{30} \cdot \frac{2^7\pi^8}{8!},$$

$$1 + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \cdots = \frac{\pi^{10}}{93555} = \frac{5}{66} \cdot \frac{2^9\pi^{10}}{10!},$$

. . . .

The numbers $\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}, \cdots$ are the so-called *Bernoulli numbers* (cf. Staudt, *Crelle's Journ.*, Bd. 21, p. 372).

* See Biermann, *Theorie der analytischen Functionen*, p. 326; Jordan, *Traité d'Analyse*, t. I, p. 360.

THE GENERAL TRIGONOMETRIC FUNCTIONS.

ART. 24. We know that $\sin 2u = \dfrac{2\cot u}{1+\cot^2 u}$ and

$$\cos 2u = \frac{\cot^2 u - 1}{\cot^2 u + 1}.$$

Further, since any rational function of a trigonometric function may be expressed rationally in terms of the sine and cosine, we may consider as the general case any rational function of $\sin u$ and $\cos u$ which in turn is a rational function of $\cot\frac{u}{2}$. These functions remain unchanged when we add to the argument u any positive or negative multiple of 2π. We say that 2π is a *primitive period* of these functions. Writing $\cot\frac{u}{2} = \tau$, we have here to consider any rational function of τ. Such a function is consequently a one-valued function of τ and has only polar singularities.

As in the case of the rational functions we shall find two forms for the representation of the trigonometric functions, the one corresponding to the decomposition of rational functions into partial fractions and the other corresponding to the expression of a rational function as a quotient of linear factors.

ART. 25. *First form.* — Write

$$f(u) = \frac{F(\sin u, \cos u)}{G(\sin u, \cos u)}$$

where the numerator and denominator are integral functions of $\sin u$ and $\cos u$.

Further, since

$$\sin u = \frac{e^{iu} - e^{-iu}}{2i} \quad \text{and} \quad \cos u = \frac{e^{iu} + e^{-iu}}{2},$$

it is seen that

$$f(u) = e^{\nu iu}\frac{A_0(e^{2iu})^m + A_1(e^{2iu})^{m-1} + \cdots + A_{m-1}e^{2iu} + A_m}{B_0(e^{2iu})^n + B_1(e^{2iu})^{n-1} + \cdots + B_{n-1}e^{2iu} + B_n},$$

where ν is zero or is an integer and where the A's and B's are constants or zero. Through division we may express $f(u)$ in the form*

$$f(u) = P(e^{iu}) + Q(e^{iu}),$$

where $P(e^{iu})$ is composed of integral (positive or negative) powers of e^{iu}. But in $Q(e^{iu})$ the degree of the numerator is *not* greater than that of the denominator and this denominator does not contain e^{iu} as a factor. Hence $Q(e^{iu}) = \Phi(u)$, say, remains finite when $u = \infty$ and also when $u = -\infty$.

* Cf. Hermite, "*Cours*," *loc. cit.*, p. 121; and also Hermite, *Cours d'Analyse de l'École Polytechnique*, p. 321.

We shall next study the function $\Phi(u)$.

Consider the integral $\int \Phi(u)du$, where the integration is taken over the contour of the rectangle $ABCD$ in which

$$OM = x_0,\ MN = 2\pi,\ AN = NB = a.$$

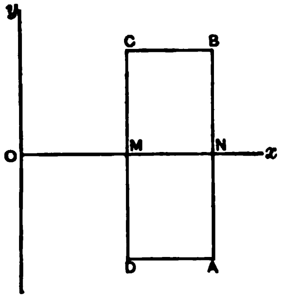


Fig. 1.

If we denote by (AB) the value of this integral taken over the line AB, we have by Cauchy's Theorem (see Art. 96)

$$(AB) + (BC) + (CD) + (DA) = 2\pi i\Sigma,$$

where Σ denotes the sum of the residues of $\Phi(u)$ corresponding to the poles that are situated on the interior of this rectangle. Since x_0 is an arbitrary length, the sides of the rectangle may always be so taken that they are free from the infinities of $\Phi(u)$.

For any point along the line DC we may write

$$u = x_0 + i\tau,$$

where τ is a real quantity that varies from $-a$ to $+a$. We may therefore write

$$(DC) = i\int_{-a}^{+a} \Phi(x_0 + i\tau)d\tau \text{ and similarly}$$

$$(AB) = i\int_{-a}^{+a} \Phi(x_0 + 2\pi + i\tau)d\tau.$$

These two integrals are equal since $\Phi(u) = \Phi(u + 2\pi)$. It follows that

$$(AB) + (CD) = 0, \quad \text{and consequently}$$

$$(DA) + (BC) = 2i\pi\Sigma; \quad \text{or}$$

$$\int_0^{2\pi} \Phi(x_0 - ia + \tau)d\tau - \int_0^{2\pi} \Phi(x_0 + ia + \tau)d\tau = 2i\pi\Sigma. \qquad (1)$$

Next let the constant a become very large and let the corresponding values of

$$\Phi(x_0 - ia + \tau) = Q[e^{i(x_0 - ia + \tau)}] = Q[e^{+a + i(x_0 + \tau)}] \quad \text{and}$$
$$\Phi(x_0 + ia + \tau) = Q[e^{i(x_0 + ia + \tau)}] = Q[e^{-a + i(x_0 + \tau)}]$$

be respectively G and H.

Formula (1) becomes then

$$G - H = i\Sigma \quad \text{or} \quad \Sigma = i(H - G),$$

an expression which gives the sum of the residues of $\Phi(u)$ for all the poles that are situated between the parallels AB and CD when indefinitely produced.

We apply this result to the product

$$\cot\left(\frac{t-u}{2}\right)\Phi(u).$$

Note that

$$\cot\left(\frac{t-u}{2}\right) = i\frac{e^{i(t-u)}+1}{e^{i(t-u)}-1} = i\frac{e^{it}+e^{iu}}{e^{it}-e^{iu}},$$

and that this quantity is equal to $-i$ for $u = \infty$ and to $+i$ for $u = -\infty$.

Hence the sum of the residues of $\cot\left(\frac{t-u}{2}\right)\Phi(u)$ that are situated between the two parallel lines above, is equal to $-G-H$.

We may next compute these residues and equate the sum of the residues computed to $-G-H$.

Let the poles of $\Phi(u)$ be a_1 of order n_1,

a_2 of order n_2,

$\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot$

a_ν of order n_ν.

We know that the residue with respect to a pole a_1 is, if we put $h = u - a_1$, the coefficient of $\frac{1}{h}$ in the development according to ascending increasing powers of h of the expression

$$\cot\left(\frac{t-a_1-h}{2}\right)\Phi(a_1+h).$$

By Taylor's Theorem

$$\cot\frac{t-a_1-h}{2} = \cot\frac{t-a_1}{2} - \frac{h}{1!}\frac{d}{dt}\cot\frac{t-a_1}{2} + \cdots$$
$$\pm\frac{h^{n_1-1}}{(n_1-1)!}\frac{d^{n_1-1}}{dt^{n_1-1}}\cot\frac{t-a_1}{2} \mp \cdots.$$

Further, the expansion of $\Phi(a_1+h)$ in the neighborhood of a_1 is of the form

$$\Phi(a_1+h) = \frac{A}{h} + \frac{A_1}{h^2} + \cdots + \frac{A_{n_1-1}}{h^{n_1}} + A_0 + \text{positive powers of } h.$$

If we put $A = C_{11}$; $A_1 = \frac{C_{21}}{1!}$, $A_2 = \frac{C_{31}}{2!}$; $\ldots$; $A_{n_1-1} = \frac{C_{n_11}}{(n_1-1)}$, it is seen that the coefficient of $\frac{1}{h}$ in the above quotient is

$$C_{11}\cot\frac{t-a_1}{2} - C_{21}\frac{d}{dt}\cot\frac{t-a_1}{2} + \cdots \pm C_{n_11}\frac{d^{n_1-1}}{dt^{n_1-1}}\cot\frac{t-a_1}{2}.$$

The sum of the residues which correspond to the poles of $\Phi(u)$ is therefore represented by

$$\sum_{i=1}^{i=\nu}\left[C_{1i}\cot\frac{t-a_i}{2}-C_{2i}\frac{d}{dt}\cot\frac{t-a_i}{2}+\cdots\pm C_{n_i i}\frac{d^{n_i-1}}{dt^{n_i-1}}\cot\frac{t-a_i}{2}\right].$$

Further, with respect to the pole $u=t$, if we write $u=t+h$ in the quotient

$$\frac{\cos\frac{t-u}{2}}{\sin\frac{t-u}{2}}\Phi(u),$$

it is seen that the coefficient of $\frac{1}{h}$, when h is very small, is $-2\,\Phi(t)$.

We thus have

$$-G-H=\sum_{i=1}^{i=\nu}\left[C_{1i}\cot\frac{t-a_i}{2}-C_{2i}\frac{d}{dt}\cot\frac{t-a_i}{2}+\cdots\pm C_{n_i i}\frac{d^{n_i-1}}{dt^{n_i-1}}\cot\frac{t-a_i}{2}\right]-2\,\Phi(t)$$

or

$$\Phi(t)=\frac{G+H}{2}+\frac{1}{2}\sum_{i=1}^{i=\nu}\left[C_{1i}\cot\frac{t-a_i}{2}-C_{2i}\frac{d}{dt}\cot\frac{t-a_i}{2}+\cdots\pm C_{n_i i}\frac{d^{n_i-1}}{dt^{n_i-1}}\cot\frac{t-a_i}{2}\right],$$

a formula which is similar to the decomposition of a rational function into its simple fractions (see Art. 11).

ART. 26. *Second form.* — If the function $f(u)$ becomes zero on the points $c_1, c_2, \ldots, c_m$ and infinite on the points $b_1, b_2, \ldots, b_n$, it follows at once from the expression of $f(u)$ above that

$$f(u)=Ce^{\nu ui}\frac{(e^{2iu}-e^{2ic_1})(e^{2iu}-e^{2ic_2})\ldots(e^{2iu}-e^{2ic_m})}{(e^{2iu}-e^{2ib_1})(e^{2iu}-e^{2ib_2})\ldots(e^{2iu}-e^{2ib_n})}$$

$$=Ae^{\mu ui}\frac{\sin(u-c_1)\sin(u-c_2)\ldots\sin(u-c_m)}{\sin(u-b_1)\sin(u-b_2)\ldots\sin(u-b_n)},$$

where μ is an integer and C and A are constants.

We shall see later (Arts. 373, 380) that there are analogous representations of the general elliptic function.

REMARK. — The functions which we have just considered admit the period 2π, so that

$$f(u+2\pi)=f(u).$$

If we change the variable by writing $u = \frac{\pi \bar{u}}{\omega}$, so that $f(u) = f\left(\frac{\pi \bar{u}}{\omega}\right) = f_1(\bar{u})$, it is seen that

$$f_1(\bar{u} + 2\,\omega) = f_1(\bar{u}),$$

and consequently $2\,\omega$ is the period of the new function; and further all rational functions of e^{iu} are now rational functions of $e^{\pi i \frac{\bar{u}}{\omega}}$.

In the next Chapter we shall show that any trigonometric function $f(u)$ has an algebraic addition-theorem; or, in other words, $f(u + v)$ may be expressed algebraically through $f(u)$ and $f(v)$.

Analytic Functions.

Art. 27. We have already referred to certain expressions as being *analytic*. The general notion of an analytic function may be had as follows.* Suppose that the function $f(u)$ has a finite number of singular points p_1, p_2, . . . , p_n in the finite portion of the u-plane.†

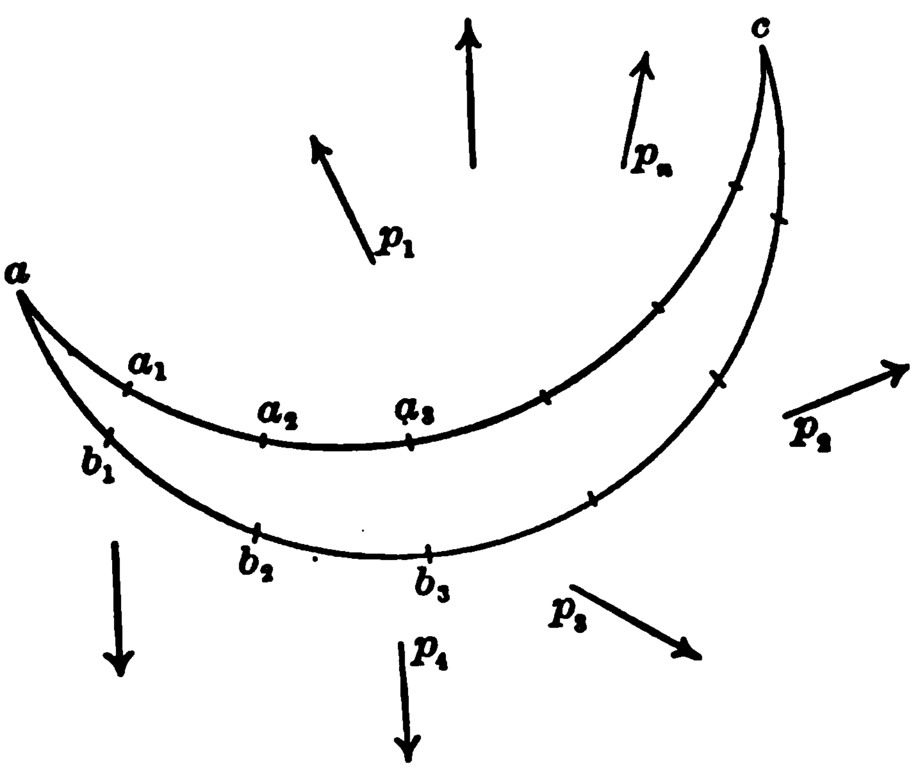


Fig. 2.

From each of these points we suppose a line drawn toward infinity, the only restriction being that no two of the lines intersect or approach each other asymptotically.‡ These lines we may consider replaced by canals which can never be crossed. The canals we suppose infinitesimally broad, so that all the points of the u-plane excepting p_1, p_2, . . . , p_n are either on or outside of the banks of the canals, the points p being the sources from which the canals flow.

We suppose that the function $f(u)$ may be expanded in convergent power series in positive integral powers of the variable at all points except

* Weierstrass, *Abhand. aus der Functionenlehre*, pp. 1 *et seq.;* Werke, 2, p. 135. See also Vivanti-Gutzmer, *loc. cit.*, pp. 334 *et seq.;* Goursat, *Cours d'analyse*, t. 2; Forsyth, *Theory of Functions*, pp. 54 *et seq.;* Harkness and Morley, *Theory of Functions*, p. 105. Osgood (*Funktionentheorie*, p. 189) defines a function as *analytic* in a fixed realm when it has a *continuous* derivative at any point within this realm. It is then *regular* at all points within this realm.

† We have supposed the function defined for the whole plane; it may, however, be restricted to any portion of this plane.

‡ Mr. Mittag-Leffler's "star-theory" suggests that the plane be cut so as to have a starlike appearance before the initial Mittag-Leffler star is formed. See references and remarks at the end of this Chapter.

$p_1, p_2, \ldots, p_n$. Let a be any such point and let $P(u - a)$ denote the power series by which the function $f(u)$ may be represented in the neighborhood of a. The *domain* of the absolute convergence of this series is a circle having a as center and with a radius that extends to the nearest of the points p (see Osgood, *loc. cit.*, p. 285). There may be a point c in the u-plane which lies without this domain and at which the function has a definite value. The function $f(u)$ may also at c be expressed in the form of a power series which has its own domain of convergence.

The question is: *What connection is there between the two power series?*

Suppose next that the points a and c are connected by any line which does not cross a canal. Take any point a_1 on this line which lies within the circle of convergence about a. The value of the function $f(u)$ at the point a_1 is therefore given by $P(a_1 - a)$, and also the derivatives of $f(u)$ at the point a_1 are had from the derivatives of this power series after we have written a_1 for u. It is thus seen that the values of $f(u)$ and of its derivatives at a_1 involve both a and a_1.

Next draw the circle of convergence about a_1 where the arbitrary point a_1 has been so chosen that the circle about a and the circle about a_1 intersect in such a way that there are points common to both circles and also points that belong to either circle but not to both.

For all points u in the domain of a_1 the function $f(u)$ may be represented by a power series, say $P_1(u - a_1)$.

We may show as follows that the coefficients of this power series involve both a and a_1:

For the domain about a we have the series

$$\text{(I)}\qquad f(u) = f(a) + \frac{u-a}{1!}f'(a) + \frac{(u-a)^2}{2!}f''(a) + \cdots = P(u-a);$$

and for points common to the domains of both a and a_1 we have

$$\begin{aligned} P_1(u - a_1) = P(u - a) &= P(a_1 - a + u - a_1) \\ &= P(a_1 - a) + \frac{u - a_1}{1!} P'(a_1 - a) + \cdots \\ &= \sum_{\mu=0}^{\mu=\infty} P^{(\mu)}(a_1 - a)\frac{(u - a_1)^\mu}{\mu!}. \end{aligned}$$

In the domain about a_1 we have

$$\text{(II)}\quad f(u) = f(a_1) + \frac{u-a_1}{1!}f'(a_1) + \frac{(u-a_1)^2}{2!}f''(a_1) + \cdots = P_1(u-a_1),$$

where in this domain

$$f^{(\mu)}(a_1) = P^{(\mu)}(a_1 - a), \quad \mu = 1, 2 \ldots,$$

which quantities are known from (I).

Since the coefficients of $P_1(u - a_1)$ involve both a and a_1, the power series $P_1(u - a_1)$ is sometimes written $P_1(u - a_1, a)$.

At a point u situated within the domains common to both a and a_1 the two series $P(u - a)$ and $P_1(u - a_1)$ give the same value for the function $f(u)$. Hence the second series gives nothing new for such points. But for a point u situated within the domain of a_1 but without the domain of a, the series $P_1(u - a_1, a)$ gives a value of $f(u)$ which cannot be had from $P(u - a)$. The new series gives an additional representation of the function. It is called a *continuation* * of the series which represents the function in the initial domain of a.

Next take a point a_2 situated within the domain of a_1 and upon the line joining a and c. This point a_2 is to be so chosen that its domain coincides in part with the domain of a_1, the other portion of the domain of a_2 lying without that of a_1. The values of $f(u)$ and its derivatives at a_2 are offered by the power series $P_1(u - a_1, a)$ and its derivatives when for u we have written a_2. It is seen that for all points common to the domains a_1 and a_2

$$P_2(u - a_2) = P_1(u - a_1) = P_1(a_2 - a_1 + u - a_2)$$

$$= P_1(a_2 - a_1) + \frac{u - a_2}{1!} P_1'(a_2 - a_1) + \cdots$$

$$= \sum_{\mu=0}^{\mu=\infty} P_1^{(\mu)}(a_2 - a_1) \frac{(u - a_2)^\mu}{\mu!} .$$

In the domain about a_2

$$\text{(III)} \quad f(u) = f(a_2) + \frac{u - a_2}{1!} f'(a_2) + \frac{(u - a_2)^2}{2!} f''(a) + \ldots = P_2(u - a_2),$$

where in this domain

$$f^{(\mu)}(a_2) = P_1^{(\mu)}(a_2 - a_1), \quad \mu = 1, 2, \ldots ,$$

which quantities are known from (II).

It is thus seen that the coefficients of the new power series $P_2(u - a_2)$ which represents $f(u)$ in the neighborhood of a_2 involve the quantities a and a_1, and it may consequently be written $P_2(u - a_2) = P_2(u - a_2, a, a_1)$.

At those points u in the domain of a_2 which do not lie within either of the two earlier circles the series $P_2(u - a_2, a, a_1)$ gives values of $f(u)$ which cannot be derived from either $P(u - a)$ or $P_1(u - a_1)$. Thus the new series is a continuation of the older ones.

Proceeding in this way we may reach all the points of the u-plane where the function behaves regularly. In an indefinitely small neighborhood of those points p which are *essential* singularities of the function $f(u)$, the

* Weierstrass, Werke, Bd. I, p. 84, 1842, employed the word *Fortsetzung;* Méray, who also did much towards the foundation of the theory of functions by means of integral power series, used the expression *cheminement, a series of circles* (see Méray, *Leçons nouvelles sur l'analyse infinitésimale et ses applications géométriques*. Paris, 1894–98).

function can take any arbitrary value (Art. 3); consequently the function may be continued up to this neighborhood but not to the points themselves; while it may be continued up to those p's which are polar singularities (cf. Stolz, *Allgemeine Arithmetik*, Bd. II., p. 100).

The combined aggregate of all the domains is called the *region of continuity* of the function. With each domain of the region of continuity constructed so as to include some portion not included in an earlier domain, a series is associated which is a continuation of the earlier series and gives at certain points values of the function that are not deducible from the earlier series. Such a continuation is called an *element* * of the function. It is seen from above that any later element may be derived from the earlier elements by a definite process of calculation. The aggregate of all the distinct elements is called an *analytic function*, or more correctly a *monogenic* analytic function, the word *monogenic* meaning that the function has a definite derivative. As only functions occur in the present treatise that have definite derivatives, the word *monogenic* will be omitted as superfluous.

ART. 28. We may note that there are functions which although finite and continuous have no definite derivatives. Weierstrass (*Crelle's Journ.*, Bd. 79, p. 29; Werke, Bd. II., p. 71) shows this by means of the function †

$$f(u) = \Sigma a^n \cos b^n u,$$

which, although always finite and continuous, never has a definite derivative, if b is an odd integer and

$$\text{(1st)}\quad ab > 1 + \tfrac{3}{2}\pi \quad\text{or}\quad \text{(2d)}\quad ab^2 > 1 + 3\,\pi^2,$$

where in the first case $ab > 1$ and in the second case ab must be $\geqq 1$.

ART. 29. If c is any point in the region of continuity but not necessarily in the circle of convergence of the initial element about a, it is evident that a value of the function at c may be obtained through the continuations of the initial element. In the formation of each new domain (and therefore of each new element) a certain amount of arbitrary choice is possible; and as a rule there may be different sets of domains (for example in the figure of p. 26 along another path $ab_1\, b_2\, \ldots\, c$), which domains taken together in a set lead to c from the initial point a. So long as we do not cross a canal and consequently do not encircle any of the singular points p, the same value of the function at c is had, whatever be the method of continuation from the initial point a. The function is *one-valued* in the plane where the canals have been drawn.

* Weierstrass, Werke 2, p. 208.

† See also Jordan, *Traité d'Analyse*, t. 3, p. 577; Dini, *Fondamenti per la teorica delle funzioni di variabili reali*, § 126; Wiener, *Crelle*, Bd. 90, p. 221; Picard, *Traité d'Analyse*, t. 2, p. 70; Forsyth, *Theory of Functions*, p. 138; Hadamard's Thesis, *Journ. de Math.*, 1872; Darboux, *Mémoire sur l'approximation, etc.*, *Liouv. Journ.*, 1877; Osgood, *Lehrbuch der Funktionentheorie*, p. 89; Pringsheim, *Ency. der Math. Wiss.*, Bd. II,[1] Heft 1, pp. 36 *et seq.*

In Chapter VI it will be seen that if the crossing of a canal is allowed we may have different values of the function at c; in fact, the function has at c just as many values * as there are different elements $P(u - c)$ which lead back to the same initial element at a.

ART. 30. The whole process given above is reversible when the function is one-valued. We can pass from any point to an earlier point by the use if necessary of intermediate points. We thus return to the point a with a certain functional element, which has an associated domain. From this the original series $P(u - a)$ can be deduced. As this result is quite general, any one of the continuations of a one-valued analytic function represented by a power series can be derived from any other; and consequently the expression of such a function is potentially given by any one element. This subject is treated more fully in Chapter VI.

To effect the above representation of an analytic function it is often necessary to calculate a number of analytic continuations, for each of which we must find the radius of the circle of convergence. Thus (cf. also Mr. Mittag-Leffler,† one of the greatest exponents of Weierstrass's Theory of Functions) it is seen that the manner given above of representing a function by means of its analytic continuations is an extremely complicated one. It seems that Weierstrass scarcely regarded the analytic continuation other than as a mode of definition of the analytic function. As a definition it has great advantages.

But the theory of Cauchy (cf. again Mittag-Leffler), which is founded upon quite different principles, has in most other respects greater advantages.

The representation of a function by means of the integral

$$f(u) = \frac{1}{2\pi i}\int_S \frac{f(z)}{z-u}\,dz,$$

the integration being taken over a closed contour S situated within the region for which $f(u)$ is defined, is fundamental in the derivation of Taylor's Theorem for a function of the complex argument.

Mr. Mittag-Leffler ‡ gives an extension of Taylor's Theorem in his "star-theory" by means of which he treats the "*prolongation of a branch of an analytic function*" in a very comprehensive manner.

General methods of representing an analytic function in the form of

* Vivanti (see Vivanti-Gutzmer, *loc. cit.*, p. 109) gives a method by which a *many-valued* function may be considered as a combination of one-valued functions. See also Weierstrass, *Abel'sche Transcendenten*, Werke, 4, p. 44.

In the sequel we shall by means of canals so arrange our plane or surface on which the function is represented, that the function may be always regarded as *one-valued*.

† *Sur la représentation analytique, etc., Acta Math.*, Bd. 23, p. 45.

‡ Mittag-Leffler, *Sur la représentation, etc., Seconde note, Acta Math.*, Bd. 24, p. 157; *Troisième note, Acta Math.*, Bd. 24, p. 205; *Quatrième note, Acta Math.*, Bd. 26, p. 353; *Cinquième note, Acta Math.*, Bd. 29, p. 101.

an arithmetical expression are given by Hilbert, Runge, and Painlevé (see Vivanti-Gutzmer, *loc. cit.*, pp. 349 *et seq.;* Osgood, *Encyklopädie der Math. Wiss.*, Bd. II², Heft 1, pp. 80 *et seq.*).

ART. 31. *Algebraic addition-theorems.* — We have seen that the rational functions are characterized by the properties of being one-valued and of having no other singularities than poles. These functions possess algebraic addition-theorems.

We have also seen that the general trigonometric functions (rational functions of $\sin u$ and $\cos u$ or of $\cot u/2$) have only polar singularities in the finite portion of the plane. These functions have periods which are integral multiples of *one primitive period* 2π. These properties, however, do not characterize the trigonometric functions; for they belong also to the function $e^{\sin u}$ which is not a trigonometric function. To characterize the trigonometric functions, it is necessary to add the further condition that they have algebraic addition-theorems, as is shown in the next Chapter.

We shall call an *elliptic function* * a one-valued analytic function which has only polar singularities in the finite portion of the plane and which has periods composed of integral (positive or negative) multiples of two primitive periods, say 2ω and $2\omega'$; for example,

$$f(u + 2\omega) = f(u), \qquad f(u + 2\omega') = f(u)$$

and

$$f(u + 2m\omega + 2n\omega') = f(u),$$

where m and n are integers.

A further condition is that these functions have algebraic addition-theorems. Weierstrass characterized as an *elliptic function* any one-valued analytic function as defined above which has only polar singularities in the finite portion of the plane and which possesses an algebraic addition-theorem, the trigonometric functions being limiting cases where one of the primitive periods becomes infinite, as are also the rational functions which have both primitive periods infinite.

EXAMPLES

1. Prove that

$$\cos \pi u = \prod_m \left\{\left(1 - \frac{u}{m + \frac{1}{2}}\right) e^{\frac{u}{(m+\frac{1}{2})}}\right\},$$

where m takes all integral values, negative, zero, and positive.

2. Show that

$$\prod_{m=-\infty}^{m=+\infty} \left\{\left(1 - \frac{u}{m - a}\right) e^{\frac{u}{m-a}}\right\} = \frac{\sin \pi(u + a)}{\sin \pi a} e^{-u\pi \cot \pi a}.$$

* To be more explicit, such a function is an elliptic function in a restricted sense. The more general elliptic functions include also the many-valued functions (see Chapter XXI).

3. Show that

$$\sum_{m=-\infty}^{m=+\infty}\left\{\frac{1}{u+a-m}-\frac{1}{a-m}\right\}=\pi[\cot\pi(u+a)-\cot\pi a].$$

4. Show that

$$\frac{1}{\Gamma(u)}=\prod_{m=1}^{m=\infty}\left(1+\frac{u}{m}\right)\left(\frac{m+1}{m}\right)^{u}=\prod_{m=1}^{m=\infty}\left\{\left(1+\frac{u}{m}\right)e^{-u\log\frac{m+1}{m}}\right\}.$$

[Gauss.]

5. Show that

$$(3,x)=\sum\frac{1}{(x+m)^3}=\pi^3\frac{\cos\pi x}{\sin^3\pi x},$$

$$(4,x)=\sum\frac{1}{(x+m)^4}=\pi^4\left(-\frac{2}{3}\frac{1}{\sin^2\pi x}+\frac{1}{\sin^4\pi x}\right);$$

and that

$$(2g,x)=\sum\frac{1}{(x+m)^{2g}}=\pi^{2g}\left[\frac{a_1}{\sin^2\pi x}+\frac{a_2}{\sin^4\pi x}+\cdots+\frac{a_g}{\sin^{2g}\pi x}\right],$$

$$(2g+1,x)=\sum\frac{1}{(x+m)^{2g+1}}=\pi^{2g+1}\left[\frac{b_1\cos\pi x}{\sin^3\pi x}+\frac{b_2\cos\pi x}{\sin^5\pi x}+\cdots+\frac{b_g\cos\pi x}{\sin^{2g+1}\pi x}\right],$$

where the coefficients $a_1, a_2, \ldots; b_1, b_2, \ldots$ are connected with the Bernoulli numbers in a simple manner and may be found by successive differentiation.

Eisenstein, *Crelle*, Bd. 35, p. 198;
Euler, *Introductio in analysin infinitorum.*

6. Prove that

$$3(4,x)=(2,x)^2+2(1,x)(3,x);$$
$$3(2,0)=\pi^2.$$

CHAPTER II

FUNCTIONS WHICH HAVE ALGEBRAIC ADDITION-THEOREMS

Characteristic properties of such functions in general. The one-valued functions. Rational functions of the unrestricted argument u. Rational functions of the exponential function $e^{\frac{\pi i u}{\omega}}$.

ARTICLE 32. The simplest case of a function which has an algebraic addition-theorem is the exponential function

$$\phi(u) = e^u.$$

It follows at once that

$$e^{u+v} = e^u \cdot e^v,$$

or

$$\phi(u + v) = \phi(u)\phi(v).$$

Such an equation offers a means of determining the value of the function for the sum of two quantities as arguments, when the values of the function for the two arguments taken singly are known.

It is called an *addition-theorem.*

In the example just cited the relation among $\phi(u)$, $\phi(v)$ and $\phi(u + v)$ is expressed through an algebraic equation, and consequently the addition-theorem is called an *algebraic addition-theorem.*

The theorem is true for all values of u and v, real or complex. The exponential function e^u is perhaps best studied by deriving its properties from its addition-theorem.

The sine function has the algebraic addition-theorem

$$\begin{aligned}\sin(u + v) &= \sin u \cos v + \cos u \sin v \\ &= \sin u \sqrt{1 - \sin^2 v} + \sin v \sqrt{1 - \sin^2 u}.\end{aligned}$$

The root signs may be done away with by squaring.

We also have

$$\tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v}, \text{ etc.}$$

We note in the above algebraic addition-theorems that the coefficients connecting $\phi(u)$, $\phi(v)$, and $\phi(u + v)$ are constants, that is, quantities independent of u and v.

With Weierstrass * *the problem of the theory of elliptic functions is to*

* Cf. Schwarz, *Formeln und Lehrsätze zum Gebrauche der elliptischen Functionen,* pp. 1 *et seq.* The Berlin lectures of Prof. Schwarz have been of service in the preparation of this Chapter.

determine all functions of the complex argument for which there exists an algebraic addition-theorem.

Every function for which there exists an algebraic addition-theorem is an elliptic function or a limiting case of one, those limiting cases being the rational functions, the trigonometric and the exponential functions.

ART. 33. We may represent a function of the complex argument by

$$\phi(u) = \xi;$$

and further we shall write

$$\phi(v) = \eta, \quad \phi(u + v) = \zeta.$$

We may assume either that the function ϕ is defined for all real, imaginary, and complex values of the argument, or that this function is defined for a definite region, which, however, must lie in the neighborhood of the origin. Further it is assumed that ϕ has an algebraic addition-theorem. We therefore have, if G represents an integral function with constant coefficients,

$$G(\xi, \eta, \zeta) = 0.$$

We may now derive other properties of such a function from the property that there exists an algebraic addition-theorem.

ART. 34. If we differentiate the function G with respect to u, then, since ξ is independent of η, we have

$$\frac{\partial G}{\partial \xi}\frac{d\xi}{du} + \frac{\partial G}{\partial \zeta}\frac{d\zeta}{du} = 0, \text{ and similarly}$$

$$\frac{\partial G}{\partial \eta}\frac{d\eta}{dv} + \frac{\partial G}{\partial \zeta}\frac{d\zeta}{dv} = 0.$$

Write $u + v = h$ and note that $\dfrac{d\zeta}{du} = \dfrac{d\phi}{dh}\cdot 1 = \dfrac{d\zeta}{dv}$.

We consequently have by subtraction

$$\frac{\partial G}{\partial \xi}\frac{d\xi}{du} - \frac{\partial G}{\partial \eta}\frac{d\eta}{dv} = 0.$$

There are two cases possible:

First. The quantity ζ may appear in the coefficients $\dfrac{\partial G}{\partial \xi}, \dfrac{\partial G}{\partial \eta}$; or

Second. The quantity ζ does *not* appear in these coefficients.

Consider the *first case*. We have the two equations

$$G(\xi, \eta, \zeta) = 0,$$

$$\frac{\partial G}{\partial \xi}\frac{d\xi}{du} - \frac{\partial G}{\partial \eta}\frac{d\eta}{dv} = 0.$$

The first of these equations may be written

$$G(\xi, \eta, \zeta) = a_0\zeta^m + a_1\zeta^{m-1} + \cdots + a_{m-1}\zeta + a_m = 0,$$

where the a's are integral functions of ξ and η; the second equation may be written

$$0 = \frac{\partial G}{\partial \xi}\frac{d\xi}{du} - \frac{\partial G}{\partial \eta}\frac{d\eta}{dv} = A_0\zeta^k + A_1\zeta^{k-1} + \cdots + A_{k-1}\zeta + A_k,$$

where the A's are integral functions of ξ, η, $\frac{d\xi}{du}$ and $\frac{d\eta}{dv}$. If ζ is eliminated from these two expressions, we have

$$H\left(\xi, \frac{d\xi}{du}, \eta, \frac{d\eta}{dv}\right) = 0.$$

In the *second case* where ζ does *not* appear in the coefficients $\frac{\partial G}{\partial \xi}$ and $\frac{\partial G}{\partial \eta}$, we have at once an equation connecting ξ, $\frac{d\xi}{du}$, η and $\frac{d\eta}{dv}$.

This case is, however, the very exceptional one. We have by the above considerations put into evidence a new property of the function ϕ, viz.:

If the function ϕ has an algebraic addition-theorem, there is always an equation of the form

$$H\left(\xi, \frac{d\xi}{du}, \eta, \frac{d\eta}{dv}\right) = 0,$$

where H represents an integral function of its arguments with constant coefficients. The equation is true for all values of u and v which lie within the ascribed region.

This equation being true for all such values of v, we may give to v a special value, and have consequently between ξ and $\frac{d\xi}{du}$ an equation of the form

$$f\left(\xi, \frac{d\xi}{du}\right) = 0,$$

where f denotes an integral function of its arguments.

This equation we shall call the *eliminant equation.** We may write it in the form

$$f[\phi(u), \phi'(u)] = 0.$$

We have therefore proved that *if for the analytic function $\phi(u)$ there exists an algebraic addition-theorem, we also have an algebraic equation between the function and its first derivative, the equation being an ordinary*

* The equation is due to Méray, see Briot et Bouquet, *Théorie des Fonctions Elliptiques*, p. 280; Picard, *Traité d'Analyse*, t. 2, p. 510; Daniels, *Amer. Journ. Math.*, Vol. VI, pp. 254–255.

differential equation of the first order. The argument u does not appear explicitly in the equation.

ART. 35. As the above theorem is made fundamental in many of the following investigations, it is of great importance to note that it is true *without exception.*

In the equation $H = 0$ we may write any arbitrary value for v which belongs to the region considered. If after the substitution of this value of v there remains an equation between ξ and $\frac{d\xi}{du}$, then our conclusions above are correctly drawn; but if after the substitution of this value of v the equation were to vanish in all its coefficients, the theorem remains yet to be established. We take the following method to prove that the theorem is *always* true:

Develop the function H in powers of ξ and $\frac{d\xi}{du}$. The coefficients in this development are either zero or functions of η and $\frac{d\eta}{dr}$, including constants. It is evident that all of the coefficients are not zero, for then the function H would be identically zero.

We represent one of the coefficients which is not zero by

$$f_1\left(\eta, \frac{d\eta}{dr}\right).$$

There must be such a coefficient which contains η and $\frac{d\eta}{dr}$; for otherwise all the coefficients would be independent of these quantities, which therefore would not enter the function H. But since $\frac{\partial G}{\partial \eta}$ is not always zero, these quantities must appear.

In this coefficient $f_1\left(\eta, \frac{d\eta}{dr}\right)$ we give r a definite value, and if the value resulting of the coefficient is different from zero, then in the above development we have an equation connecting ξ and $\frac{d\xi}{du}$.

But if this value of r causes $f_1\left(\eta, \frac{d\eta}{dr}\right)$ to be zero, we try another value and continue until we find a value of r that causes this coefficient to be different from zero, if this be possible.

If, however, the function $f_1\left(\eta, \frac{d\eta}{dr}\right)$ is zero for every value of r, we have an equation of the form

$$f_1[\phi(r), \phi'(r)] = 0,$$

where f_1 is an integral function of its arguments. This equation, however, expresses the same thing as the equation

$$f[\phi(u), \phi'(u)] = 0.$$

only in the first case the argument is v and not u, which of course makes no difference.

If any of the coefficients in the development of the function H contained η alone, $f_2(\eta)$ being such a coefficient, then since f_2 is an integral function of finite degree it can vanish only for a finite number of values of η, and we have only to give η a value such that $f_2(\eta) \neq 0$.

The theorem is therefore true without exception for every analytic function for which there exists an algebraic addition-theorem with constant coefficients; and conversely, as will be shown in Chapters VI and VII, *if a one-valued analytic function $\phi(u)$ has the property that between the function $\phi(u)$ and its first derivative $\phi'(u)$ there exists an algebraic equation whose coefficients are independent of the argument u, the function has an algebraic addition-theorem.*

This eliminant equation (see also Forsyth, *Theory of Functions*, p. 309) must be added as a *latent test* to ascertain whether or not an algebraic equation connecting ξ, η, ζ is one necessarily implying the existence of an algebraic addition-theorem. We must not suppose that every algebraic equation

$$G(\xi, \eta, \zeta) = 0$$

necessarily exacts the existence of an algebraic addition-theorem; neither does the relation

$$\phi(u + v) = F\{\phi(u), \phi'(u), \phi(v), \phi'(u)\},$$

where F denotes a rational function of its arguments, always indicate the existence of such a theorem. (See Art. 46.)

ART. 36. If we solve the equation

$$f\left(\xi, \frac{d\xi}{du}\right) = 0$$

with respect to $\frac{d\xi}{du}$, we have

$$\frac{d\xi}{du} = \psi(\xi),$$

where $\psi(\xi)$ is an algebraic function of ξ.

This equation may be written

$$\frac{d\xi}{\psi(\xi)} = du,$$

or

$$u - u_0 = \int_\alpha^\xi \frac{d\xi}{\psi(\xi)},$$

where u_0 and α denote constants.

It is thus seen that *in the case of every analytic function $\xi = \phi(u)$, for which there exists an algebraic addition-theorem with constant coefficients, the quantity u may be expressed through the integral of an algebraic function of ξ.*

We may so choose the initial value α that $u_0 = 0$, thus having

$$u = \int_{\alpha_0}^{\xi} \frac{d\xi}{\psi(\xi)} = \int_{\alpha_0}^{\xi} \frac{dt}{\psi(t)}.$$

In a similar manner

$$v = \int_{\alpha_0}^{\eta} \frac{dt}{\psi(t)} \quad \text{and} \quad u + v = \int_{\alpha_0}^{\zeta} \frac{dt}{\psi(t)}.$$

On the other hand we have

$$u + v = \int_{\alpha_0}^{\xi} \frac{dt}{\psi(t)} + \int_{\alpha_0}^{\eta} \frac{dt}{\psi(t)}.$$

We thus have the equation

$$\int_{\alpha_0}^{\xi} \frac{dt}{\psi(t)} + \int_{\alpha_0}^{\eta} \frac{dt}{\psi(t)} = \int_{\alpha_0}^{\zeta} \frac{dt}{\psi(t)},$$

a formula which is of fundamental importance.

To illustrate the significance of the above formula consider the following examples:

1. Let $\xi = \phi(u) = e^u$; $\phi'(u) = e^u$.

We therefore have as the eliminant equation

$$\xi = \frac{d\xi}{du},$$

and also

$$\psi(\xi) = \xi.$$

Since $\xi = 1$ when $\alpha_0 = 0$, we may write

$$u = \int_1^{\xi} \frac{dt}{t}.$$

On the other hand, $\zeta = \phi(u+v) = e^{u+v} = e^u \cdot e^v = \phi(u)\,\phi(v) = \xi \cdot \eta$. It follows that

$$\int_1^{\xi} \frac{dt}{t} + \int_1^{\eta} \frac{dt}{t} = \int_1^{\xi\cdot\eta} \frac{dt}{t},$$

or

$$\log \xi + \log \eta = \log \xi\cdot\eta.$$

2. Let $\xi = \phi(u) = \sin u$; $\phi'(u) = \cos u = \sqrt{1 - \sin^2 u} = \sqrt{1 - \xi^2}$.

It follows that $\psi(\xi) = \sqrt{1 - \xi^2}$, and consequently since $u = 0$ for $\xi = 0$,

$$u = \int_0^{\xi} \frac{dt}{\sqrt{1 - t^2}}.$$

Further, since

$$\zeta = \phi(u + v) = \xi\sqrt{1 - \eta^2} + \eta\sqrt{1 - \xi^2},$$

we have

$$\int_0^{\xi} \frac{dt}{\sqrt{1 - t^2}} + \int_0^{\eta} \frac{dt}{\sqrt{1 - t^2}} = \int_0^{\xi\sqrt{1-\eta^2}+\eta\sqrt{1-\xi^2}} \frac{dt}{\sqrt{1 - t^2}},$$

or

$$\sin^{-1} \xi + \sin^{-1} \eta = \sin^{-1} [\xi\sqrt{1 - \eta^2} + \eta\sqrt{1 - \xi^2}].$$

3. If $\xi = \tan u = \phi(u)$, we have

$$\int_0^\xi \frac{dt}{1+t^2} + \int_0^\eta \frac{dt}{1+t^2} = \int_0^{\frac{\xi+\eta}{1-\xi\eta}} \frac{dt}{1+t^2},$$

or
$$\tan^{-1}\xi + \tan^{-1}\eta = \tan^{-1}\left[\frac{\xi+\eta}{1-\xi\eta}\right].$$

ART. 37. We have seen that for every function $\xi = \phi(u)$ for which there exists an algebraic addition-theorem, there exists without exception a differential equation of the form

$$f[\phi(u), \phi'(u)] = 0, \text{ or } f\left(\xi, \frac{d\xi}{du}\right) = 0,$$

where f denotes an integral function of its arguments and where u does not appear explicitly in the equation.

If $\xi = \phi(u)$ is known for a definite value of u, then from the above equation we may determine $\frac{d\xi}{du}$, there being one or more values according to the degree of the equation in $\frac{d\xi}{du}$.

We may now prove the following theorem: *If the function $\xi = \phi(u)$ has an algebraic addition-theorem, the values of all the higher derivatives of $\phi(u)$ with respect to u may be expressed as rational functions with constant coefficients of the function itself and its first derivative; so that if the values of the function and its first derivative are known, the higher derivatives are uniquely determined.*

There are exceptions to the theorem which are noted in the following proof: If we write $\frac{d\xi}{du} = \xi'$, the equation above becomes

$$f(\xi, \xi') = 0, \quad \text{or, say,}$$
$$a_0\xi'^n + a_1\xi'^{n-1} + \cdots + a_{n-1}\xi' + a_n = 0,$$

where n is a positive integer and the a's are integral functions of ξ.

We may assume that $f(\xi, \xi')$ is an irreducible function, that is, it cannot be resolved into two integral functions of ξ, ξ'; for if this were the case, one of the factors put equal to zero might be regarded as the integral equation connecting ξ and ξ'.

We form the derivative $\frac{\partial f(\xi, \xi')}{\partial \xi'}$, which is an integral function in ξ, ξ'. The degree of this derivative in ξ' is one less than the degree of $f(\xi, \xi')$ in ξ'.

Further, the equation $\frac{\partial f(\xi, \xi')}{\partial \xi'} = 0$ is *not* satisfied for all pairs of values ξ, ξ' which satisfy the equation $f(\xi, \xi') = 0$. For if this were the case,

the two equations would have a greatest common divisor, this divisor appearing as a factor of both functions. But by hypothesis $f(\xi, \xi')$ is irreducible. The two equations

$$f(\xi, \xi') = 0,$$
$$\frac{\partial f(\xi, \xi')}{\partial \xi'} = 0,$$

are satisfied by only a finite number of pairs of common values ξ, ξ'. For their discriminant with respect to ξ' is an integral function in the a's; and as this discriminant put equal to zero is the condition of a root common to both equations, we have an integral equation in the a's, that is, in ξ. There are consequently only a finite number of values of ξ which satisfy this condition.

These common roots constitute the exceptional case mentioned at the beginning of the article and are excluded from the further investigation. They may be called the *singular roots*.

We next consider a value $u = u_0$ of the argument, for which $\phi(u_0) = \xi_0$, $\phi'(u_0) = \xi_0'$, where ξ_0, ξ_0' satisfy the equation $f(\xi, \xi') = 0$ but *not* the equation $\frac{\partial f(\xi, \xi')}{\partial \xi'} = 0$.

By differentiation we have

$$\frac{\partial f}{\partial \xi} d\xi + \frac{\partial f}{\partial \xi'} d\xi' = 0.$$

We further assume that the point in question is such that the function has for it a definite derivative.

We may write

$$d\xi = \xi' du,$$
$$d\xi' = \xi'' du.$$

It then follows that

$$\frac{\partial f}{\partial \xi} \xi' + \frac{\partial f}{\partial \xi'} \xi'' = 0$$

or

$$\xi'' = -\frac{\frac{\partial f}{\partial \xi} \xi}{\frac{\partial f}{\partial \xi'}}.$$

ı it is seen that $\xi'' = \phi''(u)$ is rationally expressed through ɜ the singular roots have been excluded, the denominator

ır manner it may be shown that $\xi''' = \phi'''(u)$ may be expressed ɔf a fraction whose denominator is a power of the denominator

which appears in the expression for ξ'' and consequently is different from zero. The same is true for all higher derivatives.

ART. 37a. Suppose that u_1 is a value of the argument u different from u_0 and such that

$$\phi(u_0) = \xi_0 = \phi(u_1),$$
$$\phi'(u_0) = \xi_0' = \phi'(u_1).$$

Further let $\phi(u)$ be an analytic function with an algebraic addition-theorem, and in the neighborhood of u_0 and u_1 let the function $\phi(u)$ be regular. Finally, it is assumed that

$$\left| \left(\frac{\partial f(\xi, \xi')}{\partial \xi'} \right)_{\xi = \xi_0, \xi' = \xi'_0} \right| > 0;$$

that is, ξ_0' does not belong to the singular roots of $f(\xi, \xi') = 0$.

*We assert that $\phi(u)$ under these conditions is a periodic function and that $u_1 - u_0$ is a period of the argument.**

Since the function $\phi(u)$ is regular in the neighborhood of u_0, it may be developed by Taylor's Theorem in the form

$$\phi(u) = \phi(u_0) + \frac{u - u_0}{1!}\phi'(u_0) + \frac{(u - u_0)^2}{2!}\phi''(u_0) + \cdots.$$

In a similar manner we also have

$$\phi(u) = \phi(u_1) + \frac{u - u_1}{1!}\phi'(u_1) + \frac{(u - u_1)^2}{2!}\phi''(u_1) + \cdots.$$

By hypothesis we have

$$\phi(u_0) = \phi(u_1) = \xi_0,$$
$$\phi'(u_0) = \phi'(u_1) = \xi_0'.$$

The derivative $\phi''(u_0)$ may be expressed as a rational function of $\phi(u_0)$, $\phi'(u_0)$ with constant coefficients; $\phi''(u_1)$ has the same form in $\phi(u_1)$, $\phi'(u_1)$.

It follows that

$$\phi''(u_0) = \phi''(u_1), \text{ and in a similar manner}$$
$$\phi'''(u_0) = \phi'''(u_1),$$
$$\cdots\cdots$$
$$\phi^{(n)}(u_0) = \phi^{(n)}(u_1),$$
$$\cdots\cdots$$

Let $u_0 + v$ be a point that lies within the region of convergence of the first of the above series and let $u_1 + v$ be a point situated within the region of convergence of the second.

* Cf. Biermann, *Theorie der analytischen Funktionen*, p. 392.

Instead of u write $u_0 + v$ and $u_1 + v$ in the two series respectively. They become

$$\phi(u_0 + v) = \phi(u_0) + v\phi'(u_0) + \frac{v^2}{2}\phi''(u_0) + \cdots,$$

$$\phi(u_1 + v) = \phi(u_1) + v\phi'(u_1) + \frac{v^2}{2}\phi''(u_1) + \cdots.$$

Consequently, owing to the relations above,

$$\phi(u_0 + v) = \phi(u_1 + v).$$

Next write $u_1 - u_0 = 2\omega$ or $u_1 = u_0 + 2\omega$, and we have

$$\phi(u_0 + v) = \phi(u_0 + v + 2\omega).$$

The quantity v may be regarded as an arbitrary complex quantity, and must satisfy the condition that $u_0 + v$ belongs to the region for which $\phi(u)$ has been defined.

The quantity 2ω is called the period of the argument of the function, less accurately the period of the function.

We may therefore conclude that a function $\phi(u)$ is *periodic*, if it has an algebraic addition-theorem and if there are two points, u_0 and u_1, that are not the singular roots of $f[\phi(u), \phi'(u)] = 0$, for which

$$\phi(u_0) = \phi(u_1) \quad \text{and} \quad \phi'(u_0) = \phi'(u_1).$$

ART. 38. If we have only the one condition that $\phi(u_0) = \phi(u_1)$, we cannot without further data draw the same conclusions about periodicity. If the equation connecting $\phi(u)$ and $\phi'(u)$ is of the first degree in $\phi'(u)$, as is the case of the exponential function, then the second condition, viz., $\phi'(u_0) = \phi'(u_1)$ follows at once. In general this is not the case.

We may, however, effect a conclusion if the assumptions are somewhat changed: Suppose that n is the degree of the equation $f[\phi(u), \phi'(u)] = 0$ with respect to $\phi'(u)$. To every value of $\phi(u)$ there belong at most n values of $\phi'(u)$.

Suppose next that $n + 1$ points $u_0, u_1, \ldots, u_n$ may be found, at which

$$\xi_0 = \phi(u_0) = \phi(u_1) = \cdots = \phi(u_n);$$

and suppose also that $\phi(u)$ is regular in the neighborhood of each of these points, and further suppose that ξ_0 is not a singular root of $f(\xi, \xi') = 0$.

Write

$$\phi'(u_0) = \omega_0,$$

$$\phi'(u_1) = \omega_1,$$

$$\cdots\cdots$$

$$\phi'(u_n) = \omega_n.$$

These $n + 1$ values of $\phi'(u)$ belong to one value of $\xi_0 = \phi(u_0) = \phi(u_1) = \cdots = \phi(u_n)$. But as there can only be n values of $\phi'(u)$ belonging to one

value of $\phi(u)$, it follows that two of the above values of $\phi'(u)$ must be equal, and consequently

$$\phi'(u_\alpha)=\phi'(u_\beta),$$

where α and β are to be found among the integers $0, 1, 2, \ldots, n$. But by hypothesis we also had

$$\phi(u_\alpha)=\phi(u_\beta).$$

It follows from the theorem of the preceding article that $\phi(u)$ is periodic, $u_\alpha - u_\beta$ being a period of $\phi(u)$. We have then the following theorem: *

If it can be shown that a function having an algebraic addition-theorem takes the same value on an arbitrarily large number of positions in the neighborhood of which the function is regular, the function is periodic.

ART. 39. We have seen that in the equation connecting ξ and $\frac{d\xi}{du}$, viz.,

$$f\left(\xi, \frac{d\xi}{du}\right)= 0,$$

the quantity u does not explicitly appear.

Suppose that $\xi = \phi(u)$ is a particular solution of this differential equation. As this differential equation is of the first order, the general solution must contain one arbitrary constant.

We may introduce this constant by writing

$$\zeta = \phi(u + v),$$

the arbitrary constant v being added to the argument. It makes no difference whether we differentiate with regard to u or with regard to $u + v$ since u does not enter the equation *explicitly*.

We consequently have

$$\frac{d\xi}{du}=\frac{d\xi}{d(u+v)}=\phi'(u+v),$$

from which it is seen that the differential equation is satisfied by $\phi(u + v)$. We may therefore write

$$f[\phi(u+v),\ \phi'(u+v)]=0.$$

Further, since by hypothesis $\phi(u)$ has an algebraic addition-theorem, there exists an equation of the form

$$G[\phi(u),\ \phi(v),\ \phi(u+v)]=0.$$

As $\phi(v)$ is a constant, we may determine $\phi(u + v)$ as an algebraic function of $\phi(u)$ from this equation. It is thus shown that the general integral of the differential equation

$$f[\phi(u),\ \phi'(u)]=0$$

* See Daniels, *loc. cit.*, p. 256.

is an algebraic function of the particular solution $\phi(u)$. We note that this theorem is not true for every differential equation in which the argument does *not* enter explicitly, but only for those functions for which there exists an algebraic addition-theorem.

If one succeeds in integrating the differential equation in two ways, the one being by the addition of a constant to the argument of the function and the second in any other way, the addition-theorem is at once deduced by equating the two integrals. (See Chapter XVI.)

The Discussion Restricted to One-valued Functions.

Art. 40. We proceed next with the consideration of the two equations of Art. 34:

$$\frac{\partial G}{\partial \xi}\frac{d\xi}{du} - \frac{\partial G}{\partial \eta}\frac{d\eta}{dv} = 0, \tag{1}$$

$$G(\xi, \eta, \zeta) = 0. \tag{2}$$

The first of these equations may be written in the form

$$A_0\zeta^k + A_1\zeta^{k-1} + \cdots + A_{k-1}\zeta + A_k = 0,$$

where the A's are integral functions of ξ, η, ξ', η', while the second equation has the form

$$a_0\zeta^m + a_1\zeta^{m-1} + \cdots + a_{m-1}\zeta + a_m = 0,$$

the a's being integral functions of ξ, η.

By the application of Euler's method for finding the Greatest Common Divisor of these functions, it is seen that this divisor is an integral function of the A's and a's and ζ, say

$$\bar{g}(\zeta, \xi, \eta, \xi', \eta'). \tag{3}$$

This function equated to zero is the simplest equation in virtue of which equations(1) and (2) are true. If $\bar{g}$ is to be a one-valued function of its arguments and if ξ, η, ξ', η' have each a definite value for a definite value of u, then ζ also must have a definite value, so that the equation (3) must be of the first degree in ζ. Hence ζ must have the form

$$\zeta = F\left(\xi, \frac{d\xi}{du}, \eta, \frac{d\eta}{dv}\right),$$

where F is a rational function of its arguments.

We shall leave for a later discussion (Chapter XXI) the determination of all analytic functions which have algebraic addition-theorems. At present we shall only seek among such functions those which have the property that $\zeta = \phi(u + v)$ may be expressed *rationally* in terms of $\phi(u)$, $\phi'(u)$, $\phi(v)$, $\phi'(v)$. All these functions have the property of being

one-valued analytic functions of the independent variable. The reciprocal theorem is also true: *All analytic functions for which there exists an algebraic addition-theorem and which at the same time are one-valued functions of the independent variable, have the property that* $\phi(u+v)$ *may be expressed rationally through* $\phi(u)$, $\phi'(u)$, $\phi(v)$, $\phi'(v)$. Much emphasis is put upon this theorem, which is proved in Art. 158.

Thus while the general problem has been restricted, we have in fact only limited the discussion in that one-valued analytic functions are treated.

It may be remarked here that the rationality of $\phi(u+v)$ in terms of $\phi(u)$, $\phi'(u)$, $\phi(v)$, $\phi'(v)$ is *not* characteristic of all analytic functions with algebraic addition-theorems, but only of one-valued analytic functions. To such functions for example the remarks of Prof. Forsyth at the conclusion of Chapter XIII of his *Theory of Functions* must be restricted.*

ART. 41. We shall show (cf. Schwarz, *loc. cit.*, p. 2) that

I. All rational functions of the argument u, and

II. All rational functions of an exponential function $e^{\frac{u\pi i}{\omega}}$, where ω is different from zero or infinity, have algebraic addition-theorems and have the property that $\phi(u+v)$ may be expressed rationally in terms of $\phi(u)$, $\phi'(u)$, $\phi(v)$, $\phi'(v)$.

These functions are (cf. Art. 293) limiting cases of elliptic functions; those under heading I are *not* periodic and those under II are *simply* periodic. Finally, we have

III. The elliptic functions, which are *doubly* periodic. These functions have the properties just mentioned under I and II.

We shall see in Art. 78 that there do not exist one-valued functions which have more than two periods. Hence every function for which there exists an algebraic addition-theorem is an elliptic function or a limiting case of one.

ART. 42. Let $\phi(u)$ be a rational function of finite degree and let

$$\xi = \phi(u), \quad \eta = \phi(v), \quad \zeta = \phi(u+v).$$

By means of these three equations we may eliminate u and v and then have an equation of the form

$$\text{(A)} \qquad G(\xi, \eta, \zeta) = 0,$$

where G denotes an integral function of its arguments.

Writing

$$(1) \quad \xi = \phi(u), \qquad (2) \quad \frac{d\xi}{du} = \phi'(u),$$

* Cf. also Biermann, *Theorie der analytischen Funktionen*, p. 393, and *Phragmen, Act. Math.*, Bd. 7, p. 33.

we note that both of these expressions are algebraic in u, and by the elimination of u we have the *eliminant equation*

$$\text{(B)} \qquad f\left(\xi,\ \frac{d\xi}{du}\right) = 0,$$

which is an ordinary differential equation in which the variable u does not appear explicitly.

The equation (A) and the latent test (B) are sufficient to show that *every rational function has an algebraic addition-theorem.*

We shall next show that in the case of the rational functions the argument u may be expressed rationally in terms of ξ and $\frac{d\xi}{du}$.

We assume first that the two equations

$$\xi = \phi(u) \quad \text{and} \quad \frac{d\xi}{du} = \phi'(u)$$

have only one common root, which may be a multiple root. By the method of Art. 40 we derive an equation which is either of the first degree in u, in which case we may solve with respect to u and thus have u rationally expressed through ξ and $\frac{d\xi}{du}$; or it is of a higher degree in u, of the form, say

$$a_0u^m + a_1u^{m-1} + a_2u^{m-2} + \cdot \cdot \cdot + a_m = 0,$$

where the a's are functions of ξ and $\frac{d\xi}{du}$.

Since this equation must represent the multiple root, it must be of the form

$$a_0(u - u_0)^m = 0.$$

This expression developed by the Binomial Theorem becomes

$$a_0u^m - ma_0u^{m-1}u_0 + \cdot \cdot \cdot .$$

It follows from the theory of indeterminate coefficients that

$$a_1 = -ma_0u_0 \quad \text{or} \quad u_0 = -\frac{a_1}{ma_0}.$$

Since a_1 and a_0 are integral functions of ξ and $\frac{d\xi}{du}$, it is seen that u_0 may be expressed rationally through these quantities. We may therefore write

$$u = R\left(\xi,\ \frac{d\xi}{du}\right),$$

where R denotes a rational function.

We thus see that for the case where the equations

$$\xi = \phi(u) \quad \text{and} \quad \frac{d\xi}{du} = \phi'(u)$$

have only one common root, we have

$$\phi(u+v) = \phi\left[R\left(\xi, \frac{d\xi}{du}\right) + R\left(\eta, \frac{d\eta}{dv}\right)\right].$$

Further, since ϕ and R both denote rational functions, it is seen that

$$\phi(u+v) = F\left\{\xi, \frac{d\xi}{du}, \eta, \frac{d\eta}{dv}\right\},$$

where F denotes a rational function.

ART. 43. We shall next show that the two equations

$$\xi = \phi(u), \quad \frac{d\xi}{du} = \phi'(u)$$

cannot have more than one common root. For assume that they have the common roots u_1 and u_2.

It follows that

$$\phi(u_1) = \xi = \phi(u_2), \tag{1}$$

$$\phi'(u_1) = \frac{d\xi}{du} = \phi'(u_2). \tag{2}$$

Since these two expressions exist for continuous values of ξ and $\frac{d\xi}{du}$, we may regard u_1 and u_2 as two variable quantities.

Taking the differential of (1) it follows that

$$\phi'(u_1)du_1 = \phi'(u_2)du_2.$$

If we exclude as *singular* all values of u for which

$$\phi'(u_1) = 0 = \phi'(u_2),$$

then owing to the relation (2) we have

$$du_1 = du_2,$$

or,

$$u_1 = u_2 + C,$$

where C is a constant.

If therefore the two equations have two common roots, these roots can differ only by a constant.

We thus have

$$\phi(u_1) = \phi(u_1 + C).$$

This expression is true for an arbitrarily large number of values of u_1, and since the degree of $\phi(u)$ is finite we must have the identical relation

$$\phi(u_1) \equiv \phi(u_1 + C).$$

Further, for u_1 we may write any arbitrary value in the identity, say $u_1 + C$, and we thus have

$$\phi(u_1 + C) \equiv \phi(u_1 + 2\,C) \equiv \phi(u_1).$$

Hence the roots of the identity are $u_1, u_1 + C, u_1 + 2C, \cdots$. If then $C \neq 0$, the equation

$$\xi = \phi(u)$$

has an infinite number of solutions. This, however, is not true, since the equation is of finite degree. If follows that the constant $C = 0$ and consequently the two equations can have only one common root.

We have thus shown that *every rational function of the argument u has an algebraic addition-theorem and has the property that $\phi(u+v)$ may be rationally expressed through $\phi(u)$, $\phi'(u)$, $\phi(v)$, $\phi'(v)$.*

ART. 44. We shall next show that the theorem of the last article is also true for all functions that are composed rationally of the exponential function $e^{\frac{u\pi i}{\omega}}$.

Let μ be a real or complex quantity different from 0 and ∞ and write

$$t = e^{\mu u}, \text{ and}$$

$$\psi(t) = \phi(u), \tag{1}$$

where ψ denotes a rational function.

Further, let

$$s = e^{\mu v} \text{ and}$$

$$\psi(s) = \phi(v). \tag{2}$$

It follows that

$$e^{\mu(u+v)} = e^{\mu u} \cdot e^{\mu v} = t \cdot s, \text{ and}$$

$$\psi(t \cdot s) = \phi(u+v). \tag{3}$$

From the three equations (1), (2) and (3) we may eliminate ξ and η, and have

$$\text{(A)} \qquad G\{\phi(u), \phi(v), \phi(u+v)\} = 0,$$

where G denotes an integral function.

We have under consideration a group of one-valued analytic functions which have everywhere the character of an integral or fractional function and which are simply periodic, the period of the argument being $\frac{2\pi i}{\mu} = 2\omega$, say.

We further have

$$\xi = \phi(u) = \psi(t),$$

$$\frac{d\xi}{du} = \phi'(u) = \psi'(t)\mu t.$$

If t is eliminated from these equations, we have the eliminant equation

$$\text{(B)} \qquad f\left(\xi, \frac{d\xi}{du}\right) = 0,$$

where f denotes an integral function.

It follows from equations (A) and (B) that the function $\phi(u)$ has an algebraic addition-theorem.

ART. 45. It may be shown as in the case of the rational functions that when the equations

$$\xi = \psi(t) \quad \text{and} \quad \frac{d\xi}{du} = \psi'(t)\mu t$$

have *one* common root * in t, then we may express t in the form

$$t = R\left(\xi, \frac{d\xi}{du}\right),$$

where R denotes a rational function. It also follows that

$$\phi(u + v) = F[\phi(u), \phi'(u), \phi(v), \phi'(v)],$$

where F is a rational function.

Suppose next that the two equations

$$\xi = \phi(u), \quad \frac{d\xi}{du} = \phi'(u)$$

have *more* than one common root.

Suppose that t_1 and t_2 are two roots that are common to both equations, so that

$$\xi = \psi(t_1) = \psi(t_2), \tag{1}$$

$$\frac{d\xi}{du} = \psi'(t_1)\mu t_1 = \psi'(t_2)\mu t_2. \tag{2}$$

If we consider t_1 as the independent variable, then t_2 is an algebraic function of t_1, since

$$\psi(t_1) = \psi(t_2),$$

and ψ is a rational function.

From equation (1) it follows that

$$\psi'(t_1)dt_1 = \psi'(t_2)dt_2,$$

which divided by the expression (2) becomes

$$\frac{dt_1}{t_1} = \frac{dt_2}{t_2},$$

or

$$\log t_1 = \log t_2 - \log C, \quad \text{so that}$$

$$t_2 = Ct_1.$$

It is thus seen that if the two given equations have two common roots, these roots can differ only by a multiplicative constant. Since

$$\psi(t_1) = \psi(t_2), \text{ it follows that}$$

$$\psi(t_1) = \psi(Ct_1),$$

which is an algebraic equation of finite degree.

* By equating the discriminant to zero, we may always effect the condition that there is one common root.

As this equation can be satisfied by an infinite number of values of t_1, it must be an identical equation and consequently

$$\psi(t_1) \equiv \psi(Ct_1).$$

It follows at once that

$$\psi(t_1) \equiv \psi(Ct_1) \equiv \psi(C^2t_1) \equiv \cdots \equiv \psi(C^n t_1) \equiv \cdots.$$

But the equation $\psi(t_1) = \psi(t_2)$ being of finite degree with respect to t_2 can only be satisfied by a finite number of different values of t_2. It therefore follows that in the series of quantities

$$Ct_1, C^2t_1, C^3t_1, \ldots, C^p t_1, \ldots, C^q t_1, \ldots, \tag{1}$$

some must have equal values. If the degree of the equation is n in t_2 then among the first $n + 1$ of these quantities two must be equal, say

$$C^p = C^q \qquad (q > p).$$

Writing $q - p = m$, a *positive* integer, we have

$$C^m = 1. \tag{2}$$

It is thus shown that C is an *mth root of unity*, and as m is the smallest integer that satisfies this equation it is a *primitive* mth root of unity. It is easy to see that the quantities

$$C, C^2, C^3, \ldots, C^{m-1}, C^m$$

are all different. For if

$$C^i = C^j \; (i, j \leqq m),$$

then is

$$C^{i-j} = 1 \text{ (where } i - j = m' < m).$$

This, however, contradicts the hypothesis that m is the smallest integer which satisfies the equation (2). There are consequently only m different quantities in the series (1).

We may use this fact and employ the identical equation

$$\psi(t_1) \equiv \psi(Ct_1)$$

to show that the rational function $\psi(t_1)$ may be transformed into another rational function $\psi_1(t^m)$. If then we write $\tau = t^m$, we may substitute the function $\psi_1(\tau)$ in the above investigation in the place of $\psi(t)$, where the degree of the equation in τ is $\frac{n}{m}$, n being the degree of the equation in t.

The function $\psi(t)$ may be expressed as the quotient of two integral functions without common divisor in the form

$$\psi(t) = t^{\pm\mu} A \frac{(t - a_1)(t - a_2) \ldots (t - a_\rho)}{(t - b_1)(t - b_2) \ldots (t - b_\sigma)},$$

μ is an integer and where none of the a's or b's is equal to zero.

Further, since $\psi(t) = \psi(Ct)$, we must have

$$\text{(I)} \qquad t^{\pm\mu} A \frac{(t - a_1)(t - a_2) \ldots (t - a_\rho)}{(t - b_1)(t - b_2) \ldots (t - b_\sigma)}$$

$$= (Ct)^{\pm\mu} A \frac{(Ct - a_1)(Ct - a_2) \ldots (Ct - a_\rho)}{(Ct - b_1)(Ct - b_2) \ldots (Ct - b_\sigma)}.$$

The left-hand side of this equation is zero for $t = a_1$; it follows that the right-hand side must also vanish for this value of t. But $Ca_1 - a_1 \neq 0$, if we assume that $C \neq 1$. Hence one of the other factors must be such that $Ca_1 - a_\lambda = 0$, where λ is to be found among the integers $2, 3, \ldots, \rho$. As it is only a matter of notation, we may write $\lambda = 2$, so that

$$Ca_1 - a_2 = 0, \quad \text{or} \quad a_2 = Ca_1.$$

In a similar manner, since the left-hand member of the equation vanishes for a_2, one of the factors on the right-hand side must vanish for $t = a_2$, say $Ca_2 - a_\nu = 0$, where ν is to be found among the integers $1, 3, 4, \ldots, \rho$, say $\nu = 3$.

We thus have

$$Ca_2 - a_3 = 0, \quad \text{or} \quad a_3 = Ca_2 = C^2 a_1.$$

Continuing this process we derive the relations

$$a_1 = a_1,\ a_2 = Ca_1,\ a_3 = C^2 a_1,\ \ldots,\ a_m = C^{m-1} a_1.$$

Further, since $C, C^2, \ldots, C^{m-1}$ are all different, it is seen that

$$a_1, a_2, \ldots, a_m$$

are all different.

The quantities $a_1, Ca_1, C^2a_1, \ldots, C^{m-1}a_1$ form a group of roots of the equation, and after Cote's Theorem

$$(t - a_1)(t - Ca_1)(t - C^2 a_1) \ldots (t - C^{m-1} a_1) = t^m - a_1^m.$$

This factor $t^m - a_1^m$ may consequently be separated from the two sides of the equation (I). If further there remain linear factors in the numerator of equation (I), we repeat the above process until there are no such factors. The same is also done with the denominator. When all such factors have been divided out from either side of the equation (I), there remains

$$t^{\pm\mu} = (Ct)^{\pm\mu},$$

so that $C^\mu = 1$. It follows at once that μ must be a multiple of m and consequently

$$\psi(t) = A(t^m)^{\pm\frac{\mu}{m}} \frac{(t^m - a_1^m)(t^m - a_p^m) \ldots}{(t^m - b_1^m)(t^m - b_q^m) \ldots}.$$

We have thus shown that if the two equations

$$\xi = \phi(u), \quad \frac{d\xi}{du} = \phi'(u)$$

have more than one root in common, there exists an integer m, such that $\psi(t)$ may be expressed as a rational function of t^m.

Writing $t = e^{\mu u}$, it follows that $t^m = e^{m\mu u}$ and

$$\phi(u) = \psi(t) = \psi_1(t^m) = \psi_1(e^{m\mu u}).$$

In the further discussion we may use $\psi_1(t^m)$ in the place of $\psi(t)$.

It may happen that the two equations

$$\xi = \psi_1(e^{m\mu u}) \quad \text{and} \quad \frac{d\xi}{du} = \psi_1'(e^{m\mu u})\, m\mu e^{m\mu u}$$

have more than one common root. By repeating the above process we may diminish the degree of ψ_1 and replace the function $\psi_1(e^{m\mu u})$ by the equivalent function $\psi_2(e^{mm'\mu u})$, where m' is an integer, etc.

Since the original function ψ was of finite degree, a finite number of divisors must reduce the degree to unity. It therefore follows that in the process of diminishing the degrees of the functions $\psi, \psi_1, \psi_2, \ldots$, we must come to a function, say $\xi = \psi_\kappa$, such that ξ and $\frac{d\xi}{du}$ have no common root for the new variable that has been substituted in ψ_κ. Hence without exception the following theorem is true:

I. *All rational functions of the argument u; and*

II. *All rational functions of the exponential function $e^{\frac{u\pi i}{\omega}}$ have algebraic addition-theorems and are such that*

$$\phi(u + v) = F[\phi(u), \phi'(u), \phi(v), \phi'(v)],$$

where F denotes a rational function.

Example.—Apply the above theory to the examples $\sin u$, $\cos u$, $\tan u$. Write

$$\sin u = \frac{e^{iu} - e^{-iu}}{2i} = \frac{1}{2i}\frac{t^2 - 1}{t}, \quad \text{where } t = e^{iu}.$$

ART. 46. It may be shown by an example that a function $\phi(u)$ may have the property that $\phi(u + v)$ is rationally expressible through $\phi(u)$, , $\phi'(v)$ without having an algebraic addition-theorem.

function

$$\phi(u) = Ae^{au} + Be^{bu}, \tag{1}$$

, a, b are constants and $a \neq b$. It follows that

$$\phi'(u) = aAe^{au} + bBe^{bu}. \tag{2}$$

From (1) and (2) we have

$$Ae^{au} = \frac{b\phi(u) - \phi'(u)}{b - a},$$

$$Be^{bu} = \frac{-a\phi(u) + \phi'(u)}{b - a}.$$

We further have

$$\phi(u + v) = Ae^{au}e^{av} + Be^{bu}e^{bv}$$

$$= \frac{1}{A}\,\frac{b\phi(u)-\phi'(u)}{b-a}\cdot\frac{b\phi(v)-\phi'(v)}{b-a} + \frac{1}{B}\,\frac{-a\phi(u)+\phi'(u)}{b-a}\cdot\frac{-a\phi(v)+\phi'(v)}{b-a},$$

from which it is seen that $\phi(u + v)$ may be expressed rationally in terms of $\phi(u)$, $\phi'(u)$, $\phi(v)$, $\phi'(v)$.

We shall now show that $\phi(u)$ has *not* an algebraic addition-theorem.

We so choose a and b that the ratio $\frac{b}{a}$ is an irrational or complex quantity.

In Art. 35 we saw that without exception the differential equation

$$f\left(\xi, \frac{d\xi}{du}\right) = 0,$$

where f denoted an integral algebraic function, existed for all functions which had algebraic addition-theorems. If therefore we can prove that such an equation does not exist for $\phi(u)$, we may infer that $\phi(u)$ does *not* have an algebraic addition-theorem.

Suppose for the function $\phi(u)$ there exists an equation of the form

$$f[\phi(u),\ \phi'(u)] = 0,$$

where f denotes an integral function.

Since $\phi(u)$ and $\phi'(u)$ may be expressed through e^{au} and e^{bu} where only constant terms occur in the coefficients, we may write the above equation in the form

$$f_1[e^{au}, e^{bu}],$$

where f_1 like f denotes an integral function of finite degree. This equation must be satisfied for all values of u for which the function $\phi(u)$ is defined.

We give to u successively the values

$$u_0,\ u_0 + \frac{2\pi i}{a},\ u_0 + \frac{4\pi i}{a},\ \cdots .$$

The quantity e^{au} has the same value, viz., e^{au_0} for all these values of u. But corresponding to one value of e^{au_0}, the equation above being of finite degree can furnish only a finite number of different values of e^{bu}. On

the other hand there correspond to the one value e^{au_0} an infinite number of values e^{bu} of the form

$$e^{bu_0},\quad e^{bu_0+\frac{b}{a}2\pi i},\quad e^{bu_0+\frac{b}{a}4\pi i},\ \dots,$$

which are all different, since the ratio $\frac{b}{a}$ is not rational.

It follows that the eliminant equation $f\left(\xi, \frac{d\xi}{du}\right) = 0$ does not exist for the given function, and consequently this function does *not* have an algebraic addition-theorem. We have thus proved that *the existence of the relation*

$$\phi(u+v) = F\{\phi(u),\ \phi'(u),\ \phi(v),\ \phi'(v)\},$$

F denoting a rational function, does not necessarily imply the existence of an algebraic addition-theorem.

Continuation of the Domain in which the Analytic Function $\phi(u)$ has been Defined, with Proofs that its Characteristic Properties are Retained in the Extended Domain.

ART. 47. In the previous discussion we have supposed that $\phi(u)$ was defined for a certain region which contained the origin. This region we may call the *initial domain* of the function $\phi(u)$. We further assume that $\phi(u)$ has an algebraic addition-theorem and is such that $\phi(u+v)$ may be rationally expressed through $\phi(u)$, $\phi'(u)$, $\phi(v)$, $\phi'(v)$ within this initial domain.

These properties are expressed through the two equations

$$\text{(I)}\quad G[\phi(u),\ \phi(v),\ \phi(u+v)] = 0,$$

$$\text{(II)}\quad \phi(u+v) = F\{\phi(u),\ \phi'(u),\ \phi(v),\ \phi'(v)\},$$

where G denotes an algebraic function and F a rational function.

We also assume that u and v are taken so that $u+v$ lies within the initial domain.*

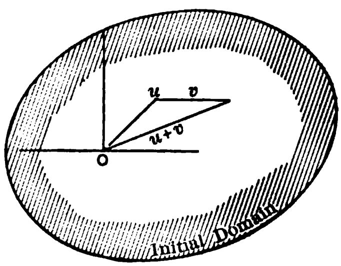


Fig. 3.

We shall now prove the following theorem: *If the function $\phi(u)$ has the properties above mentioned, it has the character of an integral or a (fractional) rational function in the neighborhood of the origin.*

In the equation (II) we write

$u+v$ in the place of u,
$-v$ in the place of v,
u in the place of $u+v$.

We thus have

$$\phi(u) = F\{\phi(u+v), \phi'(u+v), \phi(-v), \phi'(-v)\}. \tag{1}$$

* Cf. Weierstrass, *Abel'schen Functionen*, Werke 4, pp. 450 *et seq.*

Such values are chosen for v that for these values the functions $\phi(v)$ and $\phi(-v)$ belong to the initial domain. We develop $\phi(u+v)$ by Taylor's Theorem in the form

$$\phi(u+v) = \phi(v) + u\phi'(v) + \frac{u^2}{2!}\phi''(v) + \cdots,$$

a series which remains convergent so long as v takes such values that the points $u+v$ and u lie within the initial domain. The same is also true of the series

$$\phi'(u+v) = \phi'(v) + u\phi''(v) + \frac{u^2}{2!}\phi'''(v) + \cdots.$$

These series may therefore be substituted in formula (1). We thus have $\phi(u)$ expressed as a rational function of u, which as the quotient of two integral functions takes the form

$$\phi(u) = \frac{P_{10}(u)}{P_{20}(u)} = \frac{a_0 + a_1u + a_2u^2 + \cdots}{b_0 + b_1u + b_2u^2 + \cdots},$$

where the two series are convergent so long as $|u|$ is less than a certain quantity, say ρ.

If $b_0 \neq 0$, $\phi(u)$ has the character of an integral function in the neighborhood of the origin $u = 0$; if $b_0 = 0 = b_1 = \cdots = b_k$ and at the same time

$$a_0 = 0 = a_1 = \cdots = a_k,$$

then $\phi(u)$ has the character of an integral function at the origin; but if one of the a's just written is different from zero, then $\phi(u)$ becomes infinite for $u = 0$ but of a finite integral degree. It then has the character of a rational function at the origin, and its expansion by Laurent's Theorem* has a finite number of terms with negative integral exponents.

ART. 48. We may next prove the following theorem: *The domain of $\phi(u)$ may be extended to all finite values of the argument u without the function $\phi(u)$ ceasing to have the character of an integral or (fractional) rational function.*

Fundamental in the proof of this theorem is the expression of $\phi(u)$ as the quotient of two power series

$$\phi(u) = \frac{P_{1,0}(u)}{P_{2,0}(u)},$$

where the two series are convergent so long as $|u|$ does not exceed a definite limit ρ.

If we draw the circle with radius ρ about the point $u = 0$, then within

* In this connection see a proof of Laurent's Theorem by Professor Mittag-Leffler, *Acta Math.*, Bd. IV, pp. 80 *et seq.*, where the theorem is proved by the elements of the Theory of Functions without recourse to definite integrals.

this circle the function $\phi(u)$ is completely defined. In order to extend or continue this region, we may use the equation

$$\phi(u+v) = F\{\phi(u), \phi'(u), \phi(v), \phi'(v)\}.$$

We shall at first assume that we may write $u = v$ without the function F taking the form $0/0$. We then have* for $u = v$,

$$\phi(2\,u) = F\{\phi(u), \phi'(u), \phi(u), \phi'(u)\}.$$

The right-hand side of this equation is true for all values of u that lie within the circle with radius ρ. It follows then that through this expression the function ϕ on the left-hand side is defined so long as its argument lies within the circle with radius $2\,\rho$.

If then we write u in the place of $2\,u$ in this equation, we have

$$\phi(u) = F\{\phi(\tfrac{1}{2}\,u), \phi'(\tfrac{1}{2}\,u), \phi(\tfrac{1}{2}\,u), \phi'(\tfrac{1}{2}\,u)\}.$$

We express $\phi(u)$, as the quotient of two power series, $= \dfrac{P_{1,0}(u)}{P_{2,0}(u)}$. Further, $\phi'(u)$ may also be expressed as the quotient of two power series. These values substituted in F give $\phi(u)$ defined as the quotient of two new power series, say

$$\phi(u) = \frac{\overline{P_{1,1}(u)}}{\overline{P_{2,1}(u)}}.$$

Since $\frac{1}{2}\,u$ has been written for u in the two new power series, they are convergent so long as $|\,u\,| < 2\,\rho$.

We cannot apply the above method, if for $u = v$ the function $\phi(u)$ takes the form $0/0$. Nevertheless we may proceed as follows and extend the region of convergence at pleasure.

In the equation

$$\phi(u+v) = F\{\phi(u), \phi'(u), \phi(v), \phi'(v)\},$$

we write $\dfrac{u}{1+\alpha}$ instead of u,

and $\dfrac{\alpha u}{1+\alpha}$ instead of v,

where α is a real quantity such that $\frac{1}{2} < \alpha < 1$.

We have in this manner

$$\phi(u) = F\left\{\phi\left(\frac{u}{1+\alpha}\right),\ \phi'\left(\frac{u}{1+\alpha}\right),\ \phi\left(\frac{\alpha u}{1+\alpha}\right),\ \phi'\left(\frac{\alpha u}{1+\alpha}\right)\right\}.$$

ıction F being a rational function, we may express $\phi(u)$ as the ŧ of two power series in which the numerator and denominator are

* See Daniels, *Amer. Journ. Math.*, Vol. VI, p. 255.

analytic functions of u and α. The denominator cannot vanish for all values of α. We shall therefore so choose α that the denominator is different from zero. We may then express $\phi(u)$ as the quotient of two power series in the form

$$\phi(u) = \frac{P_{1,1}(u)}{P_{2,1}(u)},$$

where the series are convergent for values of u such that

$$| u | < (1 + \alpha)\rho.$$

Since $\alpha \geqq \frac{1}{2}$ the series is convergent if $| u | < \frac{3}{2} \rho$.

This process, as well as the one employed in the previous article, may be repeated as often as we wish, so that we have eventually

$$\phi(u) = \frac{P_{1,n}(u)}{P_{2,n}(u)},$$

where the power series $P_{1,n}(u)$ and $P_{2,n}(u)$ are convergent so long as

$$| u | < (\tfrac{3}{2})^n\rho.$$

Hence $\phi(u)$ may be defined for an arbitrarily large portion of the plane as the quotient of two power series which may be expanded in ascending powers of u.*

ART. 49. As an example of the above theory, consider the function

$$\phi(u) = \tan u = \frac{\sin u}{\cos u} = \frac{P_{1,0}(u)}{P_{2,0}(u)} = u + \frac{u^3}{3} + \frac{2\,u^5}{15} + \cdots = P(u),$$

where
$$P_{1,0}(u) = u - \frac{u^3}{\underline{|3}} + \frac{u^5}{\underline{|5}} - \cdots,$$

$$P_{2,0}(u) = 1 - \frac{u^2}{\underline{|2}} + \frac{u^4}{\underline{|4}} - \cdots.$$

At the points $0, \pi, 2\pi, 3\pi, \ldots$, the function $\tan u$ is zero, and is infinite at $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$.

For the point $u = \infty$, the function $\tan u$ is *not* defined, this point being an essential singularity of the function. The function is convergent for all points within a circle described about the point $u = 0$, whose radius extends up to the infinity $\frac{\pi}{2}$ of $\tan u$, so that we may take $\rho = \frac{\pi}{2}$.

* Weierstrass, Werke IV, p. 6. says that from the fact that $\phi(u)$ has an algebraic addition-theorem we may show that it is a uniquely defined function having the character of an integral or rational (fractional) function and that starting with this we may derive a complete theory of the elliptic functions.

Using the formula

$$\tan(u+v) = \frac{\tan u + \tan v}{1 - \tan u \tan v},$$

we may extend the definition of $\tan u$ to an arbitrarily large region. For writing $v = u$, then is

$$\tan 2u = \frac{2\tan u}{1 - \tan^2 u}.$$

Further, if we put $\frac{1}{2}u$ in the place of u, we have

$$\tan u = \frac{2\tan\frac{1}{2}u}{1-\tan^2\frac{1}{2}u} = \frac{2P(\frac{1}{2}u)}{1-P^2(\frac{1}{2}u)} = \frac{P_{1,1}(u)}{P_{2,1}(u)},$$

where $P_{1,1}(u)$ and $P_{2,1}(u)$ are convergent so long as

$$\tfrac{1}{2}|u| < \tfrac{1}{2}\pi \quad \text{or} \quad |u| < \pi.$$

We see that here the new circle of convergence passes through the points $+\pi$ and $-\pi$ and that the old region of convergence has been extended by a ring-formed region.

By another repetition of the same process we have

$$\tan u = \frac{2\dfrac{P_{1,1}(u)}{P_{2,1}(u)}}{1-\left(\dfrac{P_{1,1}(u)}{P_{2,1}(u)}\right)^2} = \frac{P_{1,2}(u)}{P_{2,2}(u)}.$$

The radius of convergence of the two series on the right-hand side is now 2π, so that the tangent function is defined for all points within the circle whose radius is 2π. By continuing this process we are able to define $\tan u$ for all finite values of the argument u without its ceasing to have the character of an integral or (fractional) rational function.

ART. 50. Returning to the general case we shall see whether the function which has been thus defined for all points of the plane is the same as the function $\phi(u)$ with which we started and which was defined for the interior of the circle with the radius ρ. We shall show that such is the case and that the new function is the analytic continuation of the one with which we began.* We shall first show that the two functions are identical within the circle with radius ρ.

It is seen that the expression of $\phi(u)$ as the quotient of two convergent power series is characteristic of this sort of function. We limit u to the interior of the circle with radius $\frac{1}{2}\rho$ within which v is also restricted to remain. The points u, v, $u+v$ evidently lie within the domain for which $\phi(u)$ was defined, and the property expressed through the formula

$$\phi(u+v) = F\{\phi(u), \phi'(u), \phi(v), \phi'(v)\}$$

is true for this domain.

* Weierstrass (*Definition der Abel'schen Functionen*, Werke 4, pp. 441 *et seq.*) emphasizes this fact.

On the right-hand side we again write

$$\frac{u}{1+\alpha} \text{ instead of } u$$

and

$$\frac{\alpha u}{1+\alpha} \text{ instead of } v,$$

with the limitation that the absolute values of these quantities be less than $\frac{1}{2}\rho$.

Writing *first* $\phi(u) = \dfrac{P_{1,0}(u)}{P_{2,0}(u)}$ and *then* making the formal computation as above, we have $\phi(u) = \dfrac{P_{1,1}(u)}{P_{2,1}(u)}$. These two quotients are identical* within the circle with radius $\frac{1}{2}\rho$, so that

$$\frac{P_{1,0}(u)}{P_{2,0}(u)} = \frac{P_{1,1}(u)}{P_{2,1}(u)},$$

or

$$P_{1,0}(u)P_{2,1}(u) = P_{2,0}(u)P_{1,1}(u). \tag{1}$$

If we multiply these two power series on either side of the equation, we will have the equality of two new power series, which is true for all values of u, such that $|u| < \frac{1}{2}\rho$. Now $P_{1,0}$ and $P_{1,1}$ are convergent by hypothesis within the circle of radius ρ, while $P_{2,0}$ and $P_{2,1}$ are convergent within the circle of radius $\frac{3}{4}\rho$. Within the circle with radius $\frac{1}{2}\rho$ the coefficients of u on either side of the equation are equal. But as these coefficients are constants we conclude that the two series on the right and left of equation (1) must be the same within the extended realm, the circle with radius ρ.

It follows that the representations of $\phi(u)$ through the two quotients $\dfrac{P_{1,0}(u)}{P_{2,0}(u)}$ and $\dfrac{P_{1,1}(u)}{P_{2,1}(u)}$ are the same within the interior of the circle ρ. The same process may be continued so as to extend over the whole region of convergence.

ART. 51. We shall next prove that as the definition of the function $\phi(u)$ is extended to an arbitrarily large region, the properties of the original function $\phi(u)$ that are expressed through the equations,

$$\text{(I)} \quad G\{\phi(u), \phi(v), \phi(u+v)\} = 0,$$

$$\text{(II)} \quad \phi(u+v) = F\{\phi(u), \phi'(u), \phi(v), \phi'(v)\},$$

are also retained for the extended region.† First take $|u| < \frac{1}{2}\rho$ and $|v| < \frac{1}{2}\rho$ so that $|u+v| < \rho$ and therefore lies within the initial domain.

* Weierstrass, *loc. cit.*, p. 455.

† This theorem has the same significance for the *properties* of the elliptic functions as the fact that the functions themselves may be analytically continued as emphasized in Chapter I.

In the equation $G = 0$ write $\dfrac{P_{1,0}(u)}{P_{2,0}(u)}$ for $\phi(u)$, $\dfrac{P_{1,0}(v)}{P_{2,0}(v)}$ for $\phi(v)$ and $\dfrac{P_{1,0}(u+v)}{P_{2,0}(u+v)}$ for $\phi(u+v)$. Multiply the expression thus obtained by the *least common multiple* of the denominators and we have an integral power series in u and v on the left equated to zero. This power series is convergent so long as $|u| < \frac{1}{2}\rho$ and $|v| < \frac{1}{2}\rho$. If this power series is arranged in ascending powers of u, the coefficients are functions of v which may also be arranged in ascending powers of v. Since the right-hand side is zero, the coefficients of u are all zero and consequently the power series in v are *identically zero*. Making use of equation (II) we derive the second development for $\phi(u)$, viz.,

$$\phi(u) = \frac{P_{1,1}(u)}{P_{2,1}(u)}.$$

This value and the corresponding values of $\phi(v)$, $\phi(u+v)$ are now substituted in (I). We thus make another integral power series in u and v on the left equal to zero on the right as in the previous case.

These two power series must be the same so long as $|u| < \frac{1}{2}\rho$ and $|v| < \frac{1}{2}\rho$. But as here the coefficients of u are all identically zero, this must also be true in the extended region. By repeating this process we have the theorem:

The addition-theorem while limited to a ring-formed region, exists for the whole region of convergence established for the function $\phi(u)$.

If the point $u = \infty$ is an essential singularity, the function $\phi(u)$ will have this point as a limiting position, that is, the function may be continued analytically as near as we wish to this point, but at the point ∞ the function need have the character of neither an integral nor a (fractional) rational function.

CHAPTER III

THE EXISTENCE OF PERIODIC FUNCTIONS IN GENERAL

Simply Periodic Functions. The Eliminant Equation.

ARTICLE 52. In the previous Chapter we have studied the characteristic properties of one-valued analytic functions which have algebraic addition-theorems. These properties were considered in the finite portion of the plane. The function may behave regularly at infinity or this point may be either a polar or an essential singularity of the function. In the latter case the function is quite indeterminate (Art. 3) in the neighborhood of infinity.

When the point at infinity is an essential singularity, we shall show that *the function is periodic.* To prove this we have only to show that the function may take certain values at an arbitrarily large number of points (cf. Art. 38) of the u-plane.

Suppose that m is the number of points at which $\phi(u) = \xi_0$, say, where ξ_0 is a definite constant, and denote these points by $a_1, a_2, \ldots, a_m$.

Let a_μ be any one of these points, and with a radius r_μ draw a circle C_μ about a_μ as center. Take r_μ so small that within and on the periphery of C_μ none of the other points $a_1, a_2, \ldots, a_{\mu-1}, a_{\mu+1}, \ldots, a_n$ lies, and also within and on the periphery of this circle suppose that $\phi(u)$ is everywhere regular. Next let u take a circuit around C in the u-plane; then in the plane in which $\phi(u)$ is geometrically represented $\phi(u)$ makes a closed curve S_μ, say, which does *not* pass through the point ξ_0.

We may write

$$\frac{d\phi(u)}{\phi(u) - \xi_0} = \frac{d\{\phi(u) - \xi_0\}}{\phi(u) - \xi_0},$$

and expressing $\phi(u) - \xi_0$ in the form

$$\phi(u) - \xi_0 = re^{i\theta},$$

it is seen that

$$\frac{d\phi(u)}{\phi(u) - \xi_0} = \frac{d\{re^{i\theta}\}}{re^{i\theta}} = \frac{e^{i\theta}dr}{re^{i\theta}} + \frac{rie^{i\theta}d\theta}{re^{i\theta}}$$

$$= \frac{dr}{r} + i d\theta.$$

If next we integrate around S_μ in the $\phi(u)$-plane, we have

$$\int_{S_\mu}\frac{d\phi(u)}{\phi(u)-\xi_0}=\int_{S_\mu}\frac{dr}{r}+\int_{S_\mu}id\theta.$$

The first integral on the right is $\log r$, which is here zero, since the curve returns to its initial point, making the upper and lower limits identical.

We thus have

$$\int_{S_\mu}\frac{d\phi(u)}{\phi(u)-\xi_0}=\int_{S_\mu}id\theta.$$

On the other hand,

$$\int_{S_\mu}\frac{d\phi(u)}{\phi(u)-\xi_0}=\int_{C_\mu}\frac{\phi'(u)du}{\phi(u)-\xi_0},$$

where the integration of the first integral is taken with respect to the elements $d\phi(u)$ and consequently over S_μ' in the $\phi(u)$-plane, while the integration over the second integral is with respect to du and therefore over the circle C_μ in the u-plane. The function $\phi(u)$ when developed in powers of $u-a_\mu$ is of the form

$$\phi(u)=\phi(a_\mu)+\frac{\phi'(a_\mu)}{1!}(u-a_\mu)+\frac{\phi''(a_\mu)}{2!}(u-a_\mu)^2+\cdots,$$

or, since $\phi(a_\mu)=\xi_0$,

$$\phi(u)-\xi_0=\frac{\phi'(a_\mu)}{1!}(u-a_\mu)+\frac{\phi''(a_\mu)}{2!}(u-a_\mu)^2+\cdots.$$

On the right-hand side a number of the coefficients may vanish. Let $A_k=\frac{\phi^{(k)}(a_\mu)}{k!}$ be the first of the coefficients that is different from zero and let $k=k_\mu$, say, be the order of the zero of the function $\phi(u)-\xi_0$ at the point a_μ.

We therefore have

$$\phi(u)-\xi_0=A_\mu(u-a_\mu)^{k_\mu}+\cdots,$$

and consequently

$$\int_{S_\mu}\frac{d\phi(u)}{\phi(u)-\xi_0}=\int_{C_\mu}\frac{A_\mu k_\mu(u-a_\mu)^{k_\mu-1}+\cdots}{A_\mu(u-a_\mu)^{k_\mu}+\cdots}du$$

$$=\int_{C_\mu}\frac{k_\mu du}{u-a_\mu},$$

all the remaining terms having vanished.

Since
$$\int_{C_\mu}\frac{k_\mu du}{u-a_\mu}=2\pi ik_\mu,$$
it follows that

$$2k_\mu\pi i=\int_{S_\mu}id\theta,\quad\text{or}\quad k_\mu=\frac{1}{2\pi i}\int_{S_\mu}d\theta.$$

In other words, the order of zero of the function $\phi(u) - \xi_0$ at the point $u = a_\mu$; that is, k_μ is equal to the number of circuits which the curve in the $\phi(u)$-plane makes around ξ_0 corresponding to the circle C_μ made around the point a_μ by the variable u in the u-plane. The integer k_μ is at least *unity*.

Suppose in the place of a_μ another point a_0 is written, and about this point let a circle C_0 be described with a radius so small that within and on the circumference of the circle *none* of the points $a_1, a_2, \ldots, a_n$ lies, nor any of the infinities of the function. We know then that the integral

$$\int_{C_0} \frac{\phi'(u)du}{\phi(u) - \xi_0} \text{ is zero,}$$

where the path of integration is taken over the circle C_0.

We have accordingly proved * that *the integral*

$$\frac{1}{2\pi i}\int_S \frac{d\phi(u)}{\phi(u) - \xi_0},$$

where $\phi(u)$ is a regular function for all points on and within the interior of the contour S, indicates the number of times that the function $\phi(u)$ takes the value ξ_0 within S, provided each point a_μ, say, at which $\phi(u)$ takes the value ξ_0, is counted as often as the order k_μ of the zero of $\phi(u) - \xi_0$ at the point a_μ.

ART. 53. Next in the place of ξ_0 take another value ξ_1, which also lies within S_μ, so that the corresponding value of u lies within C_μ. Then the number of circuits of the curve about ξ_0 is the same as the number of circuits about ξ_0, since all the circuits encircle both points.

It follows that

$$\frac{1}{2\pi i}\int_{S_\mu} \frac{d\phi(u)}{\phi(u) - \xi_1} = k_\mu = \int_{C_\mu} \frac{\phi'(u)du}{\phi(u) - \xi_1},$$

and consequently that $\phi(u)$ takes for values of u within C_μ the value ξ_1 at least *once;* for if this were not the case, we know that the above integral would vanish. We have shown above that it does *not* vanish.

The function $\phi(u)$ by hypothesis takes the value ξ_0 on the m different points $a_1, a_2, \ldots, a_\mu, \ldots, a_m$. Around each of these points a circle is drawn with radius sufficiently small that within the interior of the circle none of the other points of the series $a_1, a_2, \ldots, a_m$ lies. Let u make a circuit about the periphery of each of these circles; then $\phi(u)$ makes closed curves about the point ξ_0, and none of these curves passes through ξ_0. We may therefore draw a circle about ξ_0 which lies within all the other closed curves. Let ξ_1 be a point within the interior of this last circle; then it follows from above that the function $\phi(u)$ takes the value ξ_1 at least m times. There are consequently an infinite number

* See footnote to Art. 92.

of values in the neighborhood of ξ_0 which are taken by the function at least m times.

Consider next the function $\dfrac{1}{\phi(u) - \xi_0}$.

It has at the point $a_1, a_2, \ldots, a_m$ the character of a (fractional) rational function, and may therefore be expanded by Laurent's Theorem in the form *

$$\frac{1}{\phi(u) - \xi_0} = G_1\left(\frac{1}{u - a_1}\right) + G_2\left(\frac{1}{u - a_2}\right) + \cdots + G_m\left(\frac{1}{u - a_m}\right)$$
$$+ \textit{ an integral function in } u,$$

where $G_1, G_2, \ldots, G_m$ denote integral functions of finite degree of their respective arguments.

It follows that

$$\frac{1}{\phi(u) - \xi_0} - G_1\left(\frac{1}{u - a_1}\right) - G_2\left(\frac{1}{u - a_2}\right) - \cdots - G_m\left(\frac{1}{u - a_m}\right) = P(u),$$

where $P(u)$ is a power series with positive integral exponents.

The function $P(u)$ cannot reduce to a constant, for then $\phi(u)$ would be a rational function and the point $u = \infty$ would not be an essential singularity. It follows that the absolute value of the above difference exceeds any limit if we take values of u sufficiently distant from the origin. We may therefore by taking u sufficiently large make $\phi(u) - \xi_0$ as small as we wish.

If further the point ξ_1 is taken very near the point ξ_0, the value ξ_1 is certainly taken by the function $\phi(u)$ as u is made to increase. Hence the function $\phi(u)$ takes the value ξ_1 at least m times in the finite portion of the plane and another time towards infinity. Since by hypothesis $\phi(u)$ is *indeterminate* for $u = \infty$, it appears that $\phi(u) - \xi_0$ is zero for some value of u such that $u < \infty$. Call this value a_{m+1}. By repeating the above process it may be shown that *we may find such values of the function $\phi(u)$ which may be taken arbitrarily often by that function.*

ART. 54. We may derive the above results in a somewhat more explicit manner by means of our eliminant equation

$$f\left(\xi, \frac{d\xi}{du}\right) = 0.$$

We have excluded as being *singular* all values of the function $\xi = \phi(u)$ which satisfy the equation

$$\frac{\partial f(\xi, \xi')}{\partial \xi'} = 0.$$

* See Weierstrass, *Zur Theorie der eindeutigen analytischen Functionen*, Werke, Bd. II, pp. 77 *et seq.*; Weierstrass, *Zur Functionenlehre*, pp. 1 *et seq.*; Hermite, *Sur quelques points de la théorie des fonctions*, *Crelle*, Bd. 91, and "*Cours*," *loc. cit.*, p. 98; Mittag-Leffler, *Sur la représentation analytique*, etc., *Acta Math.*, Bd. IV, p. 8.

In the present discussion we shall also exclude the roots of the equation $f(\xi, 0) = 0$. In other words, the function $\xi = \phi(u)$ is not allowed to take those values of u which make $\xi' = \phi'(u) = 0$.

If we denote by ξ_0 any finite value that $\phi(u)$ can take, then all the points at which $\phi(u)$ can take this value ξ_0 are *simple roots* of the equation $\phi(u) - \xi_0 = 0$; for this difference can only become infinitesimally small of the first order since

$$\phi(u) = \xi_0 + \frac{\xi_0'}{1!}(u - u_0) + \frac{\xi_0''}{2!}(u - u_0)^2 + \cdots,$$

and by hypothesis $\xi_0' \neq 0$.

It follows that the quantities $a_1, a_2, \ldots, a_m$ above are simple roots of the equation $\phi(u) - \xi_0 = 0$, and consequently

$$\frac{1}{2\pi i}\int \frac{d\phi}{\phi(u) - \xi_0} = 1,$$

if the integration is taken over a closed curve in the $\phi(u)$-plane that corresponds to a circle made by u about any of the points $a_1, a_2, \ldots, a_m$.

We also saw that the above integral indicates the number of circuits made by the function $\phi(u)$ about ξ_0 in the $\phi(u)$-plane. As this integral equals unity, we see that there is one circuit made in the positive direction about ξ_0 corresponding to the circle made in the u-plane about any one of the points a. All values ξ_1 which belong to the surface included by the circuit about ξ_0 are therefore taken *once* by the function $\phi(u)$ if u takes all values within the corresponding circle C_μ about a_μ. We describe about ξ_0 as center a circle C with so small a radius that it lies totally within the above circuit S_μ about ξ_0. We shall show that every value ξ_1 within this circle is taken once and only once by the function $\phi(u)$ when u takes all possible values within the circle C_μ.

We saw that the integral

$$\frac{1}{2\pi i}\int_C \frac{d\phi(u)}{\phi(u) - \xi_1},$$

where $\phi(u)$ is regular on and within the contour C, is equal to the number of points at which the value ξ_1 is taken within C, provided each point is counted as often as the order of the zero of $\phi(u) - \xi_1$ at this point. It follows under the given hypotheses that the above integral is always a positive integer.

If then ξ is considered as a variable complex quantity on the interior of the given circle, the integral

$$\frac{1}{2\pi i}\int \frac{d\phi(u)}{\phi(u) - \xi}$$

is an analytic function of ξ. For since the denominator does not vanish for any point on the periphery of the circle, the elements of the integral vary in a continuous manner when ξ varies. On the other hand, we know

that the integral is equal to a constant. This integral considered as a function of ξ must also be equal to a constant. If we let ξ coincide with ξ_0, the integral is equal to unity. It follows that every value ξ_1 which lies sufficiently near ξ_0 is taken *once* and only once if u remains within the circle described about a_μ.

We draw circles as indicated above around all the points $a_1, a_2, \dots, a_m$. These points are the values of u which cause $\phi(u)$ to be equal to ξ_0. In the $\phi(u)$-plane we draw the corresponding circuits around the point ξ_0. Further we draw a circle around ξ_0 as a center with a radius so small that it lies wholly within the circuits made about this point. Let ξ_1 be a point within this circle. Then the value ξ_1 is taken by $\phi(u)$ for values of u *once* in each of the circles around $a_1, a_2, \dots, a_m$ respectively and consequently at least m times.

We consider the quantity

$$\frac{1}{\phi(u) - \xi_0},$$

where $\phi(u)$ takes the value ξ_0 at the points $u = a_1, a_2, \dots, a_m$.

By hypothesis $\phi(u) - \xi_0$ is zero of the first order on each of these points. By Laurent's Theorem we may develop $\dfrac{1}{\phi(u) - \xi_0}$ in the neighborhood of each of these points; and, if the first term of the development in the neighborhood of a_μ is denoted by $\dfrac{A_\mu}{u - a_\mu}$, it is seen that

$$\frac{1}{\phi(u) - \xi_0} - \sum_{\mu=1}^{\mu=m} \frac{A_\mu}{u - a_\mu} = g(u),$$

where $g(u)$ has the character of an integral function for all finite values of the argument.

Since $g(u)$ cannot be a constant, as otherwise $\phi(u)$ would be a rational and not a transcendental function, it is seen by taking values of u sufficiently removed from the origin that $\phi(u) - \xi_0$ may be made arbitrarily small.

Suppose that ξ_1 is a value of $\phi(u)$ which lies within the interior of the circle above. It is clear that for values of u sufficiently distant from the origin the function $\phi(u)$ is equal to ξ_1. We have also shown that besides this value of u the function $\phi(u)$ takes the value ξ_1 at m other points and consequently $\phi(u)$ takes the value ξ_1 at $m + 1$ points. By continuing this process it may be shown that *there are an indefinite number of values which do not belong to the singular values of the function $\phi(u)$, and which may be taken by $\phi(u)$ an arbitrarily large number of times.**

It follows from what we saw in Art. 38 that $\phi(u)$ is a *periodic* function.

* In this connection see Picard, *Mémoire sur les fonctions entières* (*Ann. Éc. Norm.* (2), 9, (1880), pp. 145–166), where it is shown that an integral transcendental function when put equal to any arbitrary constant has an indefinite number of roots which are isolated points on the u-plane.

ART. 55. If the function $\phi(u)$ has the properties expressed through the equations

$$f[\phi(u),\ \phi'(u)] = 0,$$
$$G\{\phi(u),\ \phi(v),\ \phi(u+v)\} = 0,$$
$$\phi(u+v) = F[\phi(u),\ \phi'(u),\ \phi(v),\ \phi'(v)],$$

we have seen that the region of u may be extended by analytic continuation to the whole plane without the function $\phi(u)$ ceasing to have the character of an integral or (fractional) rational function for all values of the argument.

If $\phi(u)$ has at infinity the character of an integral or (fractional) rational function, then $\phi(u)$ is a *rational function of u;* but if the point at infinity is an essential singularity, then $\phi(u)$ is a *periodic function.* It may happen that all the periods may be expressed as positive or negative integral multiples of the same quantity. In this case the function is *simply periodic* and the quantity in question is *the primitive period of the argument of the function.* If all the periods of a function can be expressed through integral multiples of several quantities, the function is said to be *multiply periodic.* The functions with *two primitive* periods are called *doubly periodic,* the two periods constituting a *primitive pair* of periods.

THE PERIOD-STRIPS.

ART. 56. Consider the simple case of the exponential function e^u. We shall first show that $e^{u+2\pi i} = e^u$ for all values of u. Writing $u = x + iy$, it is seen that

$$e^u = e^{x+iy} = e^x(\cos y + i\sin y) = e^x\cos y + ie^x\sin y.$$

If now we increase u by $2\pi i$, then y is increased by 2π, and consequently

$$e^{u+2\pi i} = e^x\cos(y+2\pi) + ie^x\sin(y+2\pi)$$
$$= e^x\cos y + ie^x\sin y = e^{x+iy} = e^u.$$

It follows that if we wish to examine the function e^u, then clearly we need not study this function in the whole u-plane but only within a strip which lies above the X-axis and has the breadth 2π. For we see at once that to every point u_0 which lies without this *period-strip* * there corresponds a point u_1 within the strip and in such a way that the function e^u has the same value at u_0 as at u_1. For example in the figure $e^{u_0} = e^{u_1+4\pi i} = e^{u_1}$.

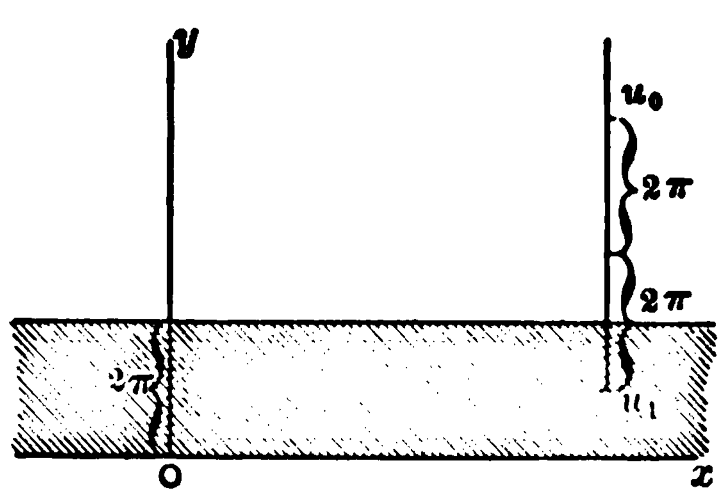


Fig. 4.

* Cf. Koenigsberger, *Elliptische Functionen*, p. 210. The lines including a period-strip need not be *straight*, if only the difference between corresponding points is a period.

Suppose that $p = \alpha + i\beta$ is an arbitrary complex quantity, and consider the equation

$$e^u = p = \alpha + i\beta.$$

Let us first see whether this equation can always be solved with respect to u; and in case it is always possible to solve it, let us see how many values of u there are within the period-strip which satisfy it.

We have

$$e^u = e^x \cos y + ie^x \sin y = p = \alpha + i\beta,$$

and consequently

$$e^x \cos y = \alpha, \qquad e^x \sin y = \beta.$$

It follows that

$$e^{2x} = \alpha^2 + \beta^2, \quad \text{or} \quad e^x = \sqrt{\alpha^2 + \beta^2}.$$

Since x is a real quantity, the positive sign is to be taken with the root. This equation determines x *uniquely*, since we have at once

$$x = \log \sqrt{\alpha^2 + \beta^2}.$$

To determine y, we have $\tan y = \dfrac{\beta}{\alpha}$.

Suppose that y_0 is a value of y situated between 0 and π which satisfies this equation (we know that there is always one such value and indeed only one).

It follows also that

$$\tan (y_0 + \pi) = \tan y_0.$$

It appears then as if $y_0 + \pi$ satisfies the conditions required of y. This, however, is not the case, since we have

$$\cos (y_0 + \pi) = -\cos y_0,$$
$$\sin (y_0 + \pi) = -\sin y_0,$$

and consequently the equations $e^x \cos y = \alpha$, $e^x \sin y = \beta$ are not satisfied by the value $y_0 + \pi$.

Hence within the period-strip the equation

$$e^u = \alpha + i\beta$$

is satisfied by only one value of $u = x + iy$, and this value of u is

$$u = \log \sqrt{\alpha^2 + \beta^2} + iy_0.$$

On the outside of the period-strip, however, the equation is satisfied by an indefinite number of values of u. These values are had if we increase or diminish by integral multiples of $2\pi i$ that value of u which satisfies the equation within the period-strip, that is, if we keep x unchanged and increase or diminish the value y_0 by 2π.

Art. 57. We shall next study two other simple functions, $\cos u$ and $\sin u$. These functions may be defined through the equations

$$\cos u = \tfrac{1}{2}(e^{iu} + e^{-iu}),$$
$$\sin u = \frac{1}{2i}(e^{iu} - e^{-iu}).$$

It follows at once that

$$\cos(u + 2\pi) = \cos u, \qquad \sin(u + 2\pi) = \sin u.$$

Both functions have the period 2π. We may therefore limit the study of these functions to a period-strip with breadth 2π measured along the lateral axis.

It is evident that to every point u_0 lying without this period-strip there is a corresponding point u_1 within the strip at which $\cos u$ and $\sin u$ have the same values as at u_0. For example in the figure

$$\cos u_0 = \cos(u_1 + 6\pi) = \cos u_1,$$
$$\sin u_0 = \sin(u_1 + 6\pi) = \sin u_1.$$

Suppose next that p is an arbitrary complex quantity, and let us see whether for the equation

$$\cos u = p$$

Fig. 5.

there is always a solution. If there is one, there is an indefinite number. For if u_1 satisfies the equation, then from the above it is also satisfied by the values $u_1 + 2\pi$, $u_1 + 4\pi$, $\cdots$.

We shall show that there are always *two* values of u within the period-strip which satisfy the equation.

For writing

$$\cos u = \tfrac{1}{2}(e^{iu} + e^{-iu}) = p,$$

we have

$$e^{iu} + e^{-iu} = 2p.$$

Writing $e^{iu} = t$, this equation becomes

$$t^2 - 2pt + 1 = 0. \tag{1}$$

From this it follows that

$$t = p \pm \sqrt{p^2 - 1}.$$

We thus have two values of $t = e^{iu}$. Let the corresponding values of u be u_1 and u_2, so that therefore

$$t_1 = e^{iu_1}, \quad t_2 = e^{iu_2}.$$

It follows that we have for iu_1 and iu_2 values of the form

$$iu_1 = \eta_1 + k_1 2\pi i,$$
$$iu_2 = \eta_2 + k_2 2\pi i,$$

where k_1 and k_2 are positive or negative integers.

Dividing by i we have at once

$$u_1 = -i\eta_1 + k_1 2\pi,$$
$$u_2 = -i\eta_2 + k_2 2\pi.$$

Hence clearly there are two solutions of the equation $\cos u = p$ within the period-strip, and these solutions are different from each other.

From the quadratic equation (1) it follows that

$$t_1 \cdot t_2 = 1, \text{ or } e^{iu_1} e^{iu_2} = 1.$$

We therefore have

$$e^{i(u_1+u_2)} = 1,$$

and consequently

$$i(u_1 + u_2) \equiv 0 \pmod{2\pi i},$$

or

$$u_1 + u_2 \equiv 0 \pmod{2\pi}.$$

It follows that the two values of u which satisfy the equation $\cos u = p$ within the period-strip are such that their sum is equal to 2π.

We may derive similar results for the function $\sin u$. It is thus seen that *the two functions $\cos u$ and $\sin u$ take any arbitrary value within the period-strip twice, while the function e^u takes such a value only once within its period-strip.*

ART. 58. The period a of a simply periodic function $f(u)$ is in general a complex quantity. We have

$$f(u + a) = f(u),$$

and if we write $u = 0$, it follows that

$$f(a) = f(0),$$

that is, the function $f(u)$ has at the origin the same value as it has at the point a in the u-plane; and also at the points $\ldots, 3a,\ 2a,\ a,\ -a,\ -2a, \ldots$ it has the same value as at the origin.

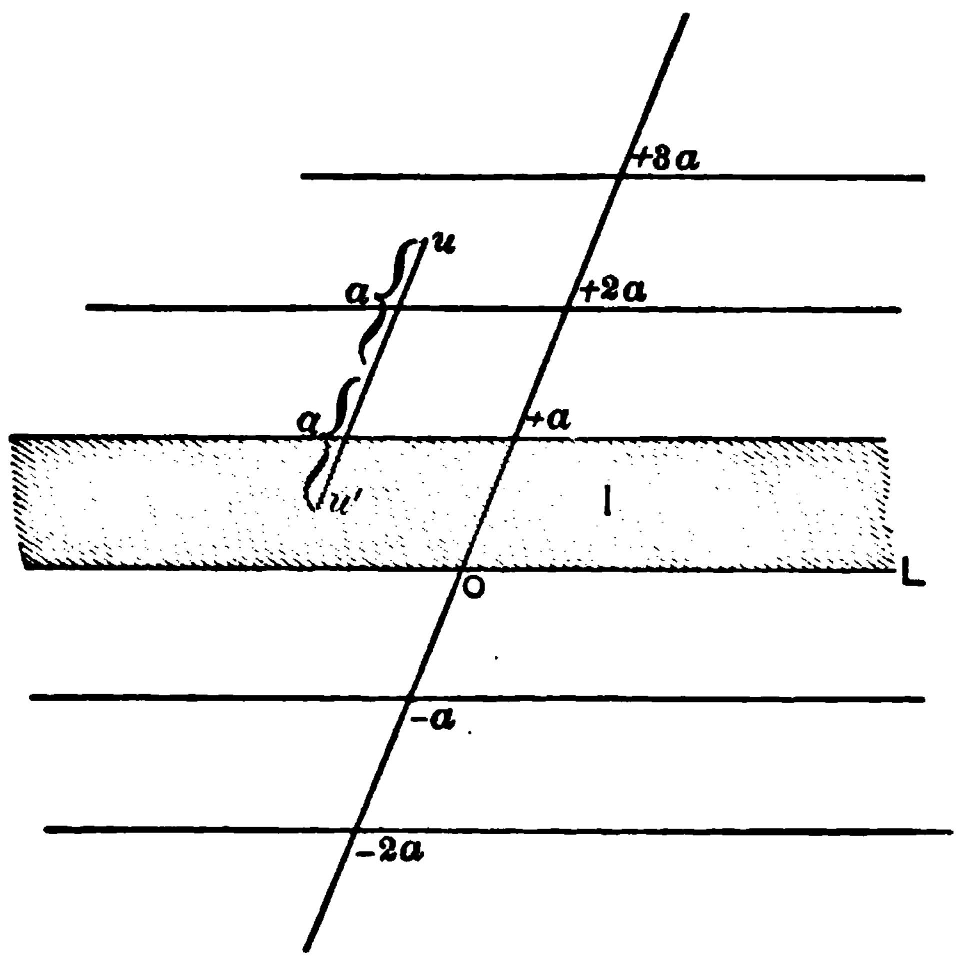


Fig. 6.

We draw through the origin an arbitrary straight line OL, and through the points $a,\ 2a,\ 3a, \ldots, -a,\ -2a, \ldots$ we draw lines parallel to OL. The entire u-plane is thus distributed into an indefinite number of strips.

That strip which is made by OL and the straight line through $+a$ parallel to OL we call the *initial strip.*

Let u be a point in any strip. There is always a point u' in the initial strip at which $f(u)$ has the same value as at u. For if through the point u we draw a line parallel to the line that goes through the points 0, a, $2a, \ldots$, and on this line measure off distances a until we come within the initial strip and call u' the end-point of the last distance measured off, then u and u' differ only by integral multiples of a, so that the function $f(u)$ has the same value at both points. In the above figure, for example, $u = u' + 2a$, so that $f(u) = f(u' + 2a) = f(u')$. Hence every value that the function can take in the u-plane is had also in each single strip. We therefore need investigate every simply periodic function only within a single period-strip. This we have done above for the simple cases of e^u, $\sin u$, $\cos u$.

ART. 59. If a represents any complex quantity, we saw in Art. 26 that a simply periodic function with a as a period may be readily formed. Such a function was $e^{\frac{2\pi i}{a}u}$.

Consider next the series

$$f(u) = \sum_{k=-\infty}^{k=+\infty} c_k e^{k\frac{2\pi i}{a}u},$$

where the constants c_k may always be so chosen that the series is convergent.* It is clear that the function just written has the period a; and, since the constants c_k may be determined in different ways, it is clear that an arbitrarily large number of such functions may be formed, all of which have the period a.

Such a function is

$$w = \frac{\sum_{k=-\infty}^{k=+\infty} c_k e^{k\frac{2\pi i}{a}u}}{\sum_{k=-\infty}^{k=+\infty} d_k e^{k\frac{2\pi i}{a}u}} = \phi(u), \quad \text{say},$$

where the d_k's are also constants.

All such functions have the property that there is no essential singularity in the finite part of the plane and they are indeterminate for no finite value of u.

For the point $u = \infty$ the exponential function is indeterminate (Art. 21), and for all other values of u it is seen that the function $\phi(u)$ is one-valued.

ART. 60. Suppose that $f(u)$ is a one-valued simply periodic function with period $a = 2\omega$, and which has only polar singularities in the finite portion of the plane.

* Cf. Briot et Bouquet, *Fonctions Elliptiques*, p. 161.

If we put

$$e^{\frac{\pi i u}{\omega}} = t = re^{i\theta},$$

it is seen that

$$u = -\frac{i\omega \log r}{\pi} + \frac{\omega\theta}{\pi}.$$

Hence in the t-plane, when θ varies from 0 to 2π, the variable t describes a circle about the origin with radius r, while in the u-plane the variable u describes the straight line AA', where $A = -i\omega \log r$ and $A' = -i\omega \log r + 2\omega$. Further, when θ varies from 2π to 4π, u varies from A' to A'', where again $A'A'' = 2\omega$, etc.

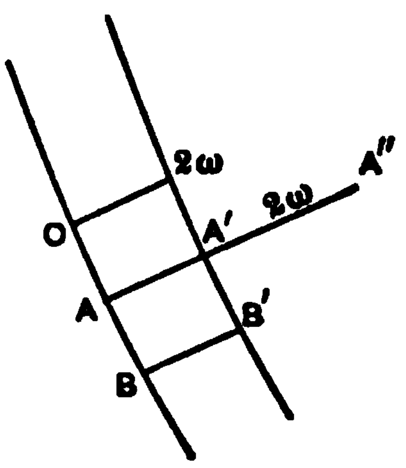


Fig. 7.

Next if we give to t the value $se^{i\theta}$, it is seen that when t describes a circle about the origin in the t-plane with radius s, u describes the straight line BB', where $B = -i\omega \log s$ and $B' = -i\omega \log s + 2\omega$.

It follows that in the u-plane the rectangle $AA'BB'$ corresponds to the ring included between the two circles with radii r and s in the t-plane, and corresponding to the initial period-strip in the u-plane is the entire t-plane. Further, any period-strip is, as we may say, *conformally represented* on the t-plane. There being an indefinite number of these strips, it is evident that to any value of t in the t-plane corresponds an infinite number of values in the u-plane differing by integral multiples of 2ω.

Suppose that the rectangle $AA'BB'$ is taken so as not to include any of the singularities of $f(u)$. Then if $F(t) = f(u)$, it is seen that $F(t)$ is regular at all points at which $f(u)$ is regular and consequently may be expanded by Laurent's Theorem in a series of the form

$$F(t) = \sum_{n=-\infty}^{n=+\infty} c_n t^n,$$

which is convergent for all values of t situated *within* the ring-formed surface in the t-plane that corresponds to the rectangle $AA'BB'$.

It also follows that for all points *within* this rectangle, $f(u)$ may be expressed in a convergent series of the form

$$f(u) = \sum_{n=-\infty}^{n=+\infty} c_n e^{\frac{n\pi i u}{\omega}};$$

or expressed in a Fourier's Series,

$$f(u) = \sum_{n=0}^{n=\infty}\left(a_n \cos n\frac{\pi}{\omega}u + b_n \sin n\frac{\pi}{\omega}u\right),$$

where

$$a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n}).$$

Prof. Osgood, *loc. cit.*, pp. 406 *et seq.*, gives more explicitly the limits within which such series are convergent.*

ART. 61. We next propose to study all *those simply periodic functions which first are indeterminate for no finite value of u, which therefore in the finite portion of the plane have no essential singularity, while they are indeterminate for u = infinity; which secondly are one-valued; and which thirdly within a period-strip take a prescribed value a finite number of times.*

Suppose that $\phi(u)$ is such a function. The function $\phi(u)$ behaves within the period-strip in a similar manner as do the rational functions in the whole plane. For if $w = \Phi(u)$ is a rational function of u, then $\Phi(u)$ is one-valued and for every given value of w there is only a finite number of values of u. In Art. 63 it is shown that at the end-points of the period-strip the function has definite values.

It is easy to see that the function $\phi(u)$ which we are considering must be *indeterminate* at infinity in the direction of the line through $0, a, 2a, \ldots$ (see Fig. 6). For let u_0 be a point within the initial period-strip. Draw through u_0 a line parallel to the line through $0, a, 2a, \cdots$. On this line, starting from u_0, we measure off distances a an indefinitely large number of times. We thus come finally to infinity and the function takes at the end of the last distance that has been laid off the value $\phi(u_0)$. Next if we start with another point u_1 and proceed to infinity in the same way as before, the function will take for the infinitely distant point the value $\phi(u_1)$. Hence at infinity there appear all possible values which the function $\phi(u)$ can take, and the function is thus said to be *indeterminate* at infinity (cf. Art. 3).

ART. 62. Let $w = \phi(u)$ be a simply periodic function with the period a which satisfies the three postulates made above. Further, write

$$t = e^{\frac{2\pi i}{a}u},$$

so that t and w have the same period a and may consequently both be considered within the same period-strip of the u-plane. Next suppose a given value is ascribed to t. Within this period-strip there is (Art. 56) *one* definite value of u which belongs to the prescribed value of t. If we write this value of u in the function $\phi(u)$, then $w = \phi(u)$ has a definite value. It is thus shown that to every value of t there belongs a definite value of w. If next we consider not only one period-strip but the whole u-plane, then there belongs to the given value of t an infinite number of values of u, namely in each period-strip one value. And if u is one of these values then all the other values have the form $u + ka$, where k is a positive or negative integer. If we write all these values in $\phi(u)$, then $w = \phi(u)$ takes always the same value, since $\phi(u + ak) = \phi(u)$. Hence

* See also Henri Lebesgue, *Leçons sur les séries trigonométriques.*

also when we consider the whole u-plane, for every definite value of t there is *one* definite value of w. Thus we have shown that *w is a one-valued function of t.* For a definite value of w there are after the third of the above postulates only a finite number of values of the argument u in each period-strip. Let those values of u belonging to the strip in question be $u_1, u_2, \ldots, u_m$, and let the corresponding values of t be

$$t_1 = e^{\frac{2\pi i}{a}u_1}, \quad t_2 = e^{\frac{2\pi i}{a}u_2}, \quad \ldots, \quad t_m = e^{\frac{2\pi i}{a}u_m}.$$

There are no other values of t which belong to the given value of w; for if we extend our consideration to the whole u-plane, that is, if with the given value of w we also associate those values of u which differ from $u_1, u_2, \ldots, u_m$ by integral powers of a, we still have for t always one of the values $t_1, t_2, \ldots, t_m$.

We have previously seen that to each value of t there belongs only one value of w. We now see that to every value of w there belong m values of t and therefore that t is an m-valued function of w. It follows that w and t are connected by an algebraic equation which is of the first degree in w and the mth degree in t, say,

$$F(w, t) = 0.$$

Solving this equation we have

$$w = \psi(t),$$

where ψ denotes an algebraic function of t.

On the other hand we saw that w was a one-valued function of t, and since one-valued algebraic functions are the rational functions, it follows that w is a rational function of $t = e^{\frac{2\pi i}{a}u}$.

We have then the important theorem:

Every simply periodic function $\phi(u)$ which is indeterminate for no value of u, and has an essential singularity only at infinity, which is one-valued and within a period-strip can take an ascribed value only a finite number of times is a rational function of $t = e^{\frac{2\pi i}{a}u}$, where a is the period of $\phi(u)$.*

All such functions may therefore be written in the form

$$w = \frac{\sum_{k=0}^{k=m} c_k e^{k\frac{2\pi i}{a}u}}{\sum_{k=0}^{k=n} d_k e^{k\frac{2\pi i}{a}u}} = \phi(u).$$

where the c_k and d_k are constants.

* A treatment of simply periodic functions which have essential singularities elsewhere than at infinity is given by Guichard, *Théorie des points singuliers essentiels* [Thèse, Gauthier-Villars, Paris. 1883].

There are no other simply periodic functions which have the required properties.

ART. 63. We may make m and n equal in the above expression without affecting its generality. For suppose $n < m$. We have then to put all the d's in the denominator equal to zero from d_{n+1} to d_m. If $n > m$, we make the corresponding change in the numerator. It follows that all simply periodic functions belonging to the category defined above may be expressed in the form

$$\phi(u) = w = \frac{\sum_{k=0}^{k=m} c_k e^{k\frac{2\pi i}{a}u}}{\sum_{k=0}^{k=m} d_k e^{k\frac{2\pi i}{a}u}} = \frac{\sum_{k=0}^{k=m} c_k t^k}{\sum_{k=0}^{k=m} d_k t^k} = \psi(t),$$

where ψ is a rational function of t. Hence the points $t = \pm\infty$, $t = 0$ are *not* essential singularities of $\psi(t)$ and consequently also $\phi(u)$ has definite values for $u = \pm\infty$. In other words, the end-points of the period-strips of the function $\phi(u)$ are not essential singularities.

We may write the above equation in the form

$$(c_m - d_m w)t^m + (c_{m-1} - d_{m-1}\, w)t^{m-1} + \cdots + (c_0 - d_0 w) = 0,$$

where m represents the number of values which $t = e^{\frac{2\pi i}{a}u}$ can take for a given value of w, or, in other words, the number of points in each period-strip at which $w = \phi(u)$ takes a definitely prescribed value. We call m the *degree* or *order* of the simply periodic function $w = \phi(u)$ (cf. again Art. 10).

The functions $\cos u$ and $\sin u$ must be expressible in the above form, since for them $a = 2\pi$, and $t = e^{\frac{2\pi i}{2\pi}u} = e^{iu}$. Further, these functions take a prescribed value *twice* within a period-strip (cf. Art. 57) and are consequently simply periodic functions of the second degree. For them we must have $m = 2$, which, indeed, is seen from the relations

$$\cos u = \tfrac{1}{2}(e^{iu} + e^{-iu}) = \frac{1}{2}\left(t + \frac{1}{t}\right) = \frac{t^2 + 0\cdot t + 1}{2(0\cdot t^2 + t)};$$

$$\sin u = \frac{1}{2i}(e^{iu} - e^{-iu}) = \frac{1}{2i}\left(t - \frac{1}{t}\right) = \frac{1}{2i}\,\frac{t^2 + 0\cdot t - 1}{0\cdot t^2 + t}.$$

Owing to the relation $\phi(u) = \psi(t)$ many of the properties of simply periodic functions may be changed into properties of rational functions; for example, the function $\phi(u)$ has as many zeros as it has infinities in each period-strip.*

* Cf. Briot et Bouquet, *Fonctions Elliptiques*, p. 161; Forsyth, *loc. cit.*, p. 215; Osgood, *loc. cit.*, p. 409; Burkhardt, *Analyt. Funktionen einer komplexen Veränderlichen*, p. 161.

The Eliminant Equation.

Art. 64. In the case of the function e^u it is seen that if

$$w = e^u, \text{ then } \frac{dw}{du} - u = 0;$$

and if $$w = \cos u, \text{ then } \left(\frac{dw}{du}\right)^2 - (1 - w^2) = 0,$$

the latter differential equation being satisfied also if $w = \sin u$. We note that these three functions have the characteristic that each of them satisfies a differential equation in which the independent variable u does not explicitly appear.

From the previous article we saw that, if w is a simply periodic function, then

$$w = \phi(u) = \psi(t),$$

where ψ is a rational function.

Further, since $e^{\frac{2\pi i}{a}u} = t$, we have

$$\frac{dw}{du} = \psi'(t)\frac{2\pi i}{a}t = \psi_1(t),$$

where ψ_1 is also a rational function.

By eliminating t from the two expressions we have the eliminant equation (Art. 34)

$$f\left(w, \frac{dw}{du}\right) = 0,$$

where f denotes an integral algebraic function.

In Art. 41 we said that if there existed an eliminant equation for a one-valued function $w = \phi(u)$, then $\phi(u)$ had an algebraic addition-theorem and belonged to one of the categories of functions

I. Rational function of u, or

II. Rational function of $e^{\frac{2\pi i}{a}u}$ (simply periodic), or

III. Doubly periodic function.

In his *Cours d'Analyse à l'École Polytechnique*, in 1873, Hermite observed that if the equation

$$f\left(w, \frac{dw}{du}\right) = 0$$

admits a one-valued integral (that is, if w is a one-valued function of u), we may express w and $\frac{dw}{du}$ rationally in terms of an auxiliary variable t, if the integral w is a rational function of u, or if it is a simply periodic function of u; and that w and $\frac{dw}{du}$ may be expressed through formulas

which include no other irrationalities than the square root of a polynomial of the fourth degree, if w is a doubly periodic function.*

ART. 65. The following question arises: *What further conditions must be satisfied in order that an integral of the equation* $f\left(w, \frac{dw}{du}\right) = 0$, *belong to the category of functions defined in Art.* 61?

Such a function must, as we have already seen, be expressible as a rational function of t, say $\psi(t)$, and its derivative is also a rational function $\psi_1(t)$.

If we put $\frac{dw}{du} = v$, the above equation is

$$f(w, v) = 0.$$

We may regard this integral algebraic equation as the equation of a curve. Strictly speaking, however, this can only be done if w and v are real quantities; still we may speak of a curve, for the sake of a graphical representation, even if as here w and v are complex quantities. From what was shown above, if we write for w a certain rational function $\psi(t)$ and for v a rational function $\psi_1(t)$, the equation $f(w, v) = 0$ must be identically satisfied for all values of t. We may therefore express w and v rationally through a parameter t in the form of the equations $w = \psi(t)$, $v = \psi_1(t)$. Curves in which such a rational representation of the variable t is possible are known as *unicursal*.†

If then an integral of the differential equation

$$f\left(w, \frac{dw}{du}\right) = 0$$

is to belong to the category of functions which we are studying, the equation

$$f(w, v) = 0$$

must represent a unicursal curve.

But this condition is not sufficient. For if $f(w, v) = 0$ represents a unicursal curve, there is an infinite number of ways in which w and v may be expressed rationally in terms of t. But among these ways there is one which is such that t for every prescribed pair of values of w and v takes only one definite value. Further, if w is a function of our category, it must be a *one-valued* function of u, and consequently $v = \frac{dw}{du}$ is also a one-valued function of u.

But if w and v are given, there is (as we have just seen) only one value of t which can be associated with them. Hence if w is a function of our

* Cf. Cayley, *Lond. Math. Soc.*, Vol. IV (1873), pp. 343–345.

† The name is due to Cayley, *Comptes rendus*, t. 62, who derived the fundamental properties of these curves. See also Clebsch, *Ueber diejenigen ebenen Curven deren Coördinaten rationale Funktionen eines Parameters sind, Crelle*, Bd. 64.

category, the parameter t must be a *one-valued* function of u. Further, since

$$\frac{dw}{du} = \psi'(t)\frac{dt}{du} = \psi_1(t),$$

it follows that

$$\frac{dt}{du} = \frac{\psi_1(t)}{\psi'(t)} = R(t),$$

where R is a rational function of t.

We have consequently established the following: *The integrals of the differential equation*

$$f\left(w, \frac{dw}{du}\right) = 0$$

may be functions of our category, first, if the equation $f(w, v) = 0$ represents a unicursal curve; second, if w and v are such rational functions of a parameter t that to every pair of values of w, v there belongs only one value of t; and third, if the parameter t, as determined through the equation $f\left(w, \frac{dw}{du}\right) = 0$, is a one-valued function of u.* It does not, then, necessarily follow that these integrals are simply periodic, for they may be *rational functions* of u.

ART. 66. The parameter t determined from the differential equation

$$\frac{dt}{du} = R(t)$$

must be a one-valued function of u.

We are thus led to the question: *What is the nature of the function $R(t)$, that t be a one-valued function of u?*

If we consider first the differential equation

$$g_0\left(\frac{dt}{du}\right)^m + g_1\left(\frac{dt}{du}\right)^{m-1} + g_2\left(\frac{dt}{du}\right)^{m-2} + \cdots + g_m = 0,$$

where the g's are integral functions of t, the condition that an integral t of this equation be a one-valued function of u is that g_0 be of the 0 degree, g_1 of the 2d degree, g_2 of the 4th, . . . , g_m of the $2m$th degree.†

We shall derive these results for the case $m = 2$ in Chapter V, and from this it will be seen in the simple case before us, viz.,

$$\frac{dt}{du} = R(t),$$

* A simple method of representing w and v as rational functions of a parameter t, when this can be done, is given by Nöther, *Math. Ann.*, Bd. III; see also Lüroth, *Math. Ann.*, Bd. IV.

† Cf. Forsyth, *loc. cit.*, p. 481, where other references are given.

that t is a one-valued function of u, if $R(t)$ is a rational integral function of the 2d degree in t. It then we write $R(t) = a_0 + a_1t + a_2t^2$, it follows that

$$u + \gamma = \int \frac{dt}{a_0 + a_1t + a_2t^2},$$

where γ is a constant.

We have *four cases* to consider:

(1) Suppose that $a_2 \neq 0$ and that the roots of the equation $a_0 + a_1t + a_2t^2 = 0$ are *not* equal.

We may then write the above integral in the form

$$u + \gamma = \int \frac{dt}{a_2(t - \alpha)(t - \beta)} = \frac{1}{a_2(\alpha - \beta)} \int \left[\frac{1}{t - \alpha} - \frac{1}{t - \beta}\right] dt$$
$$= \frac{1}{a_2(\alpha - \beta)} \log \frac{t - \alpha}{t - \beta}.$$

It follows that

$$\frac{t - \alpha}{t - \beta} = e^{a_2(\alpha - \beta)(u + \gamma)},$$

and consequently t may be determined rationally in terms of an exponential function of u. Since $w = \psi(t)$, where ψ is a rational function, it is seen that in this case w is a rational function of an exponential function and therefore belongs to our category of functions.

(2) Suppose that $a_2 \neq 0$ and that the roots of the equation $a_0 + a_1t + a_2t^2 = 0$ are equal.

We then have

$$u + \gamma = \int \frac{dt}{a_2(t - \alpha)^2} = \frac{-1}{a_2(t - \alpha)}.$$

It is seen that in this case t is a rational function of u, and since w is a rational function of t, w is a rational function of u and does *not* belong to our category of simply-periodic functions.

(3) Suppose that $a_2 = 0$. We then have

$$u + \gamma = \int \frac{dt}{a_0 + a_1t} = \frac{1}{a_1} \log (a_0 + a_1t).$$

It follows that $a_0 + a_1t = e^{a_1(u+\gamma)}$, so that w belongs to our category of functions.

(4) Suppose that $a_2 = 0 = a_1$. It is evident then that

$$u + \gamma = \int \frac{dt}{a_0} = \frac{t}{a_0},$$

or

$$t = a_0(u + \gamma).$$

In this case w is *not* a simply-periodic function.

EXAMPLES

1. Consider the differential equation

$$f\left(w, \frac{dw}{du}\right) = w^4 - \left(w - \frac{dw}{du}\right)^3 = 0;$$

or, if

$$v = \frac{dw}{du},$$

$$f(w, v) = w^4 - (w - v)^3 = 0.$$

We must first determine whether this equation represents a unicursal curve.

If we write

$$w - v = tw,$$

then is

$$w^4 - w^3t^3 = 0,$$

or

$$w = t^3 = \psi(t);$$

and

$$v = w(1 - t) = t^3(1 - t) = \psi_1(t).$$

It is thus seen that w and v may be rationally expressed through t.

We must next see whether t, as thus determined, has a definite value when w and v have prescribed values.

Since $w = t^3$, to one value of w there correspond *three* values of t, but only *one* of these can satisfy the relation $v = w(1 - t)$, when a fixed value is given to v. Hence to every pair of values w, v there belongs a single definite value of t. We further have $\psi'(t) = 3t^2$ and

$$\frac{dt}{du} = \frac{\psi_1(t)}{\psi'(t)} = R(t) = \frac{t^3(1 - t)}{3t^2} = \frac{1}{3}t(1 - t).$$

It follows that

$$u + \gamma = \int \frac{dt}{\frac{1}{3}t(1 - t)}.$$

This is the *first* case considered above where $a_2 = -\frac{1}{3}$, $\alpha = 0$, $\beta = 1$.

Integrating we have

$$\frac{t}{t - 1} = e^{\frac{1}{3}(u+\gamma)}, \quad \text{or} \quad t = \frac{-e^{\frac{1}{3}(u+\gamma)}}{1 - e^{\frac{1}{3}(u+\gamma)}},$$

and

$$w = t^3 = \frac{-e^{(u+\gamma)}}{[1 - e^{\frac{1}{3}(u+\gamma)}]^3}.$$

It is thus shown that w belongs to the category of functions considered.

2. Determine the integrals of the differential equation

$$f\left(w, \frac{dw}{du}\right) = \left(\frac{dw}{du} - 2\right)^2 + \left(\frac{dw}{du} - 1\right)w^2 = 0,$$

that is, of

$$f(w, v) = (v - 2)^2 + (v - 1)w^2 = 0, \quad \text{where } v = \frac{dw}{du}.$$

It follows that

$$\frac{\partial f}{\partial w} = 2w(v - 1) = 0,$$

$$\frac{\partial f}{\partial v} = 2(v - 2) + w^2 = 0,$$

and consequently $w = 0$, $v = 2$ is the double point.

Hence, if we write $w = (v - 2)t$, it follows from the equation of the curve that

$$1 + (v - 1)t^2 = 0,$$

or

$$v = 1 - \frac{1}{t^2} = \psi_1(t)$$

and

$$w = - t - \frac{1}{t} = \psi(t).$$

The curve is therefore unicursal; and further to every value of v there belong two values of t, but of these only one can satisfy the equation for w when to v a fixed value is given.

It is also seen that

$$R(t) = \frac{dt}{du} = \frac{\psi_1(t)}{\psi'(t)} = - 1,$$

and consequently we have the fourth case.

It follows that

$$t = -(u + \gamma)$$

and

$$w = \frac{1}{u + \gamma} + u + \gamma,$$

and being a rational function of u does *not* belong to our category of functions.

3. Show that the integrals of the differential equation

$$f\left(w, \frac{dw}{du}\right) = \left(\frac{dw}{du} - 2\right)^2 \left(\frac{dw}{du} + 1\right) = (3\,w^2 + 2)^2$$

are simply periodic functions.

Note that the equation

$$f(w, v) = (v - 2)^2\,(v + 1) = (3\,w^2 + 2)^2$$

is satisfied by

$$v = 3(1 + t^2)\,(1 + 3\,t^2),$$
$$w = 3(t + t^3).$$

4. Show that the integrals of

$$\left(\frac{dw}{du}\right)^3 - \left(\frac{dw}{du}\right)^2 + 4\,w^3 - 27\,w^6 = 0$$

are rational functions of u. [Briot et Bouquet.]

5. Show that the integrals of

$$\left(\frac{dw}{du}\right)^3 - \left(\frac{dw}{du}\right)^2 - 4\,w^2 - 27\,w^4 = 0$$

are simply periodic functions of u. [Briot et Bouquet.]

CHAPTER IV

DOUBLY PERIODIC FUNCTIONS. THEIR EXISTENCE. THE PERIODS

ARTICLE 67. Returning to the exponential function $e^{\mu u}$, we know that $\dfrac{2\pi i}{\mu} = 2\,\omega$, say, is its period.

The constant μ is taken real or complex and different from zero or infinity.

Write $t = e^{\mu u} = e^{\frac{u\pi i}{\omega}}$, and consider the function $\phi(u) = \psi(t)$, where *here* ψ is *not* necessarily a rational function.

Draw the period-strip as in the figure and let u be any point within or on the boundaries of this strip.

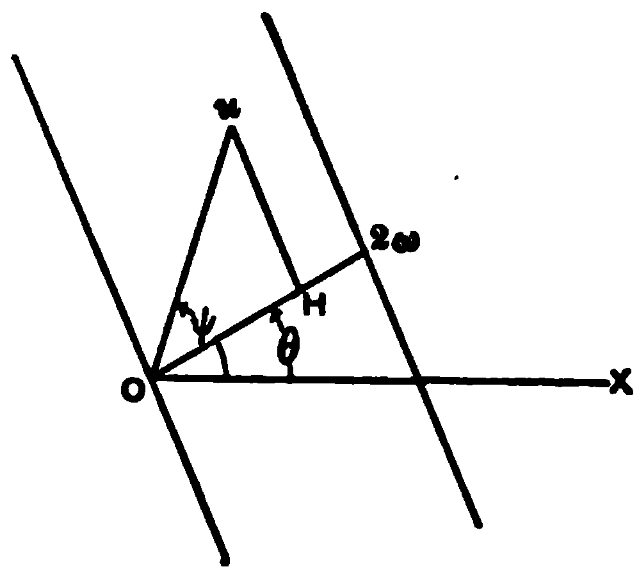


Fig. 8.

Let $|\,u\,|$ be r and $|\,2\,\omega\,|$ be s, so that

$$\frac{u}{2\,\omega} = \frac{re^{i\psi}}{se^{i\theta}} = \frac{r}{s}e^{i(\psi-\theta)}$$

$$= \frac{r}{s}[\cos(\psi-\theta) + i\sin(\psi-\theta)].$$

If R denotes the real part of the complex quantity after it, then is

$$\mathsf{R}\left(\frac{u}{2\,\omega}\right) = \frac{r}{s}\cos(\psi-\theta) = \frac{OH}{s}.$$

Hence for all values u within the period-strip we have

$$0 \leqq \mathsf{R}\left(\frac{u}{2\,\omega}\right) \leqq 1.$$

We assume that $\phi(u) = \phi(u + 2\,\omega)$ and that $\phi(u)$ has the character of an integral or (fractional) rational function for all points within the period-strip except the two points $\pm\,\infty$.

We shall show (cf. Art. 62) that if $\phi(u)$ is a one-valued function of u, it is also a one-valued function of t. Let u_1 be a point within the period-strip. We therefore have in the neighborhood of u_1

$$\phi(u) = G\left\{\frac{1}{u - u_1}\right\} + P(u - u_1), \tag{A}$$

where G denotes an integral function of finite degree (including the 0th degree) and where P is a power series with positive integral exponents.

Let $$t_1 = e^{\frac{u_1\pi i}{\omega}}, \text{ so that } e^{(u-u_1)\frac{\pi i}{\omega}} = \frac{t}{t_1};$$

further, write $$\frac{t}{t_1} = 1 + \tau \text{ or } e^{(u-u_1)\frac{\pi i}{\omega}} = 1 + \tau.$$

It follows that

$$(u - u_1)\frac{\pi i}{\omega} = \log(1 + \tau) = \tau - \frac{1}{2}\tau^2 + \frac{1}{3}\tau^3 - \cdots.$$

This series is convergent for values of τ, such that

$$0 < |\tau| < 1.$$

But we had $$\tau = \frac{t}{t_1} - 1 = \frac{t - t_1}{t_1}.$$

If then $|t - t_1| < |t_1|$, we have the convergent series

$$(u - u_1)\frac{\pi i}{\omega} = \frac{t - t_1}{t_1} - \frac{1}{2}\left(\frac{t - t_1}{t_1}\right)^2 + \frac{1}{3}\left(\frac{t - t_1}{t_1}\right)^3 - \cdots.$$

This expression for $u - u_1$ substituted in the equation (A) shows that the function $\phi(u)$ considered as a function of t is one-valued and has the same character for $t = t_1$ as it has for $u = u_1$.

ART. 68. With regard to the function $\phi(u) = \psi(t)$ two cases may arise: (1) the two points $t = 0$, $t = \infty$ may be regular points of the function. In this case $\psi(t)$ is a rational function, as there is no essential singularity. (2) At least one of the points $t = 0$, $t = \infty$ may be an essential singularity. In this case *we shall show that the function $\phi(u)$ has another period $2\omega'$, say, and we shall prove that the ratio $\frac{2\omega}{2\omega'}$ is not a real quantity.*

We must show that within the period-strip there are values which may be taken an arbitrarily large number of times by $\phi(u)$. It follows then as in Art. 38 that there exists another period $2\omega'$.

Let ξ_0 be a value which $\phi(u)$ may take. This point may lie anywhere in the finite portion of the period-strip excepting the singular values of u defined in Art. 37.

Two cases are here possible: (1) The function $\phi(u) = \psi(t)$ may take the value ξ_0 an arbitrarily large number of times. The theorem is then proved. (2) The function $\phi(u)$ may take the value ξ_0 a finite number of times, say m, within the period-strip. Let the corresponding values of t be $t_1, t_2, \ldots, t_m$.

In the neighborhood of any one of these points develop $\dfrac{1}{\psi(t) - \xi_0}$ by Laurent's Theorem.

Then as in Art. 53 it is seen that the absolute value of this expression surpasses every limit for values of t as we approach one or the other (or

possibly both) of the points $t = 0$ or $t = \infty$. There are then values ξ_1, say, in the neighborhood of ξ_0 which are taken by $\phi(u) = \psi(t)$ at least $m + 1$ times. By continuing this process it is shown as in Art. 38 that $\phi(u)$ must have another period $2\omega'$ and consequently

$$\phi(u + 2\omega) = \phi(u),$$
$$\phi(u + 2\omega') = \phi(u).$$

ART. 69. It follows at once from the development of $\phi(u)$ in the neighborhood of u_1 in the form (Art. 53)

$$\phi(u) = G\left(\frac{1}{u - u_1}\right) + P(u - u_1),$$

that there are no points in the immediate vicinity of u_1 at which $\phi(u)$ has the same value * (Art. 8) as it has at u_1. We may therefore draw with u_1 as center a circle with radius ρ which is so small (but of finite length) that within the circle the function $\phi(u)$ does not take the same value twice. Further, since $\phi(u + 2\omega) = \phi(u)$, it is evident that $|2\omega| > \rho$, where ρ is a finite quantity.

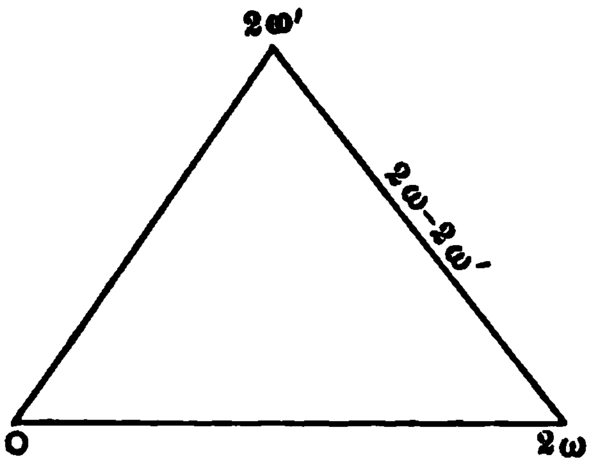


Fig. 9.

The point in the u-plane which represents 2ω we call a *period-point*. Since $2\omega'$ is also a period-point, it is evident that

$$\phi(u + 2\omega - 2\omega') = \phi(u),$$

and as above

$$|2\omega - 2\omega'| > \rho.$$

It is thus shown that the *distance between two period-points is always a finite quantity.*

It is also evident that if we bound any arbitrary but finite portion of surface (S) in the u-plane, there are only a finite number of period-points within this surface.

If A is a period-point and if B and D are the next period-points to A, then C, the other vertex of the parallelogram, is also a period-point. From what we have just seen this parallelogram has a finite area. If then there were an infinite number of period-points within (S), there would be within this area (S) an infinite number of parallelograms with finite area, which is impossible.

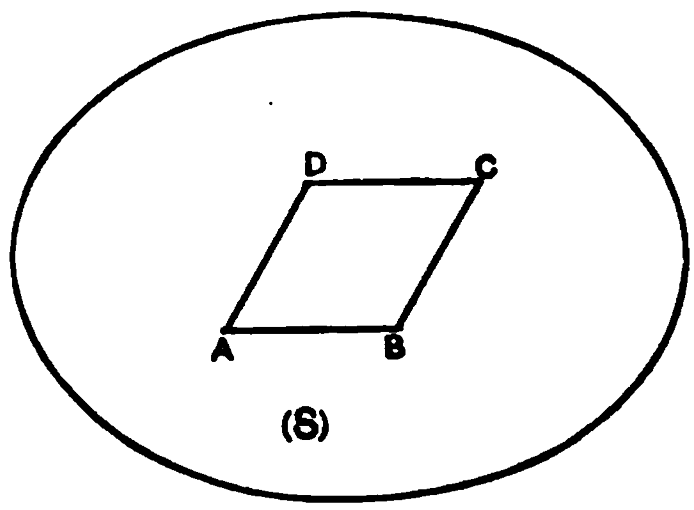


Fig. 10.

* Cf. Burkhardt, *Analyt. Funkt.*, p. 124; Forsyth, *loc. cit.*, p. 59; Osgood, *Lehrbuch der Funktionentheorie*, p. 398.

ART. 70. We consider the following question: If $2\omega = a$ and $2\omega' = b$ are periods of the function $F(u)$ and in the sense that they are not integral multiples of one and the same primitive period, is it possible for the point b to lie on the line joining the origin and the point a?

The quantities a and b may be written in the form

$$a = re^{i\theta},$$
$$b = se^{i\phi};$$

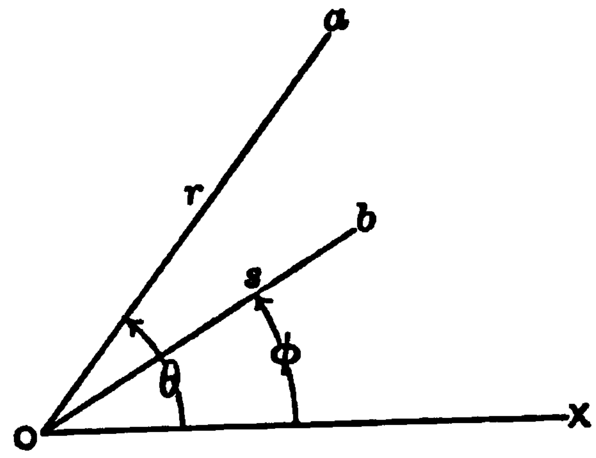


Fig. 11.

and consequently, if b lies upon the straight line Oa, then

$$\phi = \theta \text{ or } \phi = \theta + \pi.$$

We therefore have

$$\frac{a}{b} = \pm\frac{r}{s},$$

that is, the ratio $\frac{a}{b}$ is a real quantity. The above question may consequently be expressed as follows: *Can the quotient of two periods a and b be a real quantity?*

Suppose this were the case and that the point b lies upon the line Oa. The quantity a is either a primitive period or it is *not* a primitive period. If it is not, it may be written in the form $a = m\alpha$, where α *is* a primitive period and m an integer. We also know that $|\alpha| > \rho$, where ρ is a finite quantity. We measure off upon the line Oa in the direction of the point b distances α and have the points $\alpha, 2\alpha, \ldots, k\alpha, (k+1)\alpha, \cdots$. If b coincided with one of these points, for example $k\alpha$, we would have

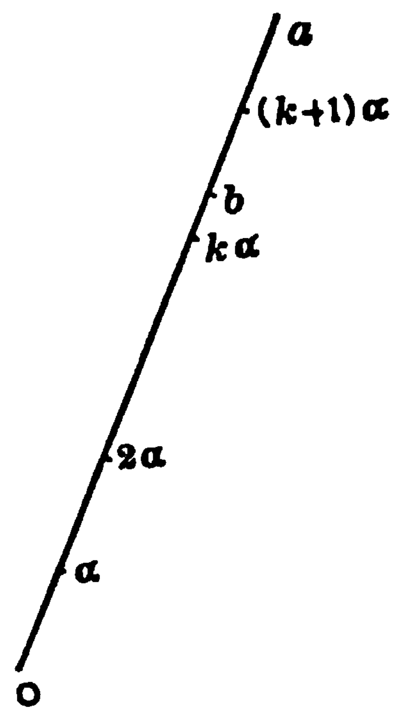


Fig. 12.

$$b = k\alpha, \quad a = m\alpha,$$

which is contrary to our hypothesis.

It follows that b must lie between two of the distances measured off, say between $k\alpha$ and $(k+1)\alpha$.

Since both b and $k\alpha$ are periods, the distance $b - k\alpha$ is also a period. We therefore have

$$|b - k\alpha| < |\alpha|.$$

Writing $b - k\alpha = \alpha'$, we measure off this new period along the line Oa and make for α the same conclusions as we did above for b. We find that $\alpha - l\alpha'$ is a new period, where l is an integer. This period is such that

$$|\alpha - l\alpha'| < \alpha'.$$

By continuing this process we come finally to periods whose absolute values are smaller than any assignable finite quantity ρ, which is a contradiction of what was proved in Art. 69.

We have thus shown the following: *If the quotient* $\frac{a}{b}$ *is real, there exists a primitive period of which a and b are integral multiples. If a and b are two different periods, as defined at the beginning of this article, then the ratio* $\frac{a}{b}$ *cannot be real, and b cannot lie upon the line Oa.*

ART. 71. The above theorem is due to Jacobi (Werke, Bd. II, pp. 25, 26), who proved it as follows: Suppose *first* that the ratio $\frac{b}{a}$ is *rational* and write $\frac{b}{a} = \frac{p_2}{p_1}$, where p_2 and p_1 are integers that are relatively prime.

It follows that

$$\frac{b}{p_2} = \frac{a}{p_1} = \alpha, \quad \text{say},$$

and consequently $b = p_2\alpha$ and $a = p_1\alpha$. To show that α is a period we determine two integers q_1, q_2, such that

$$p_1q_1 + p_2q_2 = 1.$$

We know that there are an infinite number of solutions of this equation. Multiplying by α we have

$$p_1\alpha q_1 + p_2\alpha q_2 = \alpha,$$

or

$$q_1a + q_2b = \alpha.$$

Thus α is composed of integral multiples of the periods a and b and is consequently a period. Consequently (Art. 70) a and b cannot be considered as two different periods.

Suppose *next* that the ratio $\frac{b}{a}$ is real but *irrational*. In the theory of continued fractions we know that if

$\frac{v_n}{u_n}$, $\frac{v_{n+1}}{u_{n+1}}$ are consecutive convergents, then

$$\frac{v_n}{u_n} \sim \frac{v_{n+1}}{u_{n+1}} = \frac{1}{u_nu_{n+1}} = \frac{\varepsilon}{u_n^2}, \text{ where } \varepsilon < 1.$$

Hence if we expand $\frac{b}{a}$ in a continued fraction and if $\frac{\gamma_n}{\delta_n}$ is the nth convergent, then is

$$\frac{b}{a} \sim \frac{\gamma_n}{\delta_n} = \frac{\varepsilon}{\delta_n^2}, \quad \text{or} \quad \delta_nb - \gamma_na = \frac{\varepsilon a}{\delta_n}.$$

Since δ_n may be made indefinitely large, it follows that

$$|\, \delta_nb - \gamma_na \,| < \rho, \text{ where } \rho \text{ is as small as we choose.}$$

Further, since δ_n and γ_n are integers, the left-hand side is a period. This contradicts what was given in Art. 69. It is thus seen that the ratio $\frac{b}{a}$ must be a complex quantity * (including the case of a pure imaginary).

* See Pringsheim, *Math. Ann.*, Bd. 27, pp. 151–157; Falk, *Acta Math.*, Bd. 7, pp. 197–200; W. W. Johnson, *Am. Journ.*, Vol. 6, pp. 246–253; Fuchs, *Crelle*, Bd. 83, pp. 13 *et seq.*; Méray, *Ann. de l'Ecole Norm. Sup.* (3), t. 1, pp. 177–184.

ART. 72. We may, however, prove that if the ratio of any two periods is real it is *also rational*. For let $2\omega_2$, $2\omega_1$ be any two periods whose ratio is real. The ratio $\frac{2\omega_2}{2\omega_1}$ may always be taken positive; for if it were negative we might substitute the period $-2\omega_2$ in the place of $+2\omega_2$.

We lay off the periods $2\omega_1, 4\omega_1, 6\omega_1, \ldots; 2\omega_2, 4\omega_2, 6\omega_2, \ldots$ upon the same straight line (cf. Art. 70).

It is evident that
$$2\omega_2 = 2m_1\omega_1 + 2\omega_3,$$
where m_1 is a positive or negative integer, and $2\omega_3 < 2\omega_1$. Similarly we write
$$4\omega_2 = 2m_2\omega_1 + 2\omega_4,$$
m_2 being an integer, and $2\omega_4 < 2\omega_1$.

It follows that
$$\begin{aligned} 2\omega_2 - 2m_1\omega_1 &= 2\omega_3, \\ 4\omega_2 - 2m_2\omega_1 &= 2\omega_4, \\ 6\omega_2 - 2m_3\omega_1 &= 2\omega_5, \\ \cdot \quad \cdot \quad \cdot \quad \cdot & \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$
and consequently the quantities $2\omega_3, 2\omega_4, 2\omega_5, \ldots$ are all periods.

There are two cases possible: (1) These quantities are all different; or (2) they are *not* all different. Suppose that $2\omega_3, 2\omega_4, \ldots$ are all different, and consider the n quantities $2\omega_3, 2\omega_4, \ldots, 2\omega_{n+2}$, to which we also add $2\omega_1$, in all $n + 1$ quantities.

Divide the distance between 0 and $2\omega_1$ into n equal parts; then, since each of the quantities $2\omega_3, 2\omega_4, \ldots, 2\omega_{n+2}$ is less than $2\omega_1$, two of these quantities must lie within one of the n equal intervals. Let these two quantities be $2\omega_k$ and $2\omega_l$. It is clear that $2\omega_k - 2\omega_l$ is also a period and less than $\frac{2\omega_1}{n}$.

Since n is an arbitrarily large integer, it is seen that we have here periods that are arbitrarily small, contrary to what was proved in Art. 69. It follows then that two of the above quantities must be equal (which includes now also the second case). We then have for example
$$2\omega_{q+2} = 2\omega_{p+2},$$
so that
$$2q\omega_2 - 2m_q\omega_1 = 2p\omega_2 - 2m_p\omega_1,$$
m_q and m_p being integers; and from this it is seen that $\frac{2\omega_2}{2\omega_1}$ must be a rational quantity.

ART. 73. We may prove as follows that the ratio $\frac{2\omega'}{2\omega}$ cannot be *real*. For take in the period-strip of Art. 67 two points u_2 and u_1 such that $u_2 - u_1 = 2\omega'$. In that article we saw that
$$0 \leqq \mathsf{R}\left(\frac{u_2}{2\omega}\right) \leqq 1,$$
and
$$0 \leqq \mathsf{R}\left(\frac{u_1}{2\omega}\right) \leqq 1.$$

It follows that

$$-1 < \mathsf{R}\left(\frac{u_2 - u_1}{2\,\omega}\right) < 1.$$

If now $\qquad \dfrac{u_2 - u_1}{2\,\omega} = \dfrac{2\,\omega'}{2\,\omega}$ is a real quantity,

then is $\qquad \dfrac{2\,\omega'}{2\,\omega} < 1, \quad \text{or } 2\,\omega' < 2\,\omega.$

We thus have two periods which lie along the same straight line, of which one is less than the primitive period $2\,\omega$, which contradicts the notion of a primitive period. Hence $2\,\omega$ and $2\,\omega'$ must have different directions.*

ART. 74. *There exist two primitive periods through which all other periods may be expressed.*

Geometrical Proof.

We shall first show that it is always possible to form a period-parallelogram which is free from periods. Suppose that in the period-parallelogram formed of the periods a and b there are present periods. Their number must be finite (Art. 69). Among all these periods let β be the one whose perpendicular distance on $O\alpha$ is the shortest. It is then evident that the period-parallelogram constructed on $O\alpha$ and $O\beta$ is free from periods. Of course we have assumed that $O\alpha$ is *not* an integral multiple of another period.

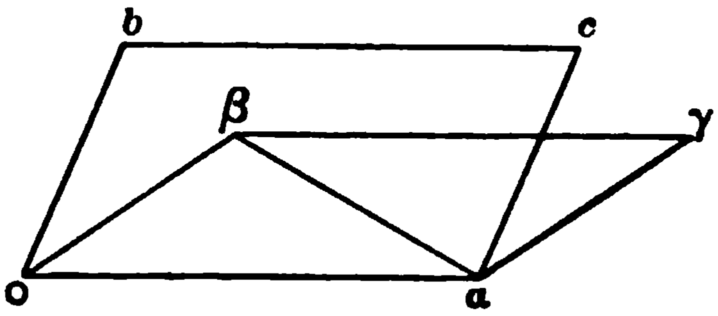


Fig. 13.

It is evident that γ is a period since $\alpha + \beta = \gamma$; and it is also evident that there can be no period-points within or on the boundaries of $\alpha\beta\gamma$.

If for example λ were a period-point on the side $\beta\gamma$, then through λ we could draw the parallel to the side $O\beta$ which cuts the line $O\alpha$ in μ. We would then have a period-point at μ, which contradicts the fact that no period-point lies on $O\alpha$.

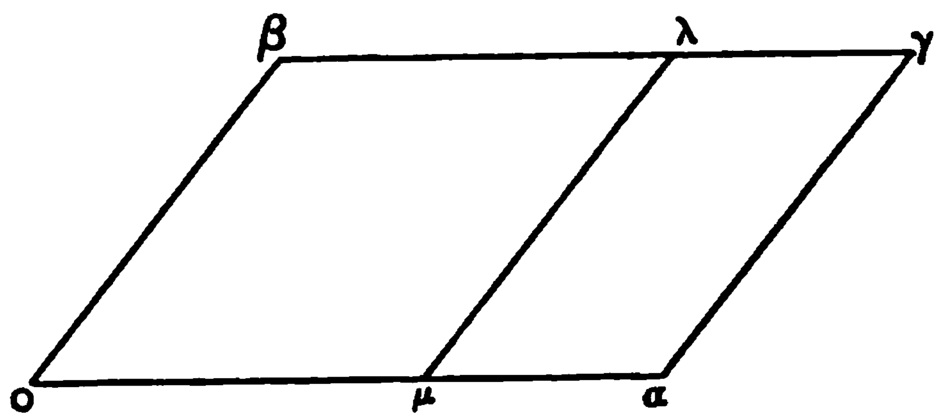


Fig. 14.

In the same way it may be shown that no period-point lies on $\alpha\gamma$.

Suppose next that a period-point ν lies within the triangle $\beta\gamma\alpha$ (Fig. 15); then by completing the parallelogram $\beta\nu\alpha\mu$ it is seen that μ is also a period-point and lies within the triangle $O\beta\alpha$, which contradicts what we saw above.

* Picard, *Traité d'Analyse*, t. 2, p. 220, gives an interesting proof of this theorem; see also other proofs in Hermite's "*Cours*" (4$^{\text{me}}$ éd.), p. 217, and Goursat, *Cours d'Analyse*, t. 2, No. 314.

We thus see that within the entire parallelogram $O\beta\gamma\alpha$, the sides included, there are situated no period-points except at the vertices. It is also evident that if the whole u-plane be filled with the *congruent* parallelograms, as indicated in Fig. 16, there is nowhere a period-point except at the vertices. If for example there were a period-point u in any of the parallelo-

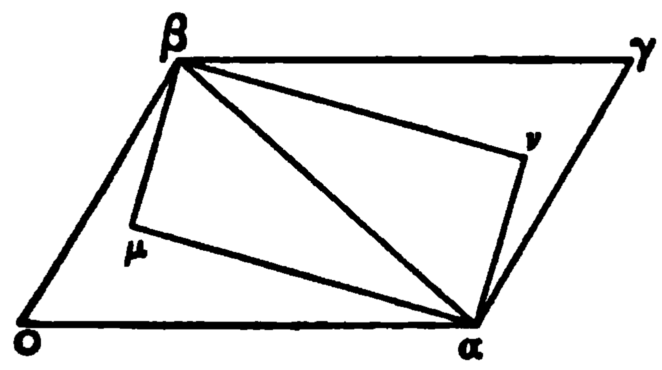


Fig. 15.

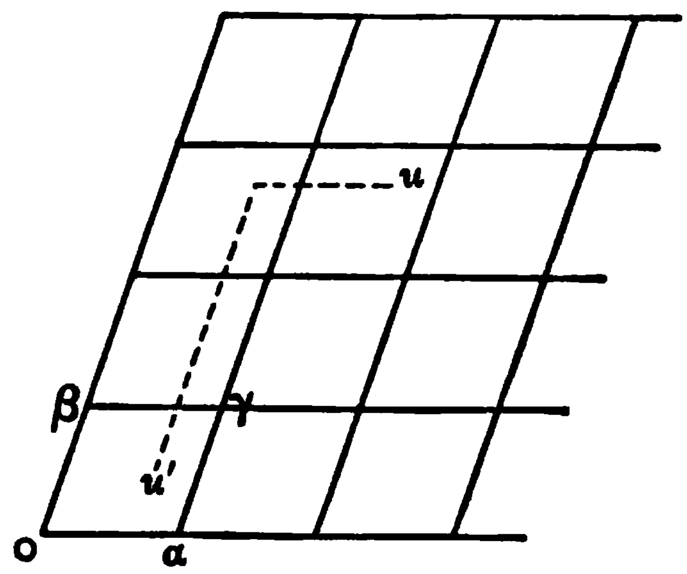


Fig. 16.

grams, there exists in the initial parallelogram $O\beta\gamma\alpha$ a point u' which differs from u only by integral multiples of a period, and contrary to hypothesis there would be a period-point within the initial parallelogram. It is also evident that the vertices of all the parallelograms are period-points since they are of the form

$$k\alpha + l\beta,$$

where k and l are integers.

It follows that a one-valued analytic function cannot have three independent periods a, b, c; for, as we have just seen, these three quantities are expressible in the form

$$a = k\alpha + l\beta,$$
$$b = k'\alpha + l'\beta,$$
$$c = k''\alpha + l''\beta,$$

where the k's and l's are integers.

We have thus shown that *a one-valued analytic function, which (in the neighborhood of at least one point) is developable in an ascending integral power series, cannot have more than two independent periods.*

We shall see later that the pairs of primitive periods may be chosen in an infinite number of different ways (see Art. 80).

ART. 75. It is evident from the foregoing that it is only necessary to consider the values of a doubly periodic function $\phi(u)$ within the initial period-parallelogram whose sides are, say, $\alpha = 2\,\omega$, $\beta = 2\,\omega'$. In this parallelogram the function $\phi(u)$ has everywhere the nature of an integral or a (fractional) rational function. We shall agree that the second period lies to the left if we look from the origin toward $2\,\omega$. (See Fig. 17).

We may write

$$\frac{2\,\omega'}{2\,\omega} = \tau = \sigma + i\rho,$$

where by hypothesis $|\,\rho\,| \neq 0$, since the ratio $\frac{2\,\omega'}{2\,\omega}$ is *not* real. All points within the interior and on the sides of this period-parallelogram may be expressed in the form

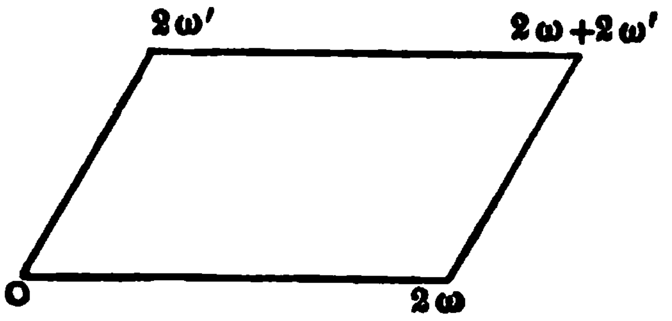


Fig. 17.

$$u = 2\,t\omega + 2\,t'\omega',$$

where $\quad 0 \leqq t \leqq 1, \quad 0 \leqq t' \leqq 1.$

The totality of all such values of u may be considered as the analytic definition of a period-parallelogram. The vertices (except the origin) are excluded from the consideration.

Further, let

$$w = 2\,m\omega + 2\,m'\omega'$$

where m and m' are *real* quantities.

It follows that

$$\frac{w}{2\,\omega} = m + m'\frac{\omega'}{\omega};$$

and since $\frac{\omega'}{\omega}$ is a complex quantity, $\frac{w}{2\,\omega}$ is also complex, $= \sigma' + i\rho'$, say.

We thus have

$$\sigma' + i\rho' = m + m'(\sigma + i\rho),$$

or

$$\sigma' = m + m'\sigma, \quad \rho' = m'\rho.$$

It follows that

$$m' = \frac{\rho'}{\rho}, \qquad m = \sigma' - \frac{\rho'}{\rho}\sigma.$$

Since ρ is different from zero, the denominator does not vanish, and consequently m and m' are determinate quantities.

It is thus seen that every complex quantity w may be *uniquely* written in the form

$$w = 2\,m\omega + 2\,m'\omega',$$

where m and m' are real quantities.

ART. 76. Two points w and w' are called congruent if

$$w - w' = 2\,k\omega + 2\,l\omega',$$

where k and l are integers. The fact that w is congruent to w' may be written

$$w \equiv w' \text{ (modd. } 2\,\omega,\, 2\,\omega');$$

or, if no confusion can arise,

$$w \equiv w'.$$

It is also clear that, when w and w' are congruent, then $w - w'$ is a period of the argument of the function.

If we write

$$w = 2\,m\omega + 2\,m'\omega',$$
$$w' = 2\,n\omega + 2\,n'\omega',$$

and if

$$w \equiv w' \text{ (modd. } 2\,\omega,\ 2\,\omega'),$$

it is evident that the quantities m and n, as also the quantities m' and n', differ only by integers, that is, $m - n$ = integer as is also $m' - n'$.

ART. 77. Suppose that the period-parallelogram formed on the two sides $0\,.\,.\,2\,\omega$ and $0\,.\,.\,2\,\omega'$ is free from period-points. We may show *analytically* that all the period-points in the u-plane are composed through addition and subtraction of $2\,\omega$ and $2\,\omega'$.

For let

$$|\,2\,\omega\,| = l.$$

Then, since

$$\frac{2\,\omega'}{2\,\omega} = \sigma + i\rho,$$

it is seen that

$$|\,2\,\omega'\,| = l' = l\sqrt{\rho^2 + \sigma^2}.$$

Further, since $2\,\omega + 2\,\omega' = 2\,\omega(1 + \sigma + i\rho)$, it follows that the length of one diagonal of the parallelogram is

$$|\,2\,\omega + 2\,\omega'\,| = l\sqrt{(1 + \sigma)^2 + \rho^2},$$

while the length of the other diagonal is

$$|\,2\,\omega' - 2\,\omega\,| = l\sqrt{(1 - \sigma)^2 + \rho^2}.$$

Represent by L the longest of the four sides

$$|\,2\,\omega\,|,\ |\,2\,\omega'\,|,\ |\,2\,\omega + 2\,\omega'\,|,\ |\,2\,\omega' - 2\,\omega\,|.$$

Next divide the two sides $0\,.\,.\,2\,\omega$ and $0\,.\,.\,2\,\omega'$ respectively into n equal parts, so that the period-parallelogram will be divided into n^2 small parallelograms. The distance between any two points situated within one of the smaller parallelograms is not greater than $\dfrac{L}{n}$.

If there are periods that *cannot* be expressed through integral multiples of $2\,\omega$ and $2\,\omega'$ and if $2\,\omega_1$ is such a period, we shall construct the congruent point which lies within the initial period-parallelogram.

We may write

$$2\,\omega_1 \equiv 2\,\mu_1\omega + 2\mu_1'\omega',$$

where

$$0 \leqq \mu_1 < 1 \quad \text{and} \quad 0 \leqq \mu_1' < 1.$$

This point must fall within or on the boundaries of one of the small parallelograms.

Admitting (Art. 69) that every period has a definite length, it may be shown as follows that μ_1 and μ_1' are rational numbers.

We have the congruence

$$2\,\omega_1 \equiv 2\,\mu_1\omega + 2\,\mu_1'\omega',$$

and in a similar manner we form

$$2\cdot 2\,\omega_1 \equiv 2\,\mu_2\omega + 2\,\mu_2'\omega',$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$2(n^2+1)\omega_1 \equiv 2\,\mu_{n^2+1}\omega + 2\,\mu_{n^2+1}'\omega',$$

where $$0 \leqq \mu_k < 1 \quad\text{and}\quad 0 \leqq \mu_k' < 1,$$

$$(k = 1, 2, \ldots, n^2+1).$$

If these $n^2 + 1$ points in the initial period-parallelogram are all different, at least two of them must fall within or on the boundaries of one of the small parallelograms, and the distance between these points is therefore less than $\frac{L}{n}$. As n can be made arbitrarily large, there are then periods that are arbitrarily small, which is contrary to our hypothesis.

It follows that at least two of the $n^2 + 1$ points must coincide, in which event we would have

$$2\,p\omega_1 \equiv 2\,\mu_p\omega + 2\,\mu_p'\omega',$$

$$2\,q\omega_1 \equiv 2\,\mu_p\omega + 2\,\mu_p'\omega',$$

and consequently $2(p-q)\omega_1 \equiv 0 \pmod{2\,\omega, 2\,\omega'}$ (modd.),

where p and q are both integers. We have thus shown that *an integral multiple of $2\,\omega_1$ is congruent to the origin.* Since $2\,\omega$, $2\,\omega'$ are a pair of primitive periods, it follows from the theorem of the next article that $2\,\omega_1$ must be congruent to the origin.

ART. 78. Jacobi (Werke, Bd. II, pp. 27–32) proves the following theorem: *If a one-valued function has three periods $\omega_1, \omega_2, \omega_3$, such that*

$$m_1\omega_1 + m_2\omega_2 + m_3\omega_3 = 0,$$

where m_1, m_2, m_3 are integers, then there exist two periods of which $\omega_1, \omega_2, \omega_3$ are integral multiple combinations.

We may assume that there is no common divisor other than unity of m_1, m_2, m_3. Let d be the common divisor of m_2 and m_3. Of course, $d = 1$ when m_2 and m_3 are relatively prime.

Then, since $\frac{m_1}{d}\omega_1 = -\frac{m_2}{d}\omega_2 - \frac{m_3}{d}\omega_3$ and the right-hand side is an integral combination of periods, it follows that $\frac{m_1}{d}\omega_1$ is a period. Since $\frac{m_1}{d}$ is a fraction in its lowest terms, when expressed as a continued fraction it may be written

$$\frac{m_1}{d} - \frac{p}{q} = \pm\frac{1}{dq},$$

where $\frac{p}{q}$ is the last convergent before the proper value. It follows that

$$\frac{qm_1}{d}\omega_1 - p\omega_1 = \pm\frac{1}{d}\omega_1 = \omega, \text{ say},$$

where ω is a period.

Let $$\frac{m_2}{d} = m_2', \; \frac{m_3}{d} = m_3',$$

so that $$m_1\omega + m_2'\omega_2 + m_3'\omega_3 = 0.$$

Change $\frac{m_2'}{m_3'}$ into a continued fraction, taking $\frac{r}{s}$ to be the last convergent before the proper value, so that

$$\frac{m_2'}{m_3'} - \frac{r}{s} = \pm \frac{1}{sm_3'}.$$

Then $r\omega_2 + s\omega_3$ being an integral combination of periods, is a period $=$ ω', say.

On the other hand,

$$\begin{aligned} \pm\,\omega_2 &= \omega_2(sm_2' - rm_3') \\ &= -\,r\omega_2 m_3' - s(m_1\omega + m_3'\omega_3) \\ &= -\,m_1 s\omega - m_3'(r\omega_2 + s\omega_3) \\ &= -\,m_1 s\omega - m_3'\omega'; \end{aligned}$$

also $$\begin{aligned} \pm\,\omega_3 &= \omega_3(sm_2' - rm_3'), \\ &= sm_2'\omega_3 + r(m_1\omega + m_2'\omega_2), \\ &= m_1 r\omega + m_2'\omega'; \end{aligned}$$

and $$\omega_1 = d\omega.$$

Hence two periods ω, ω' exist of which ω_1, ω_2, ω_3 are integral multiple combinations.*

We may conclude from the foregoing that *All one-valued analytic functions are either*

(1) *Not periodic*, or
(2) *Simply periodic*, or
(3) *Doubly periodic.*

Triply or multiply periodic one-valued functions do not exist.

ART. 79. We may next prove the following theorem: *It is possible in an infinite number of ways to form pairs of primitive periods of a doubly periodic function.*

Let $2\,\omega$, $2\,\omega'$ be a pair of primitive periods, and suppose that

$$\frac{2\,\omega'}{2\,\omega} = \sigma + i\rho,$$

where ρ is positive, that is,

$$\mathsf{R}\left(\frac{\omega'}{i\omega}\right) > 0.$$

We wish to form another pair of primitive periods $2\,\tilde{\omega}$, $2\,\tilde{\omega}'$ such that

$$\mathsf{R}\left(\frac{\tilde{\omega}'}{i\tilde{\omega}}\right) > 0.$$

* Cf. Forsyth, *Theory of Functions*, p. 202; see also Hermite in *Lacroix's Calculus*, Vol. II, p. 370.

It is evident that we must have

$$2\,\tilde{\omega} = 2\,p\omega + 2\,q\omega',$$
$$2\,\tilde{\omega}' = 2\,p'\omega + 2\,q'\omega',$$

where p, q, p', q' are integers.

Further, p and q must be relatively prime, for otherwise $2\,\tilde{\omega}$ would be the integral multiple of a period. The integers p' and q' must also be relatively prime. It follows that

$$2\,\omega = \frac{2\,q'\tilde{\omega} - 2\,q\tilde{\omega}'}{pq' - qp'}.$$

Since $2\,\tilde{\omega}$ and $2\,\tilde{\omega}'$ are to be a pair of primitive periods, the period $2\,\omega$ must be expressible integrally through them.

It follows that

$$\frac{q'}{pq' - qp'} \quad \text{and} \quad \frac{-q}{pq' - qp'}$$

must be integers.

We further have

$$2\omega' = \frac{-2\,p'\tilde{\omega} + 2\,p\tilde{\omega}'}{pq' - qp'}, \text{ and consequently}$$

$$-\frac{p'}{pq' - qp'} \quad \text{and} \quad \frac{p}{pq' - qp'} \text{ are integers.}$$

If we put $pq' - qp' = \Delta$, it is seen that the four quantities above are integers, if $\Delta = \pm 1$. For suppose that Δ is different from ± 1. It would then follow, since $\frac{q}{\Delta}$ and $\frac{p}{\Delta}$ are to be integers, that q and p have a common divisor other than unity, which is contrary to the hypothesis. The next question is: *Are both values* $\Delta = +1$ *and* $\Delta = -1$ *admissible?*

We required that

$$\mathsf{R}\left(\frac{\omega'}{i\omega}\right) > 0 \quad \text{and} \quad \mathsf{R}\left(\frac{\tilde{\omega}'}{i\tilde{\omega}}\right) > 0.$$

We have

$$\frac{2\,\tilde{\omega}'}{2\,\tilde{\omega}i} = \frac{2\,p'\omega + 2\,q'\omega'}{i(2\,p\omega + 2\,q\omega')} = \frac{p' + q'\dfrac{\omega'}{\omega}}{i\left(p + q\dfrac{\omega'}{\omega}\right)}.$$

Since $\dfrac{\omega'}{\omega} = \sigma + i\rho$, it follows that

$$\frac{2\,\tilde{\omega}'}{2\,\tilde{\omega}i} = \frac{p' + q'(\sigma + i\rho)}{i[p + q(\sigma + i\rho)]} = \frac{-(p' + q'\sigma)q\rho + (p + q\sigma)q'\rho}{(p + q\sigma)^2 + q^2\rho^2} + i[\quad];$$

and consequently

$$\left(\frac{2\,\tilde{\omega}'}{2\,\tilde{\omega}'i}\right) = \frac{pq' - qp'}{(p + q\sigma)^2 + q^2\rho^2}\,\rho.$$

As ρ is positive by hypothesis, we must have $pq' - qp'$ *positive* in order to fulfill the condition

$$\mathsf{R}\left(\frac{2\,\tilde{\omega}'}{2\,\tilde{\omega}i}\right) > 0.$$

It follows then that

$$\Delta = pq' - qp' = +1.$$

ART. 80. Using the condition just written, we may form an arbitrary number of *equivalent* pairs of primitive periods as soon as one such pair is known.*

The transition from one pair of periods to another is known as a *transformation*, and the quantity $\Delta = pq' - qp'$ is called the *degree* of the transformation. We have here to consider transformations of the *first* degree.

The quantity Δ gives the *measure* of the surface-area of the second period-parallelogram, if that of the first is denoted by *unity*.

Hence all primitive period-parallelograms have the same area, for if

$$2\,\tilde{\omega} = x + iy \quad \text{and} \quad 2\,\tilde{\omega}' = x' + iy',$$

the area of the corresponding parallelogram is

$$\pm\,(xy' - yx').$$

If further,

$$2\,\omega = \xi + i\eta \quad \text{and} \quad 2\,\omega' = \xi' + i\eta',$$

the area of the corresponding period-parallelogram is

$$\pm\,(\xi\eta' - \xi'\eta).$$

It follows that, if

$$2\,\tilde{\omega} = 2\,p\omega + 2\,q\omega' \quad \text{and} \quad 2\,\tilde{\omega}' = 2\,p'\omega + 2\,q'\omega',$$

then

$$\begin{cases} x = p\xi + q\xi', \\ y = p\eta + q\eta'; \end{cases} \quad \text{and} \quad \begin{cases} x' = p'\xi + q'\xi', \\ y' = p'\eta + q'\eta'; \end{cases}$$

and consequently

$$\pm\begin{vmatrix} x, & y \\ x', & y' \end{vmatrix} = \pm\begin{vmatrix} p, & q \\ p', & q' \end{vmatrix}\begin{vmatrix} \xi, & \eta \\ \xi', & \eta' \end{vmatrix}.$$

But here $pq' - qp' = 1$.

Hence a primitive period-parallelogram is *not* unique.

The linear substitution

$$2\,\tilde{\omega} = 2\,p\omega + 2\,q\omega',$$
$$2\,\tilde{\omega}' = 2\,p'\omega + 2\,q'\omega'$$

is denoted by

$$\begin{bmatrix} p, & q \\ p', & q' \end{bmatrix}.$$

* Cf. Briot et Bouquet, *Fonctions Elliptiques*, pp. 234, 235, and pp. 268 *et seq.*

One of the substitutions which satisfies the condition

$$\Delta = pq' - qp' = 1$$

is

$$\begin{bmatrix} 0, & 1 \\ -1, & 0 \end{bmatrix}.$$

In this case we have

$$2\tilde{\omega} = 2\omega',$$
$$2\tilde{\omega}' = -2\omega.$$

A second substitution which satisfies the same condition is

$$\begin{bmatrix} 1, & 0 \\ 1, & 1 \end{bmatrix},$$

or

$$2\tilde{\omega} = 2\omega,$$
$$2\tilde{\omega}' = 2\omega + 2\omega'.$$

It may be shown that every linear substitution with integral elements and *determinant* $\Delta = 1$ may be formed by a finite number of repetitions of these two substitutions.

ART. 81. The question arises* whether among the infinite number of equivalent pairs of periods there are those to which preference should be given. There are *one*, *two*, and sometimes *three* pairs of primitive periods which may be chosen in preference to the others. One of the periods in these selected pairs of periods has the smallest absolute value among all the periods. It is clear that such a period exists; indeed there are two such periods differing only in sign. Taking this smallest period as a radius we describe a circle about the origin. Within this circle no period can be situated, but upon the periphery there lie at least two periods (180 degrees from each other). It is also seen that the surfaces of the two circles drawn about these period-points and having the same radii as the first circle must be free of periods. Hence besides the period-points P and P' none can be situated on any part of the periphery of the first circle except the shaded arcs P_1P_2 and P_3P_4. On these arcs there may be two periods differing by 180 degrees and possibly four periods.

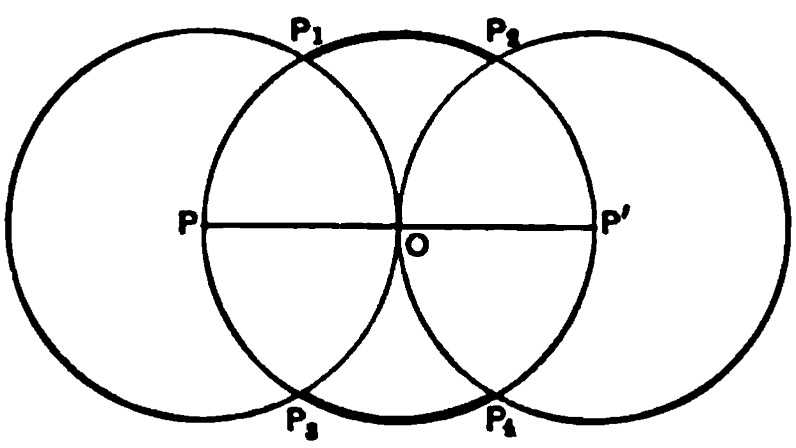


Fig. 18.

In the last case the period-points must lie at the four points of intersection of the circles, viz., P_1, P_2, P_3 and P_4, so that there may lie upon the first circle two, four, or at most six period-points; and consequently the period of smallest absolute value is either 2-ply, 4-ply, or 6-ply determined.

* Cf. Burkhardt, *Elliptische Funktionen*, p. 194.

Denote any one of these six periods by 2ω, which we use as one of the selected pair of primitive periods.

We shall impose a further condition upon the other period of this selected pair. The second period $2\omega'$ must lie to the left of $0 \,.\,.\, 2\omega$. We also know that $|2\omega'| > |2\omega|$. We cut a strip out of the plane as indicated in the figure. The second period-point may always be made to lie within this strip; for if it were situated without the strip, by the addition of $2m\omega$, where m is a positive or negative integer, it can be caused to lie within the strip, but it does not fall within that part of the strip which belongs to the two circles. Hence the triangle $0 \,.\,.\, 2\omega \,.\,.\, 2\omega'$ has only acute angles, the right angle being a limiting case.

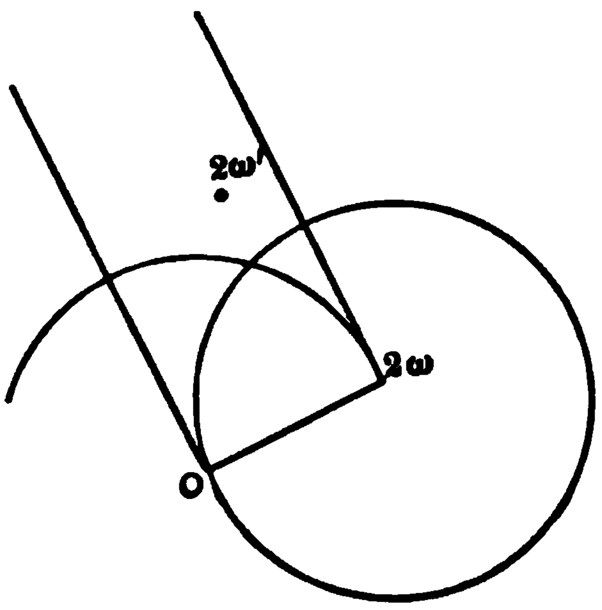


Fig. 19.

We write $\tau = \dfrac{2\omega'}{2\omega} = \alpha + i\beta,$

where $$\alpha^2 + \beta^2 \geqq 1, \qquad 0 \leqq \alpha < 1.$$

Owing to the substitution $$\begin{bmatrix} 1, & 0 \\ 1, & 1 \end{bmatrix}$$

we may so choose $2\omega'$ that $$-\tfrac{1}{2} \leqq \alpha \leqq \tfrac{1}{2}.$$

It follows that $\beta > \frac{1}{2}\sqrt{3}$.

If further we write
$$h = q = e^{\tau\pi i} = e^{(\alpha + i\beta)\pi i} = e^{-\beta\pi} e^{\alpha\pi i},$$
it is clear that
$$|q| \leqq e^{-\frac{1}{2}\sqrt{3}\pi} \leqq \tfrac{1}{15},$$
a fact which we shall find to be very important in the development of the Theta-functions (Chapter X).

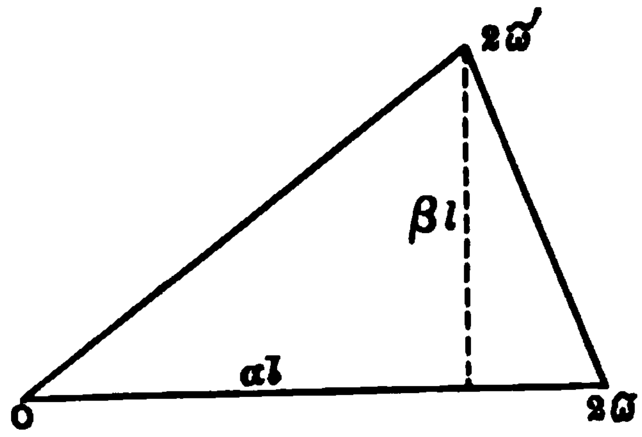


Fig. 20.

ART. 82. We have interpreted the equation $\Delta = pq' - qp' = 1$ as denoting that the parallelograms formed on pairs of primitive periods have the same area. Let $2\tilde{\omega}$, $2\tilde{\omega}'$ be a pair of primitive periods. The quantities $2\tilde{\omega}$ and $2\tilde{\omega}'$ determine a triangle, and all such period-triangles have the same area. Let $|2\tilde{\omega}| = l$.

Then if $\dfrac{2\tilde{\omega}'}{2\tilde{\omega}} = \alpha + i\beta$, the area of the triangle is $\dfrac{\beta l^2}{2}$ and that of the period-parallelogram is βl^2. This quantity being constant for all equivalent primitive pairs of periods, we have
$$\beta = \frac{\text{const.}}{l^2}.$$

From this it is seen that β is a maximum when l is a minimum. If then β is to have its greatest value, we must choose the first period $2\,\tilde{\omega}$ so that it has the smallest possible value.

If the ratio of the periods is a pure imaginary, then $\alpha = 0$ and $\beta \geqq 1$. In this case

$$|\,h\,| = |\,e^{\tau\pi i}\,| \leqq e^{-\beta\pi} < \tfrac{1}{23}.$$

EXAMPLE

If ω_1, ω_2 and ω_3 are periods of $\phi(u)$ and if

$$29\omega_3 = 17\,\omega_1 + 11\,\omega_2,$$

show that

$$\omega = 5\,\omega_1 + 3\,\omega_2 - 8\,\omega_3$$
$$\omega' = 3\,\omega_1 + 2\,\omega_2 - 5\,\omega_3$$

are a pair of primitive periods of $\phi(u)$. [Forsyth.]

CHAPTER V

CONSTRUCTION OF DOUBLY PERIODIC FUNCTIONS

Hermite's Intermediary Functions. The Eliminant Equation.

ARTICLE 83. Having established the existence of the doubly periodic functions, we shall next show how to construct such functions and naturally the simplest ones possible.

The expression

$$\phi(u) = \sum_{k=-\infty}^{k=+\infty} A_k e^{k\frac{2\pi i}{a}u}$$

is a simply periodic function which can be developed in positive ascending powers of $u - u_0$, and which is not indeterminate or infinite for any finite value of u, provided the constants A_k have been suitably chosen.

A function which is developable in a convergent power series in ascending positive integral powers and in the finite portion of the plane nowhere becomes infinite or indeterminate is an integral transcendental function (see Chapter I). Such a function is $\phi(u)$ above.

The question is asked: *Is there an integral transcendental function which has besides the period a another period b?*

Liouville [*Crelle's Journ.*, Bd. 88, p. 277] answered this question by proving the following theorem: *An integral transcendental function which is doubly periodic is a constant.*

We need only study the function within the first or initial period parallelogram, *i.e.*, the one which has the origin as a vertex and which lies to the right of this vertex. For every point u of the plane is *congruent* to a point u' in the first parallelogram, that is,

$$u = u' + ka + lb,$$

where k and l are integers. The function has therefore the same value at u and at u'.

An integral transcendental function becomes infinite for no finite value of the argument. Consequently the function remains finite in the first period-parallelogram and therefore the absolute value of the function in this parallelogram is smaller than a certain finite quantity M. Further, since the function at points *without* the first period-parallelogram always takes such values as it has *within* this parallelogram, it remains in the

whole plane less in absolute value than M. But an integral transcendental function

$$g(u) = a_0 + a_1u + a_2u^2 + a^3u^3 + \cdots$$

which remains finite for arbitrarily large values of u is a constant, since $g(u)$ can remain finite only if $a_1 = 0 = a_2 = a_3 = \cdots$.

The following is a more direct proof of Liouville's Theorem.

If

$$\Phi(u) = \sum_{k=-\infty}^{k=+\infty} A_k e^{k\frac{2\pi i}{a}u},$$

the condition that

$$\Phi(u + b) = \Phi(u)$$

is

$$\sum_{k=-\infty}^{k=+\infty} A_k e^{k\frac{2\pi i}{a}b} e^{k\frac{2\pi i}{a}u} = \sum_{k=-\infty}^{k=+\infty} A_k e^{k\frac{2\pi i}{a}u};$$

and consequently

$$A_k e^{k\frac{2\pi i}{a}b} = A_k.$$

Since $\frac{b}{a}$ is an irrational quantity, $e^{k\frac{2\pi i}{a}b} \neq 1$, and therefore

$$A_k = 0 \; (k = \pm 1, \pm 2, \ldots).$$

It follows that

$$\Phi(u) = A_0;$$

and consequently there is no integral transcendental function which is doubly periodic.

ART. 84. We shall now seek to form a doubly periodic function which has the character of a rational function and which may therefore be written in the form

$$\phi(u) = \frac{\Phi(u)}{\Psi(u)},$$

where $\Phi(u)$ and $\Psi(u)$ are integral transcendental functions. We may write

$$\Phi(u) = \sum_{k=-\infty}^{k=+\infty} A_k e^{k\frac{2\pi i}{a}u}, \qquad \Psi(u) = \sum_{k=-\infty}^{k=+\infty} B_k e^{k\frac{2\pi i}{a}u},$$

where A_k and B_k are constants, so chosen that the two series are convergent.

Since $\Phi(u)$ and $\Psi(u)$ both have the period a, their quotient $\phi(u)$ has the period a. We therefore have to bring it about that the quotient $\frac{\Phi(u)}{\Psi(u)}$ has also the period b.

We must so determine the functions $\Phi(u)$ and $\Psi(u)$ that

$$\Phi(u + b) = T(u)\,\Phi(u),$$
$$\Psi(u + b) = T(u)\,\Psi(u),$$

where $T(u)$ is a function of u. If we succeed in this, then

$$\phi(u + b) = \frac{\Phi(u + b)}{\Psi(u + b)} = \frac{T(u)\,\Phi(u)}{T(u)\,\Psi(u)} = \frac{\Phi(u)}{\Psi(u)} = \phi(u),$$

or $\phi(u)$ has also the period b.

It will be advantageous to make our choice so that $\Phi(u+b)$ has the same zero as $\Phi(u)$, and consequently

$$T(u)=\frac{\Phi(u+b)}{\Phi(u)}$$

does not vanish or become infinite for any finite value of u. This will be effected if we write

$$T(u)=e^{G(u)},$$

where $G(u)$ is an integral function in u.

We have then to seek a function $\Phi(u)$ and a function $\Psi(u)$ so that

$$\Phi(u+a)=\Phi(u), \qquad \Psi(u+a)=\Psi(u),$$
$$\Phi(u+b)=e^{G(u)}\Phi(u), \qquad \Psi(u+b)=e^{G(u)}\psi(u).$$

We shall next bring about a further limitation in that we determine $\Phi(u)$ and $\Psi(u)$ so that $G(u)$ is an integral function of the first degree in u.

We will then have

$$\Phi(u+a)=\Phi(u), \qquad \Psi(u+a)=\Psi(u),$$
$$\Phi(u+b)=e^{\lambda u+\mu}\Phi(u), \qquad \Psi(u+b)=e^{\lambda u+\mu}\Psi(u),$$

where λ and μ are constants which are at our disposal. We shall see that there is an infinite number of such functions.

Hermite * called them "*doubly periodic functions of the third sort* (*espèce*)."

If $\Phi(u+a)=\nu\Phi(u)$ and $\Phi(u+b)=\nu'\Phi(u)$, where ν and ν' are constants, one or both being different from unity, then $\Phi(u)$ is a doubly periodic function of the *second sort;* and if $\nu=1=\nu'$ we have the doubly periodic functions of the *first sort,* which are properly the doubly periodic functions.

Note that the word *sort* (*espèce*) used here in no manner connects a doubly periodic function of the first sort, say, with an elliptic integral of the *first kind* (*espèce*), a term which will be employed later.

* Hermite (*Lettre à Jacobi*; Hermite's Œuvres, t. 1, p. 18) first considered these functions. Briot and Bouquet, *Fonctions Elliptiques*, p. 236, called them "*intermediary functions.*" They are sometimes called *quasi-* or *pseudo-periodic.* See also Hermite, "*Cours*" (4me éd.), pp. 227–234; Hermite, *Note sur la théorie des fonctions* in Lacroix, *Calcul* (6me éd.), t. 2, p. 384, which is reprinted in Hermite's Œuvres, t. 2, p. 125; Hermite, *Note sur la théorie des fonctions elliptiques, Camb. and Dubl. Math. Journ.*, Vol. III (1848); Hermite, Œuvres, p. 75 of Vol. I; *Crelle*, Bd. 100; *Comptes Rendus* (1861), t. 53, pp. 214–228, and *Comptes Rendus* (1862), t. 55, pp. 11–18, 85–91; Biehler, *Thèse*, 1879; Painlevé, *Ann. de la Faculté des Sciences de Toulouse*, 1888; Appell, *Ann. de l'École Normale*, 3d Series, Vols. I, II, III and V; Picard, *Comptes Rendus*, 21 Mars, 1881. The Berlin lectures of the late Prof. L. Fuchs have also been of service in the preparation of this Chapter.

ART. 85. From the formula

$$\Phi(u)=\sum_{k=-\infty}^{k=+\infty} A_k e^{k\frac{2\pi i}{a}u},$$

it follows at once that

$$\Phi(u+b)=\sum_{k=-\infty}^{k=+\infty} A_k e^{k\frac{2\pi i}{a}(u+b)}.$$

If with Hermite we write $Q=e^{\pi i\frac{b}{a}}$, it follows that

$$\Phi(u+b)=\sum_{k=-\infty}^{k=+\infty} A_k Q^{2k} e^{k\frac{2\pi i}{a}u}. \tag{1}$$

On the other hand we had

$$\Phi(u+b)=e^{\lambda u+\mu}\Phi(u).$$

If on the right-hand side we write for $\Phi(u)$ its value and put $\lambda=\frac{2\pi i}{a}g$, we have

$$\Phi(u+b)=e^{\mu}\sum_{k=-\infty}^{k=+\infty} A_k e^{\frac{2\pi i}{a}u(k+g)}. \tag{2}$$

In this formula k is an integer and we shall choose the quantities so that g is also an integer.

If further we write $t=e^{\frac{2\pi i}{a}u}$ and equate like powers of t in formulas (1) and (2), we have for the determination of the A's the formula

$$A_m Q^{2m}=e^{\mu}A_{m-g}.$$

If we take the logarithms of both sides of this equation, we have

$$\mu+\log A_{m-g}=\log A_m+2m\log Q+n\,2\pi i, \tag{i}$$

where on the right $n\,2\pi i$ has been added, since the logarithm is an infinitely multiple-valued function.

We shall further write

$$\mu=\pi i\frac{b}{a}\nu,\quad \text{so that } e^{\mu}=e^{\pi i\frac{b}{a}\nu}=Q^{\nu},$$

or

$$\mu=\nu\log Q.$$

Since the constant μ is perfectly arbitrary, ν is also arbitrary. It follows directly from (i) that

$$\frac{\log A_{m-g}-\log A_m}{\log Q}=2m-\nu+\frac{n\,2\pi i}{\log Q}. \tag{ii}$$

We note that m, n, and g are integers, and we seek the *most general* solution of this equation.

If for brevity we put $\frac{\log A_k}{\log Q}=c_k$, the equation (ii) becomes

$$c_{m-g}-c_m=2m-\nu+n\frac{2\pi i}{\log Q}. \tag{iii}$$

To determine first a particular solution of this equation, write

$$c_k = \alpha k^2 + \beta k,$$

where the constants α and β are to be determined.

Since $$c_{m-g} = \alpha(m-g)^2 + \beta(m-g) \quad \text{and}$$

$$c_m = \alpha m^2 + \beta m,$$

we have from equation (iii)

$$-2\alpha m g + \alpha g^2 - \beta g = 2m - \nu + n\frac{2\pi i}{\log Q}.$$

Since this equation must be satisfied for every value of m, the coefficients of like powers of m on either side of it must be equal. We thus have

$$-2\alpha g = 2, \quad \alpha g^2 - \beta g = -\nu + n\frac{2\pi i}{\log Q};$$

and consequently

$$\alpha = -\frac{1}{g}, \qquad \beta = \frac{-g + \nu - n\dfrac{2\pi i}{\log Q}}{g}.$$

We may give to the arbitrary constant ν a value and we shall write $\nu = g$. It follows at once that

$$\alpha = -\frac{1}{g}, \quad \beta = \frac{-n\,2\pi i}{g\log Q}.$$

These values written in the formula

$$c_k = \alpha k^2 + \beta k$$

will give the particular solution of the equation

$$c_{m-g} - c_m = 2m - \nu + n\frac{2\pi i}{\log Q}. \qquad \text{(iii)}$$

We may write the general solution in the form

$$c_m = \alpha m^2 + \beta m + C_m,$$

where C_m is a function of m.

Writing for c_m its value, we have

$$\frac{\log A_m}{\log Q} = \alpha m^2 + \beta m + C_m, \quad \text{or}$$

$$A_m = e^{\alpha m^2 \log Q + (\beta m + C_m)\log Q}.$$

Writing for α its value from above and putting $\beta m + C_m = D_m$, we have

$$A_m = e^{-\frac{m^2}{g}\log Q} e^{D_m \log Q} = Q^{-\frac{m^2}{g}} e^{D_m \log Q}.$$

Finally, putting $D_m \log Q = \log B_m$, we have

$$A_m = Q^{-\frac{m^2}{g}} B_m,$$

where B_m is a new function of m. Here, indeed, we have not determined A_m, since B_m is not determined; but we have found a suitable form for A_m.

Returning to the original equation

$$A_m Q^{2m} = e^{\mu} A_{m-g}, \quad \text{it follows that}$$

$$Q^{2m} Q^{-\frac{m^2}{g}} B_m = Q^g Q^{-\frac{(m-g)^2}{g}} B_{m-g},$$

or

$$B_{m-g} = B_m,$$

where m and g are integers.

The integer g being arbitrary we shall write $g = -k$, where k is a positive integer. We thus have

$$B_{m+k} = B_m.$$

It follows at once that

$$\begin{aligned} B_k &= B_0, \\ B_{k+1} &= B_1, \\ B_{k+2} &= B_2, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ B_{2k-1} &= B_{k-1}, \\ B_{2k} &= B_k = B_0. \end{aligned}$$

We thus see that the constants $B_0, B_1, B_2, \ldots, B_{k-1}$ repeat themselves but are otherwise quite arbitrary.

It has thus been shown that *the function*

$$\Phi(u) = \sum_{m=-\infty}^{m=+\infty} B_m Q^{\frac{m^2}{k}} e^{m\frac{2\pi i}{a}u}$$

satisfies the functional equations

$$\Phi(u+a) = \Phi(u),$$

$$\Phi(u+b) = e^{-\frac{\pi i k}{a}(2u+b)} \Phi(u).$$

This function $\Phi(u)$ is the most general integral transcendental function which satisfies these two equations. It contains the k arbitrary constants $B_0, B_1, B_2, \ldots, B_{k-1}$.

ART. 86. It remains to be proved that the series through which the function $\Phi(u)$ has been expressed is convergent. Instead of the convergence of the series itself, we may consider the convergence of the series of moduli of the single terms, that is, of the series

$$\sum_{m=-\infty}^{m=+\infty} |B_m| |Q|^{\frac{m^2}{k}} \left| e^{\frac{2\pi i}{a}u} \right|^m.$$

In this series the coefficients $|B_0|, |B_1|, \ldots, |B_{k-1}|$ repeat themselves. We collect all those terms which contain $|B_0|$ and likewise all those which contain $|B_1|$, etc., and take $|B_0|, |B_1|, \ldots$ on the outside of the summation signs. We thus distribute the above series into k new series of which each is multiplied by one of the quantities $|B|$. If each of these series is convergent, then the product of each one of them by the corresponding $|B|$ is convergent and therefore also the sum of the products, that is, the above series of moduli, is convergent. If this series of moduli is convergent, it follows also *à fortiori* that the series which represents $\Phi(u)$ is convergent.

It therefore remains to prove the convergence of the k single series. To do this we may make use of the following well-known criterion of convergence:

Suppose we have given a series composed solely of positive terms

$$v_1 + v_2 + \cdots + v_m + \cdots.$$

This series is convergent if the mth root of the mth term, that is, $\sqrt[m]{v_m}$, *tends towards a definite value which is less than unity, with increasing values of m.*

For if $\sqrt[m]{v_m} < \rho < 1$, then is $v_m < \rho^m < 1$, and $v_{m+1} < \rho^{m+1} < 1$, etc., so that Σv_m is less than a geometrical series in which $\rho < 1$. The general term in the above series is

$$\left|Q^{\frac{m^2}{k}}\right| \left|e^{\frac{2\pi i}{a}u}\right|^m,$$

and the mth root of this quantity is

$$\left|Q^{\frac{m}{k}}\right| \left|e^{\frac{2\pi i}{a}u}\right|.$$

The second of the above factors has for all finite values of u a definite value which is independent of m. For the other factor we may write

$$\left|Q\right|^{\frac{m}{k}} = \left|e^{\pi i\frac{b}{a}}\right|^{\frac{m}{k}}.$$

If we put $\frac{b}{a} = \alpha + i\beta$ (where $\beta \neq 0$, since $\frac{b}{a}$ is *not* real), we have

$$\pi i\frac{b}{a} = \pi i\alpha - \pi\beta$$

and
$$\left|e^{\pi i\frac{b}{a}}\right| = \left|e^{-\pi\beta}\right| \left|e^{\pi i\alpha}\right| = \left|e^{-\pi\beta}\right|.$$

It follows that

$$\left|Q\right|^{\frac{m}{k}} = e^{-\pi\beta\frac{m}{k}}.$$

If β is a *positive* quantity, the quantity $e^{-\pi\beta\frac{m}{k}}$ becomes arbitrarily small for increasing values of m, which proves the convergence of each of the above series.

The condition that β be positive need not be regarded as a limitation. For if β is negative, we form the quotient

$$\frac{a}{b}=\frac{1}{\alpha+i\beta}=\frac{\alpha-i\beta}{\alpha^2+\beta^2}=\frac{\alpha}{\alpha^2+\beta^2}+\frac{-\beta}{\alpha^2+\beta^2}i,$$

where the coefficient of i on the right is *positive*. We may therefore write $\frac{a}{b}=\alpha'+i\beta'$, where β' is positive. If then the coefficient of i in $\frac{b}{a}$ is negative, we interchange b and a in the whole investigation and thus form a function $\Phi(u)$ of such characteristics that

$$\Phi(u+b)=\Phi(u),$$

$$\Phi(u+a)=e^{-\frac{\pi ik}{b}(2u+a)}\Phi(u).$$

The function $\Phi(u)$ is defined by the series

$$\Phi(u)=\sum_{m=-\infty}^{m=+\infty}B_mQ_0^{\frac{m^2}{k}}e^{m\frac{2\pi i}{b}u},$$

where $$Q_0=e^{\pi i\frac{a}{b}}.$$

ART. 87. *If* $k=1$, we have (Art. 85)

$$\Phi(u+a)=\Phi(u),$$

$$\Phi(u+b)=e^{-\frac{\pi i}{a}(2u+b)}\Phi(u),$$

which equations are satisfied by the series

$$\Phi(u)=\sum_{m=-\infty}^{m=+\infty}B_mQ^{\frac{m^2}{1}}e^{m\frac{2\pi}{a}u},$$

where $$B_{m+1}=B_m.$$

In this case, since the B's are all equal, we may write

$$\Phi(u)=B_0\sum_{m=-\infty}^{m=+\infty}Q^{m^2}e^{m\frac{2\pi i}{a}u}.$$

This is Hermite's function $X(u)$, when we make $B_0=1$. It is the *simplest* intermediary function and is called the *Chi-function.*

For $k=2$, we have

$$\Phi(u+a)=\Phi(u),$$

$$\Phi(u+b)=e^{-\frac{2\pi i}{a}(2u+b)}\Phi(u)$$

and $$\Phi(u)=\sum_{m=-\infty}^{m=+\infty}B_mQ^{\frac{m^2}{2}}e^{m\frac{2\pi i}{a}u},$$

$$B_{m+2}=B_m.$$

In this case $\Phi(u)$ contains the two arbitrary constants B_0 and B_1. We may therefore write

$$\Phi(u) = B_0\Phi_0(u) + B_1\Phi_1(u),$$

where
$$\Phi_0(u) = \sum_{\mu=-\infty}^{\mu=+\infty} Q^{2\mu^2} e^{\frac{2\pi i}{a} 2\mu u}, \qquad (m = 2\mu)$$

$$\Phi_1(u) = \sum_{\mu=-\infty}^{\mu=+\infty} Q^{\frac{(2\mu+1)^2}{2}} e^{\frac{2\pi i}{a}(2\mu+1)u}, \quad (m = 2\mu + 1).$$

The constants B_0 and B_1 being arbitrary, we choose $B_0 = 1$, and $B_1 = 0$, and thus have a particular solution $\Phi_0(u)$ of the functional equations; writing $B_0 = 0$ and $B_1 = 1$ we have another particular solution $\Phi_1(u)$.

The functions $\Phi_0(u)$ and $\Phi_1(u)$ are the remarkable functions first introduced into analysis by Jacobi and known as the Jacobi Theta-functions.* Jacobi employed a somewhat different notation, which we will have, if we write

$$Q^2 = e^{2\pi i\frac{b}{a}} = q.$$

It follows then that

$$\Phi_0(u) = \sum_{\mu=-\infty}^{\mu=+\infty} q^{\mu^2} e^{\frac{4\pi i}{a}\mu u},$$

$$\Phi_1(u) = \sum_{\mu=-\infty}^{\mu=+\infty} q^{\left(\frac{2\mu+1}{2}\right)^2} e^{\frac{2\pi i}{a}(2\mu+1)}.$$

Jacobi further wrote instead of a the quantity $4K$, and instead of b the quantity $2iK'$, and consequently

$$q = e^{-\pi\frac{K'}{K}}.$$

The above functions become

$$\Phi_0(u) = \Theta_1(u) = \sum_{\mu=-\infty}^{\mu=+\infty} q^{\mu^2} e^{\frac{\pi i\mu u}{K}},$$

$$\Phi_1(u) = \mathrm{H}_1(u) = \sum_{\mu=-\infty}^{\mu=+\infty} q^{\left(\frac{2\mu+1}{2}\right)^2} e^{\frac{\pi i u}{2K}(2\mu+1)}.$$

* In his memorial address Lejeune-Dirichlet eulogized Jacobi as follows (see Jacobi, *Ges. Werke*, I, p. 14): "Bedenkt man, dass die neue Function jetzt das ganze Gebiet der elliptischen Transcendenten beherrscht, dass Jacobi aus ihren Eigenschaften wichtige Theoreme der höhreren Arithmetik abgeleitet hat, und dass sie eine wesentliche Rolle in vielen Anwendungen spielt, von welchen hier nur die vermittelst dieser Transcendenten gegebene Darstellung der Rotationsbewegung erwähnt werden mag, so wird man dieser Function die nächste Stelle nach den längst in die Wissenschaft aufgenommenen Elementartranscendenten einräumen müssen."

ART. 88. If we put

$$\phi(u) = \frac{\Phi_1(u)}{\Phi_0(u)},$$

it is seen that $\quad \phi(u + a) = \dfrac{\Phi_1(u + a)}{\Phi_0(u + a)} = \dfrac{\Phi_1(u)}{\Phi_0(u)} = \phi(u),$

and $\quad \phi(u + b) = \dfrac{\Phi_1(u + b)}{\Phi_0(u + b)} = \dfrac{e^{-\frac{2\pi i}{a}(2u+b)}\Phi_1(u)}{e^{-\frac{2\pi i}{a}(2u+b)}\Phi_0(u)} = \dfrac{\Phi_1(u)}{\Phi_0(u)} = \phi(u).$

It is thus shown that the function $\phi(u)$ is a doubly periodic function having the periods a and b. This function $\phi(u)$ cannot be a constant, for if

$$\phi(u) = \frac{\Phi_1(u)}{\Phi_0(u)} = C,$$

then $\Phi_1(u) = C\Phi_0(u)$, which is not true since $\Phi_0(u)$ is developable in the even powers of $e^{\frac{2\pi i}{a}u}$ while $\Phi_1(u)$ is developable in the odd powers.

The functions $\Phi(u)$ which have been considered do not become infinite or indeterminate for any finite value of u; they have the character of integral functions and may be developed in power series which proceed in positive integral powers. They are integral transcendental functions (Chapter I).

ART. 89. *Historical.* — Abel (Œuvres, Sylow and Lie edition, T. I, p. 263 and p. 518, 1827–1830) showed that the elliptic functions considered as the inverse of the elliptic integrals could be expressed as the quotient of infinite products. These infinite products Jacobi [Gesam. Werke, Bd. I, p. 198, 1829] introduced into analysis under the name of Theta-functions, and by expanding them in infinite series (see Chapter X) he discovered many new properties other than those which had been previously employed in mathematical physics by French mathematicians, notably by Poisson and Fourier (*Sur la Théorie de la Chaleur*).

Jacobi [*Fund. Nova*, p. 45; Werke, Bd. I, p. 497] founded the whole theory of the elliptic functions upon these new transcendents, which made the elliptic functions remarkably simple, as well as their application, for example, to rotary motion, the swing of the pendulum, and innumerable problems of physics and mechanics; also through these Theta-functions the realms of geometry were essentially widened and many abstract properties of the theory of numbers were revealed in a new light. In the present treatise these Theta-functions are to be regarded as the fundamental elements.

ART. 90. *The intermediary functions of the kth order or degree.*—It is clear that we may write the function $\Phi(u)$ of Art. 85 in the form

$$\Phi(u) = B_0\Phi_0(u) + B_1\Phi_1(u) + \cdots + B_{k-1}\Phi_{k-1}(u),$$

where

$$\Phi_\lambda(u) = \sum_{\mu=-\infty}^{\mu=+\infty} q^{\frac{(\mu k+\lambda)^2}{2k}} e^{(\mu k+\lambda)\frac{2\pi i}{b}u}$$

$$(\lambda = 0, 1, \ldots, k-1).$$

Such functions, for reasons given in Art. 92, are said to be of the kth degree or order. We shall next prove that there *are k (and not more than k) independent intermediary functions of the kth order.*

Suppose that we have $k+1$ such functions

$$\Psi_1(u), \quad \Psi_2(u), \ldots, \Psi_{k+1}(u)$$

which satisfy the functional equations

$$\Phi(u+a) = \Phi(u),$$

$$\Phi(u+b) = e^{-\frac{\pi i k}{a}(2u+b)}\Phi(u).$$

These functions are therefore of the form

$$\Psi_\alpha(u) = B_0^{(\alpha)}\Phi_0(u) + B_1^{(\alpha)}\Phi_1(u) + \cdots + B_{k-1}^{(\alpha)}\Phi_{k-1}(u)$$

$$(\alpha = 1, 2, \ldots, k+1).$$

We have at once, if we take $p(=1)$ as the coefficient of $\Psi_\alpha(u)$,

$$0 = -p\Psi_\alpha(u) + B_0^{(\alpha)}\Phi_0(u) + B_1^{(\alpha)}\Phi_1(u) + \cdots + B_{k-1}^{(\alpha)}\Phi_{k-1}(u)$$

$$(\alpha = 1, 2, \ldots, k+1).$$

In these $k+1$ equations we may consider $p, \Phi_0, \Phi_1, \ldots, \Phi_{k-1}$ as unknown quantities; then, since the equations are homogenous, either their determinant must be zero, or all the unknown quantities are zero. The latter cannot be the case, since $p = 1$.

We must therefore have

$$\begin{vmatrix} \Psi_1(u), & B_0^{(1)}, & \ldots, & B_{k-1}^{(1)} \\ \Psi_2(u), & B_0^{(2)}, & \ldots, & B_{k-1}^{(2)} \\ \cdot & \cdot & \cdot & \cdot \\ \Psi_{k+1}(u), & B_0^{(k+1)}, & \ldots, & B_{k-1}^{(k+1)} \end{vmatrix} = 0.$$

If this determinant is expanded with reference to the terms of the first column, we have

$$C_1\Psi_1(u) + C_2\Psi_2(u) + \cdots + C_{k+1}\Psi_{k+1}(u) = 0,$$

where the C's are the constant minors (sub-determinants).

We thus see that there exists a linear homogeneous equation with constant coefficients among any $k+1$ intermediary functions of the kth degree.

ART. 91. *The zeros.* — In the initial period-parallelogram there is a congruent point u' corresponding to any point u in the u-plane, such that

$$u = u' + \lambda a + \mu b,$$

where λ and μ are integers.

We have

$$\Phi(u) = \Phi(u' + \mu b + \lambda a) = \Phi(u' + \mu b),$$

and further,

$$\Phi(u + b) = e^{-\frac{\pi i k}{a}(2u+b)}\Phi(u),$$

$$\Phi(u + 2b) = e^{-\frac{\pi i k}{a}(2u+3b)}\Phi(u + b),$$

$$\Phi(u + 3b) = e^{-\frac{\pi i k}{a}(2u+5b)}\Phi(u + 2b),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\Phi(u + \mu b) = e^{-\frac{\pi i k}{a}[2u+(2\mu-1)b]}\Phi(u + (\mu - 1)b).$$

When these equations are multiplied together, we have

$$\Phi(u + \mu b) = e^{-\frac{\pi i k}{a}[2\mu u+b(1+3+5+\cdots+2\mu-1)]}\Phi(u),$$

or
$$\Phi(u + \mu b) = e^{-\frac{\pi i k}{a}[2\mu u+\mu^2 b]}\Phi(u).$$

It follows that

$$\Phi(u) = \Phi(u' + \mu b + \lambda a) = e^{-\frac{\pi i k}{a}(2\mu u'+\mu^2 b)}\Phi(u').$$

Since the exponential factor is different from zero, it follows that $\Phi(u)$ can only vanish when $\Phi(u')$ equals zero. We may therefore limit ourselves to the discussion of $\Phi(u)$ within the initial period-parallelogram.

Since an integral transcendental function can have only a finite number of zeros* (Art. 8) within a finite surface-area, it follows that there are only a finite number of zeros of $\Phi(u)$ within the period-parallelogram. This parallelogram may be constructed in different ways. If from any point Q in the u-plane we measure off both in length and direction the quantities a and b and draw parallels through the end-points, we have a period-parallelogram of the function with the periods a and b. If starting with this parallelogram we cover the plane with similar parallelograms, it is seen that the plane is differently divided from what it was in the former distribution of parallelograms, where the first initial parallelogram had the origin as one of the vertices.

* Cf. Forsyth, *Theory of Functions*, p. 62.

It will be convenient for the following investigation if the initial period-parallelogram is so situated that there are no zeros of the function upon its sides. To effect this let $QA'C'B'$ be any period-parallelogram.

As there can be only a finite number of zeros of $\Phi(u)$ within this parallelogram, it is evident that upon the line QB' there is a point D such that there is no zero of the function on the line DE which is drawn parallel to $QA' = a$. Similarly there will be a point F on the line QA' such that there is no zero of the function on the line FG drawn parallel to $QB' = b$. The lines DE and FG intersect in a point P, say. We take P as the vertex of a new parallelogram $PACB$. We shall see that there are no zeros of the function $\Phi(u)$ on the sides of this parallelogram. On the side PE there is by construction no zero. Also upon EA there can be none owing to the relation $\Phi(u+a) = \Phi(u)$, so that $\Phi(u)$ takes the same values upon EA as upon DP. Upon PG likewise by construction there is no zero of the function $\Phi(u)$ and upon GB there is also none, since

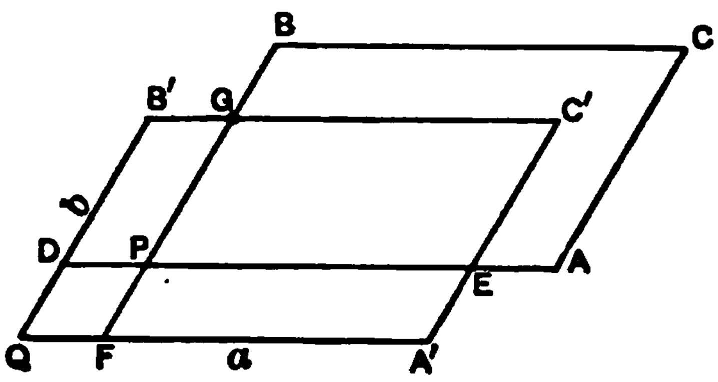


Fig. 21.

$$\Phi(u+b) = e^{-\frac{\pi i k}{a}(2u+b)}\Phi(u).$$

Hence upon the sides PA and PB there are no zeros of the function. It follows also on account of the two functional equations just written that there are no zeros on the sides AC and BC.

ART. 92. We may now apply the following well-known theorem of Cauchy:* *If a function $\Phi(u)$ within a definite region, boundaries included, is everywhere one-valued, finite and continuous, and if N denotes the number of zeros within this region, then is*

$$N = \frac{1}{2\pi i}\int \frac{\Phi'(u)}{\Phi(u)}du,$$

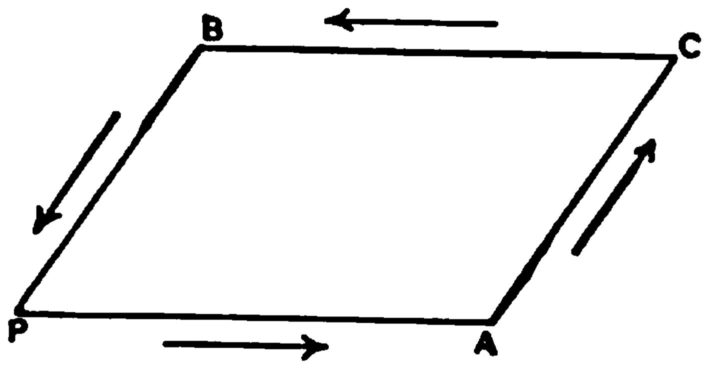


Fig. 22.

where the integration is to be taken over the boundaries of this region and in the direction such that the region is always to the left.

This theorem is applicable to our function $\Phi(u)$ which is infinite for no finite value of u. The region in question is the period-parallelogram $PACB$. We therefore have, if we write $\psi(u) = \dfrac{\Phi'(u)}{\Phi(u)}$,

$$2\pi i N = \int_{PA}\psi(u)du + \int_{AC}\psi(u)du + \int_{CB}\psi(u)du + \int_{BP}\psi(u)du.$$

* Cf. Forsyth, *loc. cit.*, p. 63; Osgood, *loc. cit.*, p. 282. Professor Osgood demands that the curve be analytic (regular) for all points within the boundaries and continuous for all points of the boundaries. See the theorem at the end of Art. 52.

We may transform these integrals of the complex variable into integrals of a real variable t. Let u take the value p at P; then, since $PA = a$, we may write all the values which u can take on this portion of line PA in the form

$$u = p + at,$$

where $0 \leqq t \leqq 1$.

It follows that

$$\int_{PA} \psi(u)du = a\int_0^1 \psi(p + at)dt.$$

Further, the variable u has at A the value $p + a$, and since $AC = b$, we have

$$\int_{AC} \psi(u)du = b\int_0^1 \psi(p + a + bt)dt.$$

Similarly u has at B the value $p + b$, and therefore all values of u on CB have the form $p + b + at$, and consequently

$$\int_{AC} \psi(u)du = a\int_1^0 \psi(p + b + at)dt = -a\int_0^1 \psi(p + b + at)dt.$$

Finally we have in the same manner

$$\int_{BP} \psi(u)du = b\int_1^0 \psi(p + bt)dt = -\int_0^1 \psi(p + bt)dt.$$

It is thus seen that

$$2\pi i N =$$

$$a\int_0^1 \Big[\psi(p + at) - \psi(p + b + at)\Big]dt + b\int_0^1 \Big[\psi(p + a + bt) - \psi(p + bt)\Big]dt.$$

Further, since $\Phi(u + a) = \Phi(u)$ and $\Phi(u + b) = e^{-\frac{\pi i k}{a}(2u+b)}\Phi(u)$, it follows at once through logarithmic differentiation that

$$\psi(u + a) = \psi(u) \quad \text{and} \quad \psi(u + b) = \psi(u) - \frac{2\pi i k}{a}.$$

These values substituted in the above integrals give

$$2\pi i N = a\int_0^1 \frac{2\pi i k}{a} dt,$$

or

$$N = k.$$

We thus see* that the number of zeros of the function $\Phi(u)$ which lie within the period-parallelogram is equal to the integer k which appears in the second functional equation which $\Phi(u)$ satisfies.

* Cf. Hermite, "*Cours*" [4th ed.], p. 224.

In algebra we say an integral rational function which vanishes for k values of u in the u-plane is of the kth degree. In a corresponding manner we say of our function $\Phi(u)$, it is of the kth *degree* or *order*, because it vanishes at k points within the period-parallelogram.

ART. 93. For $k = 1$, we had in Art. 87

$$\Phi(u + a) = \Phi(u),$$

$$\Phi(u + b) = e^{-\frac{\pi i}{a}(2u+b)}\Phi(u).$$

After the theorem just proved we know that there is one and only one zero of the function $\Phi(u)$ which satisfies these two functional equations in the period-parallelogram. We shall seek this zero in the initial period-parallelogram. We had

$$\Phi(u) = \sum_{m=-\infty}^{m=+\infty} Q^{m^2} e^{m\frac{2\pi i}{a}u} = \mathrm{X}(u).$$

Writing $m = -(n + 1)$ in this formula, it becomes

$$\begin{aligned}\mathrm{X}(u) &= \sum_{n=-\infty}^{n=+\infty} Q^{(n+1)^2} e^{-\frac{2\pi i}{a}u(n+1)} \\ &= \sum_{n=-\infty}^{n=+\infty} Q^{n^2} e^{(2n+1)\frac{b}{a}\pi i} e^{-\frac{2\pi i}{a}u(n+1)} \\ &= \sum_{n=-\infty}^{n=+\infty} Q^{n^2} e^{\frac{\pi i}{a}(2nb+b-2nu-2u)}.\end{aligned}$$

If we give to u the value $\frac{a+b}{2}$ in the above formula, it becomes

$$\begin{aligned}\mathrm{X}\left(\frac{a+b}{2}\right) &= \sum_{n=-\infty}^{n=+\infty} Q^{n^2} e^{\frac{\pi i}{a}[nb-(n+1)a]} \\ &= \sum_{n=-\infty}^{n=+\infty} Q^{n^2} Q^n e^{-(n+1)\pi i} = -\sum_{n=-\infty}^{n=+\infty} Q^{n^2+n}(-1)^n.\end{aligned}$$

If we also write $\frac{a+b}{2}$ in the original expression for $\mathrm{X}(u)$, it becomes

$$\mathrm{X}\left(\frac{a+b}{2}\right) = \sum_{m=-\infty}^{m=+\infty} Q^{m^2} Q^m e^{\pi i m} = +\sum_{m=-\infty}^{m=+\infty} Q^{m^2+m}(-1)^m.$$

Comparing the two expressions thus obtained for $\mathrm{X}\left(\frac{a+b}{2}\right)$, it is seen that they differ from each other only in sign, and consequently it necessarily follows that

$$\mathrm{X}\left(\frac{a+b}{2}\right) = 0.$$

Since the zero of the intermediary function $\Phi(u)$ of the first order, *i.e.*, of $\mathrm{X}(u)$, is the intersection of the diagonals of the initial period-parallelogram, it follows that $\mathrm{X}(u) = 0$ at all the intersections of the diagonals of the parallelograms which are congruent to this initial parallelogram.

Remark.—The question might be raised as to whether there were zeros of $\mathrm{X}(u)$ on the boundaries of the initial period-parallelogram. We saw in Art. 91 that it was always possible so to place the period-parallelogram that its boundaries were free from zeros. If, however, we consider as we do here a definite period-parallelogram, viz., the one where the origin is the vertex and which lies to the right of the origin, we do not *à priori* know that there is no zero of $\mathrm{X}(u)$ upon its boundaries.

Suppose that the period-parallelogram which has $u = p$ as one of its vertices is so drawn that there are no zeros upon its boundaries. There is one zero within the period-parallelogram, since $\Phi(u)$ is of the first degree. The value of u at this point may be expressed in the form

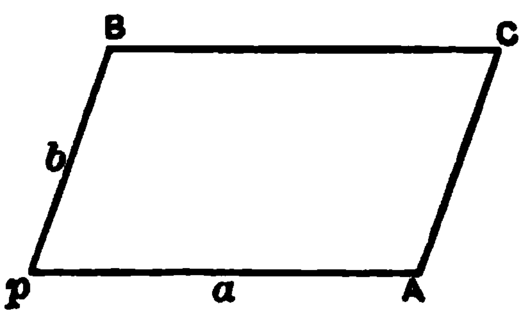


Fig. 23.

$$p + \lambda a + \nu b,$$

where λ and ν are proper fractions. If now we cover the u-plane with congruent parallelograms, there does not lie a zero of $\mathrm{X}(u)$ on any of the boundaries of these parallelograms, and within each parallelogram there is always one and only one zero. Since all the zeros are congruent one to the other and since from above $\frac{a+b}{2}$ is one of them, we must have

$$p + \lambda a + \nu b = \frac{a+b}{2} + ga + lb,$$

where g and l are integers. Every zero of $\mathrm{X}(u)$ may be expressed in this form, and therefore also the zero which we suppose may lie upon one of the boundaries of the initial period-parallelogram, say at L, where

$$L = b + \vartheta a,$$

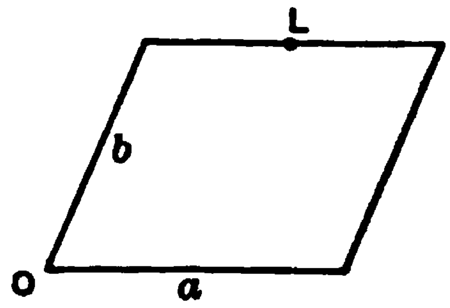


Fig. 24.

ϑ being a proper fraction.

We would then have

$$\frac{a+b}{2} + ga + lb = b + \vartheta a,$$

and consequently
$$\frac{a}{b} = -\frac{2l-1}{1-2\vartheta+2g}.$$

But the right-hand side of this expression is a rational number, which is contrary to what has been proved in Art. 71. When L lies upon any other side of the parallelogram, we may derive a similar result and thus by a *reductio ad absurdum* show that there *does not lie a zero of* $\mathrm{X}(u)$ *upon the boundary of the initial period-parallelogram.*

THE GENERAL DOUBLY PERIODIC FUNCTION EXPRESSED THROUGH A SIMPLE TRANSCENDENT.

ART. 94. We shall next consider a doubly periodic function $F(u)$ which has nowhere in the finite portion of the plane an essential singularity. Such a function has only a finite number of zeros and a finite number of infinities within a finite area. We may limit our study, as shown above, to the initial period-parallelogram. We shall assume that within this parallelogram the function $F(u)$ has the infinities $u_1, u_2, \dots, u_n$; and we shall further assume that these infinities are of the *first* order, so that in the neighborhood of any one of them, u_1 say, $F(u)$ has the form

$$F(u) = \frac{d}{u - u_1} + c_0 + c_1(u - u_1) + c_2(u - u_1)^2 + \cdots,$$

where d and the c's are constants.

We shall see that every such function may be expressed through the general intermediary functions $\Phi(u)$. We shall form such a function where the integer k is taken equal to $n + 1$ and which therefore satisfies the two functional equations

$$\Phi(u + a) = \Phi(u),$$
$$\Phi(u + b) = e^{-(n+1)\frac{\pi i}{a}(2u+b)}\Phi(u).$$

There being $n + 1$ arbitrary constants in this function, we may write it in the form

$$\Phi(u) = B_0\Phi_0(u) + B_1\Phi_1(u) + \cdots + B_n\Phi_n(u).$$

The constants $B_0, B_1, \dots, B_n$ may be so determined that the function $\Phi(u)$ becomes zero of the first order on the points $u_1, u_2, \dots, u_n$.

For write

$$\Phi(u_1) = B_0\Phi_0(u_1) + B_1\Phi_1(u_1) + \cdots + B_n\Phi_n(u_1) = 0,$$
$$\Phi(u_2) = B_0\Phi_0(u_2) + B_1\Phi_1(u_2) + \cdots + B_n\Phi_n(u_2) = 0,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\Phi(u_n) = B_0\Phi_0(u_n) + B_1\Phi_1(u_n) + \cdots + B_n\Phi_n(u_n) = 0.$$

In these equations we may consider the B's as the unknown quantities. We have then n equations with $n + 1$ unknown quantities, from which we may determine the ratios of the B's so that $\Phi(u)$ becomes zero of the first order at all the points $u_1, u_2, \dots, u_n$. By hypothesis $F(u)$ becomes infinite of the first order on all these points.

Form the product

$$f(u) = \Phi(u)\,F(u).$$

It is seen that

$$\phi(u+a) = \Phi(u+a)\,F(u+a) = \Phi(u)\,F(u) = f(u);$$

and also

$$f(u+b) = \Phi(u+b)\,F(u+b)$$

$$= e^{-(n+1)\frac{\pi i}{a}(2u+b)}\Phi(u)\,F(u),$$

or

$$f(u+b) = e^{-(n+1)\frac{\pi i}{a}(2u+b)}f(u).$$

From this it is seen that $f(u)$ is also one of the intermediary functions which satisfies the same functional equations as does $\Phi(u)$. Further, since $\Phi(u)$ becomes zero of the first order at the same points at which $F(u)$ is infinite of the first order, the product $f(u) = \Phi(u)\,F(u)$ is nowhere infinite in the finite portion of the plane. A one-valued analytic function which does not have an essential singularity in the finite portion of the plane and in this portion of plane is nowhere infinite, is an integral transcendental function; and, as there are only $n+1$ such functions that are linearly independent (cf. Art. 90), it follows that

$$f(u) = C_0\Phi_0(u) + C_1\Phi_1(u) + C_2\Phi_2(u) + \cdots + C_n\Phi_n(u),$$

where the C's are constant.

It is also seen that

$$F(u) = \frac{f(u)}{\Phi(u)}.$$

We consequently have the theorem: *Any arbitrary doubly periodic function which has only infinities of the first order may be expressed as the quotient of two integral transcendental functions, both of which satisfy the same functional equations.*

ART. 95. By means of the $X(u)$-function we can make the above theorem more general in that the order of the infinities of $F(u)$ is not restricted.

We have noted in Art. 93 that $X(u)$ is zero for the value $u = \frac{a+b}{2} = c$, say. Hence $X(u+c) = 0$ for $u = \lambda a\,(\lambda = 0, 1, 2, \ldots)$.

If we write $X(u+c) = X_1(u)$, it is seen that $X_1(u) = 0$ for $u = 0$. We also observe that the function $X_1(u)$ satisfies the two functional equations

$$X_1(u+a) = X_1(u),$$

$$X_1(u+b) = e^{-\frac{\pi i}{a}(2u+a+b+b)}X_1(u).$$

We have immediately the following relations:

$$X_1(u-u_1+b) = e^{-\frac{\pi i}{a}(2u-2u_1+a+b+b)}X_1(a-u_1),$$

$$X_1(u-u_2+b) = e^{-\frac{\pi i}{a}(2u-2u_2+a+b+b)}X_1(u-u_2),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$X_1(u-u_k+b) = e^{-\frac{\pi i}{a}(2u-2u_k+a+b+b)}X_1(u-u_k).$$

If we put $\Psi(u) = \mathrm{X}_1(u - u_1)\,\mathrm{X}_1(u - u_2)\ldots\mathrm{X}_1(u - u_k)$, it is seen that

$$\Psi(u + a) = \Psi(u),$$

$$\Psi(u + b) = e^{-k\frac{\pi i}{a}(2u+b)}\Psi(u),$$

provided that

$$k(a + b) - 2(u_1 + u_2 + \cdots + u_k) = 2\,ma;$$

that is, if

$$u_1 + u_2 + \cdots + u_k = kc - ma, \tag{1}$$

where m is any integer.

Hence if $k = n + 1$, the function $\Psi(u)$ becomes zero on any n arbitrary points $u_1, u_2, \ldots, u_n$, while the other zero must satisfy equation (1). As some of the points $u_1, u_2, \ldots, u_n$ may be made equal to one another, it is seen that the zeros are not restricted to being of the first order in $\Psi(u)$. We may therefore let $\Psi(u)$ take the place of $f(u)$ in the preceding article and *mutatis mutandis* have the same result as stated there.

ART. 96. It is convenient to form here a function which becomes infinite of the first order for $u = 0$, $u = a$, $u = 2\,a$, $\cdots$. Such a function is the Zêta-function (see Art. 97),

$$\mathrm{Z}_0(u) = \frac{\mathrm{X}'(u + c)}{\mathrm{X}(u + c)}.$$

This function $\mathrm{Z}_0(u)$ is one-valued in the entire u-plane and has an essential singularity only at infinity. By means of this fundamental element Hermite* has *given a general method of expressing any one-valued doubly periodic function which in the finite portion of the plane has no essential singularity.*

We shall so choose the period-parallelogram that $F(u)$ does not become infinite on its boundaries. If the function $F(u)$ is infinite of the λth order say at u_1, the development in the neighborhood of this point is

$$F(u) = \frac{b_\lambda}{(u - u_1)^\lambda} + \frac{b_{\lambda-1}}{(u - u_1)^{\lambda-1}} + \cdots + \frac{b_1}{u - u_1} + P(u - u_1),$$

the b's being constants.

We shall now give a method of representing this function when for every infinity the complex of all the negative powers is known. This complex of negative powers we have called (in Chapter I) the *principal part* of the function. We introduce a new variable ξ and form

$$f(\xi) = F(\xi)\,\mathrm{Z}_0(u - \xi),$$

where now u is to play the rôle of a parameter, being a point within the initial parallelogram, while ξ is the variable. We consider in the ξ-plane

* Hermite, *Ann. de Toulouse*, t. 2 (1888), pp. 1–12, and "*Cours*" (4th ed.), p. 226.

a period-parallelogram of $F(\xi)$, upon whose boundaries there is no infinity of $F(\xi)$.

The function $f(\xi)$ becomes infinite within this period-parallelogram on the points $u_1, u_2, \ldots, u_n$, the points at which $F(\xi)$ is by hypothesis infinite; and $f(\xi)$ is infinite also at the additional point $\xi = u$, since $Z_0(0) = \infty$.

We form the sum of the residues of $f(\xi)$ with regard to all the above infinities and have after Cauchy's Residue Theorem

$$\sum \operatorname{Res} f(\xi) = \frac{1}{2\pi i}\int f(\xi)d\xi,$$

where the integration is to be taken over the sides of the period-parallelogram and in such a way that the surface of the parallelogram is always to the left. We therefore have

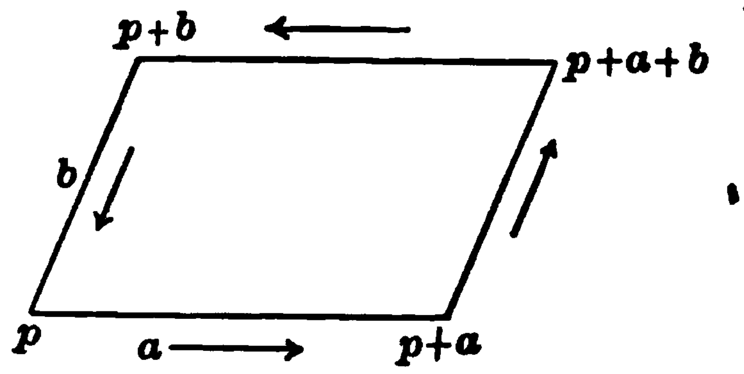


Fig. 25.

$$2\pi i \sum \operatorname{Res} f(\xi) = \int_p^{p+a} f(\xi)d\xi + \int_{p+a}^{p+a+b} f(\xi)d\xi + \int_{p+a+b}^{p+b} f(\xi)d\xi + \int_{p+b}^{p} f(\xi)d\xi;$$

or, as in Art. 92,

$$2\pi i \sum \operatorname{Res} f(\xi) = a\int_0^1 f(p + at)dt + b\int_0^1 f(p + a + bt)dt - a\int_0^1 f(p + b + at)dt - b\int_0^1 f(p + bt)dt.$$

Further, since

$$Z_0(v + a) = Z_0(v),$$
$$Z_0(v + b) = Z_0(v) - \frac{2\pi i}{a},$$
$$F(\xi + a) = F(\xi),\quad F(\xi + b) = F(\xi),$$

it follows that

$$f(\xi + a) = F(\xi + a)\,Z_0(u - \xi - a) = F(\xi)\,Z_0(u - \xi) = f(\xi)$$

and

$$f(\xi + b) = F(\xi)\left\{Z_0(u - \xi) + \frac{2\pi i}{a}\right\} = f(\xi) + \frac{2\pi i}{a}F(\xi).$$

We therefore have

$$2\pi i \sum \operatorname{Res} f(\xi) = a\int_0^1 [f(p + at) - f(p + b + at)]dt$$
$$= a\int_0^1 \left[-\frac{2\pi i}{a}F(p + at)\right]dt;$$

and consequently *

$$\sum \operatorname{Res} f(\xi) = -\int_0^1 F(p + at)dt.$$

There is no infinity of the function F on the path of integration, this being a side of the parallelogram above. Hence the integral on the right

* Cf. Hermite, *loc. cit.*, p. 226.

has a definite value, a value which is independent of u, and as it does not contain ξ, it is a definite constant.

ART. 97. We shall next determine by direct computation the sum of the above residues of $f(\xi)$.

We had

$$Z_0(v) = \frac{X'(v+c)}{X(v+c)}.$$

The function $X(v+c)$ becomes zero of the first order for $v = 0$, and is one-valued and finite for all finite values of v.

Its development is therefore of the form

$$X(v+c) = \gamma_1 v + \gamma_2 v^2 + \cdots,$$

where the γ's are constant and $\gamma_1 \neq 0$.

Through differentiation it is seen that

$$X'(v+c) = \gamma_1 + 2\gamma_2 v + \cdots,$$

and consequently

$$Z_0(v) = \frac{1}{v}\frac{\gamma_1 + 2\gamma_2 v + \cdots}{\gamma_1 + \gamma_2 v + \cdots} = \frac{1}{v} + d_0 + d_1 v + d_2 v^2 + \cdots.$$

We note that the residue of $Z(v)$ with respect to $v = 0$ is *unity*. This function, as shown in the sequel, has in regard to the doubly periodic functions the same relation as has $\cot u$ with respect to the simply periodic functions and as has $\frac{1}{v}$ to the rational functions.

If for v we substitute $u - \xi$, we have

$$\begin{aligned} Z_0(u-\xi) &= \frac{1}{u-\xi} + d_0 + d_1(u-\xi) + \cdots \\ &= -\frac{1}{\xi - u} + d_0 + d_1(u-\xi) + \cdots, \end{aligned}$$

which is the development of $Z_0(u - \xi)$ in the neighborhood of $\xi = u$.

We next form the corresponding development of $F(\xi)$. In the interior of the period-parallelogram the function $F(\xi)$ becomes infinite at the points $u_1, u_2, \ldots, u_n$ but not at u. Hence we may develop $F(\xi)$ by Taylor's Theorem in the form

$$F(\xi) = F(u) + \frac{F'(u)}{1!}(\xi - u) + \frac{F''(u)}{2!}(\xi - u)^2 + \cdots.$$

Further, since $f(\xi) = F(\xi)\, Z_0(u - \xi)$, it follows that

$$f(\xi) = -\frac{F(u)}{\xi - u} + d_0 F(u) + \cdots,$$

and consequently

$$\operatorname*{Res}_{\xi = u} f(\xi) = -F(u).$$

We saw above that $\Sigma \operatorname{Res} f(\xi)$ is independent of u, but as shown here, the single residues are dependent upon this quantity.

ART. 98. We shall next calculate the residues of $f(\xi)$ with respect to the other infinities $u_1, u_2, \dots, u_n$. Suppose that the function $F(\xi)$ becomes infinite of the λth order on the point u_1, so that $F(\xi)$ when expanded in the neighborhood of this point is of the form

$$F(\xi)=\frac{b_\lambda}{(\xi-u_1)^\lambda}+\frac{b_{\lambda-1}}{(\xi-u_1)^{\lambda-1}}+\cdots+\frac{b_1}{\xi-u_1}+c_0+c_1(\xi-u_1)+\cdots,$$

where the b's and c's are constants.

For the value $\xi=u_1$ the function $\mathrm{Z}_0(u-\xi)$ is not infinite and may be developed by Taylor's Theorem in the form

$$\mathrm{Z}_0(u-\xi)=\mathrm{Z}_0(u-u_1)-\frac{\mathrm{Z}_0'(u-u_1)}{1!}(\xi-u_1)+\frac{\mathrm{Z}_0''(u-u_1)}{2!}(\xi-u_1)^2-\cdots.$$

It follows that the coefficient of $\dfrac{1}{\xi-u_1}$ in the product $F(\xi)\mathrm{Z}_0(u-\xi)$ is

$$b_1\mathrm{Z}_0(u-u_1)-\frac{b_2}{1!}\mathrm{Z}_0'(u-u_1)+\frac{b_3}{2!}\mathrm{Z}_0''(u-u_1)+\cdots\pm\frac{b_\lambda}{(\lambda-1)!}\mathrm{Z}_0^{(\lambda-1)}(u-u_1),$$

which is the residue of $f(\xi)$ with respect to the infinity $\xi=u_1$. The residues with respect to the other infinities $u_2, u_3, \dots, u_n$ are found in the same manner. The b's and λ, of course, have different values for each of these points.

Let the orders of infinity at u_k be λ_k ($k=1, 2, \dots, n$) and in the neighborhood of the infinity u_k let the principal part of the function $F(\xi)$ be

$$\frac{b_{k,\lambda_k}}{(u-u_k)^{\lambda_k}}+\frac{b_{k,\lambda_k-1}}{(u-u_k)^{\lambda_k-1}}+\frac{b_{k,\lambda_k-2}}{(u-u_k)^{\lambda_k-2}}+\cdots+\frac{b_{k,1}}{u-u_k}.$$

It follows at once that

$$\sum_{\substack{\xi=u_k\\(k=1,2,\dots,n)}}\operatorname{Res} f(\xi)=\sum_{k=1}^{k=n}\Big[b_{k,1}\mathrm{Z}_0(u-u_k)-\frac{b_{k,2}}{1}\mathrm{Z}_0'(u-u_k)+\frac{b_{k,3}}{2!}\mathrm{Z}_0''(u-u_k)-\cdots\pm\frac{b_{k,\lambda_k}}{(\lambda_k-1)!}\mathrm{Z}_0^{(\lambda_k-1)}(u-u_k)\Big].$$

We also saw that $\operatorname*{Res}_{\xi=u} f(\xi)=-F(u)$, which must be added to the sum just written.

On the other hand we had

$$\sum\operatorname{Res} f(\xi)=-\int_0^1 F(p+at)dt=C,\text{ say,}$$

where C is a constant.

Equating these two expressions for the sum of the residues, we have

$$F(u)=C+\sum_{k=1}^{k=n}\Big[b_{k,1}\mathrm{Z}_0(u-u_k)-\frac{b_{k,2}}{1!}\mathrm{Z}_0'(u-u_k)+\frac{b_{k,3}}{2!}\mathrm{Z}_0''(u-u_k)-\cdots\pm\frac{b_{k,\lambda_k}}{(\lambda_k-1)!}\mathrm{Z}_0^{(\lambda_k-1)}(u-u_k)\Big],$$

which is the required representation of the doubly periodic function $F(u)$.

We thus see that *a doubly periodic function may be expressed through a finite sum of terms that are formed of the function* Z_0 *and its derivatives.*

EXAMPLE

Show that two doubly periodic functions with the same periods and the same principal parts differ only by an additive constant.

In Chapter XX, several methods of representing a doubly periodic function will be found and the consequences which result therefrom will be derived. All these methods, however, are little other than different interpretations of the above formula.

It is seen at once from this formula that we may represent a doubly periodic function when its principal parts are given, the function being completely determined except as to an additive constant. This expression for a doubly periodic function is the analogue of the formula for the decomposition of a rational function into its simple fractions or of the decomposition of a simply periodic function into its simple elements (see Arts. 11 and 25). It may be shown that the latter cases may be derived from the former by making *one* of the periods infinite for the case of the simply periodic functions, and by causing them *both* to be infinite for the rational functions.

ART. 99. There is a restriction with respect to the constants that appear in the above development.

We saw that

$$Z_0(v + a) = Z_0(v) \quad \text{and} \quad Z_0(v + b) = Z_0(v) - \frac{2\pi i}{a}.$$

It follows that $Z_0(v)$ is not a doubly periodic function; but all its derivatives are doubly periodic, since we have

$$Z_0'(v + a) = Z_0'(v),$$
$$Z_0'(v + b) = Z_0'(v), \text{ etc.}$$

Hence under the summation sign of the preceding article all terms except the first are doubly periodic. Further, since $F(u + b) = F(u)$, it also follows that

$$\sum_{k=1}^{k=n} b_{k,1} Z_0(u - u_k) = \sum_{k=1}^{k=n} b_{k,1} Z_0(u - u_k + b).$$

Since
$$Z_0(u - u_k + b) = Z_0(u - u_k) - \frac{2\pi i}{a},$$

it is evident from the equality of the two summations just written that

$$-\frac{2\pi i}{a}\sum_{k=1}^{k=n} b_{k,1} = 0, \quad \text{or} \quad \sum_{k=1}^{k=n} b_{k,1} = 0.$$

We thus have the very important theorem: *The sum of the residues within a period-parallelogram of a doubly periodic function with respect to all of its infinities, is equal to zero.*

If we wish to form a doubly periodic function, when its principal parts with reference to its infinities are given, the restriction just mentioned must be imposed upon the constants.

ART. 100. We may prove in a different manner that

$$\Sigma \operatorname{Res} F(u) = 0.$$

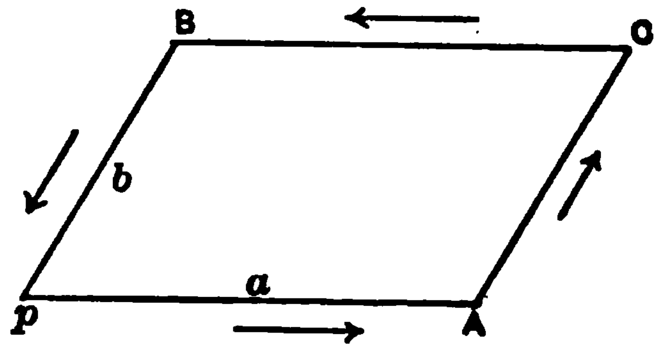


Fig. 26.

Take any period-parallelogram, upon the sides of which there are no infinities of $F(u)$. Then by Cauchy's Residue Theorem

$$2\pi i \sum \operatorname{Res} F(u) = \int_{pACB} F(u)du.$$

But from Art. 92 we have

$$\int_{pACB} F(u)du = a\int_0^1 F(p+at)dt + b\int_0^1 F(p+a+bt)dt$$
$$- a\int_0^1 F(p+b+at)dt - b\int_0^1 F(p+bt)dt.$$

Further, since
$$F(u+a) = F(u) = F(u+b),$$
it follows that

$$\Sigma \operatorname{Res} F(u) = 0.$$

ART. 101. It follows directly from the above representation of a doubly periodic function that it cannot be an integral transcendental function (cf. Art. 83). In this case all the quantities $b_{k,1}, b_{k,2}, \ldots, b_{k,\lambda_k}$ would be zero and consequently

$$F(u) = C.$$

It also follows that a doubly periodic function cannot be infinite of the first order at only one point of the period-parallelogram. For if u_1 were such a point, then is

$$F(u) = \frac{b_{1,1}}{u-u_1} + c_0 + c_1(u-u_1) + \cdots$$

in the neighborhood of this point, and consequently

$$\Sigma \operatorname{Res} F(u) = b_{1,1}.$$

But as the sum of the residues is equal to zero, it also follows that $b_{1,1} = 0$ and consequently $F(u)$ would be an integral transcendent. But an integral transcendental function with two periods is a constant (Art. 83). We have consequently the following theorem due to Liouville: *A doubly periodic function must have at least two infinities of the first order within the period-parallelogram, or it must be infinite of at least the second order on one such point.*

ART. 102. We have then two different methods which may be followed in the treatment of the doubly periodic functions, the one where the two infinities of the first order in the period-parallelogram are distinct, which is the older method employed by Jacobi, say $z = sn\, u$; while the other method where the function becomes infinite of the second order is the one followed by Weierstrass, and in this case z considered as a function of u is written $z = \wp u$. The notation in the two different cases is inserted here, as it is convenient to refer to the two methods by means of this notation before the general treatment of these particular functions is considered.

In the next Chapter it will be shown that a doubly periodic function which becomes infinite at n points (the order being finite at each point) is algebraically expressible through either one of the above simple forms $z = sn\, u$ or $z = \wp u$; and consequently the general theory of doubly periodic functions is reduced to the consideration of the two simpler cases.

THE ELIMINANT EQUATION.

ART. 103. We have shown in Chapter III that a one-valued simply periodic function which in the finite portion of the plane has no essential singularity and which takes within a period-strip any value only a finite number of times, satisfies an algebraic differential equation in which the independent variable u does not explicitly enter. In Chapter II we have seen that associated with every one-valued analytic function which has an algebraic addition-theorem there exists an equation of the form just mentioned. We shall see later in Art. 158 that every one-valued doubly periodic function has an algebraic addition-theorem, so that (see Art. 35) the notion of the doubly periodic function and of the eliminant equation is seen to be coextensive for the one-valued functions.

We wish now to show that there is an eliminant equation which is associated with every one-valued doubly periodic function. First, however, it is necessary to consider certain preliminary investigations.

ART. 104. Suppose that the doubly periodic function $F(u)$ has n infinities of the first order within a period-parallelogram, or if it becomes infinite of the λth order on any point, let this point be counted as λ infinities of the first order, so that the totality of infinities is still n. Let v be any arbitrary quantity and consider the number of solutions of the equation

$$F(u) = v$$

within a period-parallelogram.

After the same method by which we constructed a period-parallelogram which had no infinities upon its boundaries we may also construct one which has no zero of the function $F(u) - v$ upon the boundaries. We

may therefore assume that there are no zeros or infinities of the function $F(u) - v$ upon the boundaries of our period-parallelogram.

Consider next the function

$$G(u) = F(u) - v.$$

It is a doubly periodic function with the same periods as $F(u)$, viz., a and b. As it becomes infinite at the same points as $F(u)$, it has n infinities within the period-parallelogram.

Form next the logarithmic derivative of $G(u)$,

$$\frac{G'(u)}{G(u)} = H(u), \text{ say.}$$

The function $H(u)$ has the periods a and b and becomes infinite at the points where $G'(u)$ is infinite and also where $G(u)$ is zero.

Let u_1 be an infinity of $G(u)$ of the λth order, so that

$$G(u) = (u - u_1)^{-\lambda} G_1(u), \quad \text{where } G_1(u_1) \neq 0.$$

We then have (Art. 4) in the neighborhood of u_1,

$$H(u) = -\frac{\lambda}{u - u_1} + P(u - u_1),$$

so that

$$\operatorname*{Res}_{u=u_1} H(u) = -\lambda,$$

that is, *the residue of $H(u)$ with respect to u_1 is the order of the infinity of $G(u)$ at the point u_1 with the negative sign.*

Suppose next that w_1 is a zero of $G(u)$ of the μth order, so that

$$G(u) = (u - w_1)^{\mu} G_2(u), \quad \text{where } G_2(w_1) \neq 0.$$

We then have in the neighborhood of w_1

$$H(u) = \frac{\mu}{u - w_1} + P(u - w_1), \quad \text{or}$$

$$\operatorname*{Res}_{u=w_1} H(u) = \mu,$$

that is, *the residue of $H(u)$ with respect to a zero of $G(u)$ is equal to the order of the zero at this point.*

Further, since the sum of the residues of a doubly periodic function with respect to all its infinities within a period-parallelogram is zero, it follows that

$$-\Sigma\lambda + \Sigma\mu = 0,$$

where $\Sigma\lambda$ denotes the sum of the infinities of the function $G(u)$ in a period-parallelogram, an infinity of the λth order counting as λ simple infinities, and where $\Sigma\mu$ denotes the number of zeros of the first order of $G(u)$, a

zero of the μth order counting as μ zeros of the first order. Since $G(u) = F(u) - v$, it follows that the number of roots of the equation

$$F(u) - v = 0$$

within a period-parallelogram is equal to the number of infinities of the first order of the function $F(u)$ within this parallelogram.

It follows that *a doubly periodic function $F(u)$ takes within every period-parallelogram any value v as often as it becomes infinite of the first order within this period-parallelogram.**

ART. 105. Let $z = F(u)$ be a doubly periodic function of the nth order with the primitive periods a and b and let $w = G(u)$ be a doubly periodic function of the kth order with the same periods. Neither of these functions is supposed to have an essential singularity in the finite portion of the u-plane. We assert that there exists an algebraic equation with constant coefficients connecting z and w.

For if a definite value is given to z there are n values of u, say u_1, u_2, $\dots$, u_n, for which $F(u) = z$. If we write these values of u in $w = G(u)$, we have n values of $G(u)$, say $w_1 = G(u_1)$, $w_2 = G(u_2)$, $\dots$, $w_n = G(u_n)$. Hence the variable z is related to the variable w in such a way that to one value of z there correspond n values of w and similarly to one value of w there correspond k values of z, and consequently between z and w there exists an integral algebraic equation

$$G(z, w) = 0,$$

which is of the nth degree in w and of the kth degree in z.

We may next suppose that $z = \phi(u)$ is a doubly periodic function with the periods a and b, then $w = \dfrac{dz}{du} = \phi'(u)$ is a doubly periodic function having the same periods. Hence from the theorem above there is an algebraic equation connecting z and $\dfrac{dz}{du}$, say

$$f\left(z, \frac{dz}{du}\right) = 0.$$

It is easy to determine the degree of f in z and $\dfrac{dz}{du}$; for if $\phi(u) = z$ is of the nth degree then $\dfrac{dz}{du}$ occurs to the nth degree in the above equation.

If u_1 is an infinity of the λth order of $\phi(u)$, then u_1 is an infinity of the $\lambda + 1$ order of $\phi'(u)$, so that $\phi'(u)$ becomes infinite on the same points as $\phi(u)$, the order of infinity of $\phi'(u)$ being *one* greater on each of these points than is the order of $\phi(u)$ on the same point.

If all the infinities of $\phi(u)$ are of the first order and if n is the order of $\phi(u)$, it follows that $\phi'(u)$ is of the $2n$th order and consequently the

* Cf. Neumann, *Abel'schen Integrale*, p. 107.

degree of $f\left(z, \frac{dz}{du}\right)$ is at most $2n$ in z. This equation $f\left(z, \frac{dz}{du}\right)$ we have called the *eliminant equation.*

ART. 106. In Art. 104 we saw that any two doubly periodic functions that have the same periods are connected by an algebraic equation. It will therefore be sufficient, if we confine our attention to any doubly periodic function and express the others which have the same periods through this one. This function we shall take of the second order (cf. Art. 92) and consequently either $z = sn\, u$ or $z = \wp u$ (Art. 102).

Let z be a doubly periodic function of the second order ($n = 2$), so that the eliminant equation is

$$f\left(z, \frac{dz}{du}\right) = 0,$$

which is of the second degree in $\frac{dz}{du}$ and at most of the fourth degree in z.

The above equation must therefore have the form

$$\text{(I)} \qquad g_0(z)\left(\frac{dz}{du}\right)^2 + g_1(z)\frac{dz}{du} + g_2(z) = 0,$$

where the g's are integral functions of at most the fourth degree.

We saw above that z and $\frac{dz}{du}$ are infinite at the same points within the period-parallelogram and that $\frac{dz}{du}$ does not become infinite for values of u other than those which make z infinite.

But from (I) it is seen that

$$\frac{dz}{du} = \frac{-g_1(z) \pm \sqrt{g_1(z)^2 - 4g_0(z)g_2(z)}}{g_0(z)}$$

and becomes infinite for those values of z which make $g_0(z) = 0$. It follows that $g_0(z)$ must be a constant and consequently the equation (I) becomes

$$\text{(I}'\text{)} \qquad \left(\frac{dz}{du}\right)^2 + g_1(z)\frac{dz}{du} + g_2(z) = 0,$$

where the constant has been absorbed in the two functions $g_1(z)$ and $g_2(z)$.

ART. 107. If z is a doubly periodic function, then also $v = \frac{1}{z}$ is a doubly periodic function. Further, we have at once

$$\frac{dz}{du} = \frac{dz}{dv}\frac{dv}{du} = -\frac{1}{v^2}\frac{dv}{du}.$$

Making these substitutions in the above differential equation we have

$$\left(\frac{dv}{du}\right)^2\frac{1}{v^4} - g_1\left(\frac{1}{v}\right)\frac{dv}{du}\frac{1}{v^2} + g_2\left(\frac{1}{v}\right) = 0.$$

Since $g_1\left(\frac{1}{v}\right)$ and $g_2\left(\frac{1}{v}\right)$ are at most of the fourth degree in $\frac{1}{v}$, it follows

that $v^4 g_1\left(\frac{1}{v}\right)$ and $v^4 g_2\left(\frac{1}{v}\right)$ are integral functions of at most the fourth degree in v, which we denote respectively by $\bar{g}_1(v)$ and $\bar{g}_2(v)$.

The above differential equation is then

$$\left(\frac{dv}{du}\right)^2 - \frac{\bar{g}_1(v)}{v^2}\frac{dv}{du} + \bar{g}_2(v) = 0.$$

We saw above that $\frac{dz}{du}$ is finite for finite values of z; the same must also be true of $\frac{dv}{du}$ and v.

But in the differential equation just written $\frac{\bar{g}_1(v)}{v^2}$ becomes infinite for $v = 0$. It follows that $g_1(v)$ cannot be of the *fourth* but must be of the second degree in z at most.

It then follows from the equation (I′) that

$$\frac{dz}{du} = -\tfrac{1}{2}\, g_1(z) \pm \tfrac{1}{2}\sqrt{g_1(z)^2 - 4\, g_2(z)};$$

or, if we write $4\, R(z) = g_1(z)^2 - 4\, g_2(z)$,

$$\frac{dz}{du} = \tfrac{1}{2}\, g_1(z) \pm \sqrt{R(z)},$$

where $R(z)$ is an integral function of at most the fourth degree.

It follows that
$$u = \int^z \frac{dz}{-\tfrac{1}{2}\, g_1(z) \pm \sqrt{R(z)}}.$$

Our problem consists in the treatment of this integral when $R(z)$ is of the third or the fourth degree; when $R(z)$ is of the second or first degree the integral is an elementary one.

If we write
$$u = \int_0^z \frac{dz}{\sqrt{1 - z^2}},$$
we have $u = \sin^{-1} z$, where the inverse sine-function is many-valued.

We know, however, that the upper limit z considered as a function of the integral and written $z = \sin u$ is a one-valued simply periodic function of u. In the more general case above we wish to consider z as a function of u. This is the so-called "*problem of inversion.*" Possibly the clearest and simplest method of treating this problem is in connection with the Riemann surface upon which the associated integrals may be represented. Before proceeding to the problem of inversion we shall therefore consider this surface in the next Chapter.

EXAMPLE

1. If two doubly periodic functions $f(z)$ and $\phi(z)$ have only two poles *of the first order* in the period-parallelogram and if each pole of the one function coincides with a pole of the other, then is

$$\phi(z) = Cf(z) + C_1,$$

where C and C_1 are constants.

CHAPTER VI

THE RIEMANN SURFACE

ARTICLE 108. At the close of the preceding Chapter we were left with the discussion of an integral which contained a radical. Such an expression is *two-valued*, and we must now consider more closely the meaning of such functions and their associated integrals.

Take as simplest case the example

$$s = \pm \sqrt{z - a} = \pm (z - a)^{\frac{1}{2}},$$

where z is a complex variable and a an arbitrary constant. For the value $z = a$, we have $s = 0$; but for all other finite values of z there are two values of s that are equal and of opposite signs. The point a is called a *branch-point* of s. The point $z = \infty$ is also a branch-point of this function; for $\frac{1}{s} = \frac{1}{\pm \sqrt{z - a}} = 0$ for $z = \infty$. Consequently $\frac{1}{s}$ and likewise s has only *one* value for $z = \infty$.

There are other reasons why $z = a$ and $z = \infty$ are called branch-points. Corresponding to the value $z = z_0$, let $s = s_0$ be a definite value of s. Along the curve (1) from z_0 to z_1 consider the values of s at all the points of the curve which differ from one another by infinitesimally small quantities, and similarly consider the values of s along the curve (2) until we again come to z_1. The value of s at this point will be the same whether we have gone over the first or second curve, provided the branch-point a is not situated between the two curves.

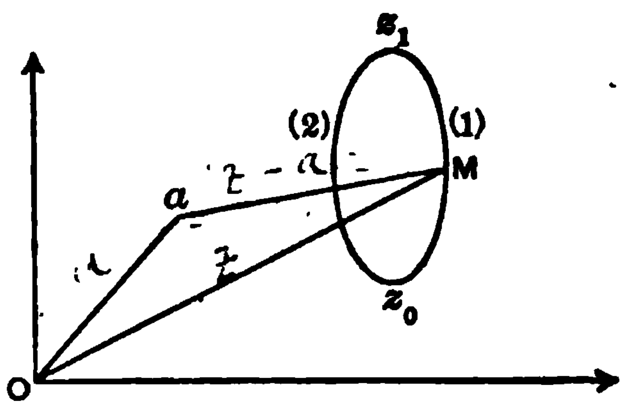


Fig. 27.

This may be shown geometrically as follows:

Let $\overline{OM} = z, \qquad \overline{Oa} = a,$

$\overline{aM} = z - a$, and $|z - a| = r$.

We therefore have

$$z - a = re^{i\phi},$$

where ϕ is the angle that $\overline{aM}$ makes with the real axis.

It follows that

$$s = r^{\frac{1}{2}} e^{\frac{i\phi}{2}} \quad \text{and} \quad s_0 = r_0^{\frac{1}{2}} e^{\frac{i\phi_0}{2}}.$$

If aM turns about a, and M starting from z_0 after making a circuit returns again to z_0, then if this circuit does not include a, the values of ϕ_0 and s_0

are the same as before the circuit and consequently s_0 has its initial value. But if the circuit includes a, the quantity r_0 is the same after the circuit, but ϕ_0 has become $\phi_0 + 2\pi$. The corresponding value of s_0 is

$$s_0' = r_0^{\frac{1}{2}} e^{\frac{i[\phi_0 + 2\pi]}{2}} = r_0^{\frac{1}{2}} e^{\frac{i\phi_0}{2}} e^{i\pi} = -s_0 .$$

We thus see that s_0 has taken the opposite sign after the circuit.

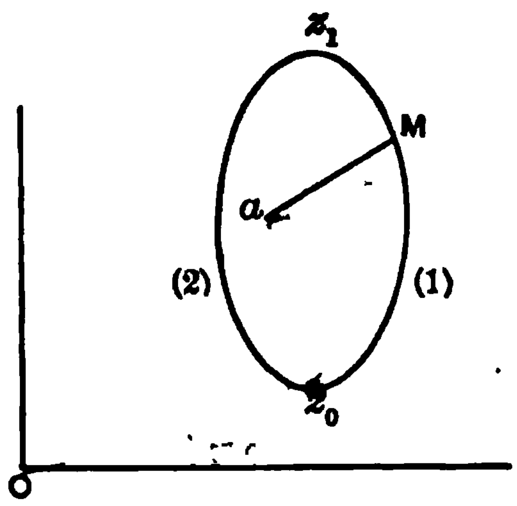


Fig. 28.

ART. 109. Consider next the expression

$$s^2 = R(z),$$

where $R(z)$ is an integral function of the fourth degree in z. We may write

$$R(z) = A(z - a_1)(z - a_2)(z - a_3)(z - a_4),$$

A being a constant.

We then have

$$s = \pm\sqrt{R(z)} = \pm A^{\frac{1}{2}}(z - a_1)^{\frac{1}{2}}(z - a_2)^{\frac{1}{2}}(z - a_3)^{\frac{1}{2}}(z - a_4)^{\frac{1}{2}}.$$

The function s has two values with opposite signs for any value of z except a_1, a_2, a_3, a_4. When z is equal to any of these values, s has the *one* value zero. The points a_1, a_2, a_3 and a_4 are *branch-points.* The value $|a_1 - z_0|$ is the radius of the circle about z_0 which goes through a_1. Suppose that z is any point situated within this circle so that

$$|z - z_0| < |a_1 - z_0|.$$

Then, since $z - a_1 = z - z_0 - (a_1 - z_0)$, we have

$$(z - a_1)^{\frac{1}{2}} = \sqrt{-(a_1 - z_0)}\left\{1 - \frac{z - z_0}{a_1 - z_0}\right\}^{\frac{1}{2}}.$$

Since $\left|\dfrac{z - z_0}{a_1 - z_0}\right| < 1$, the right-hand side may be expanded by the Binomial Theorem in the form

$$(z - a_1)^{\frac{1}{2}} = \sqrt{-(a_1 - z_0)}\left\{1 - \lambda_1\frac{z - z_0}{a_1 - z_0} + \lambda_2\left(\frac{z - z_0}{a_1 - z_0}\right)^2 - \cdots\right\}.$$

This series is uniformly convergent for all values of z within* the circle. In the same way we may develop $(z - a_2)^{\frac{1}{2}}$, $(z - a_3)^{\frac{1}{2}}$, $(z - a_4)^{\frac{1}{2}}$ in positive integral powers of $z - z_0$. All these series are convergent within circles about z_0.

We have the development of s in powers of $z - z_0$ by multiplying the

* When we say "within" we mean within any interval that lies wholly within. See Osgood, *loc. cit.*, p. 77 and p. 285.

four series together, the multiplication being possible, since the series of the moduli of the terms that constitute the four series are convergent. We thus derive the result: *We may develop $s = \sqrt{R(z)}$ in positive integral powers of $z - z_0$, if z_0 is different from the four branch-points a_1, a_2, a_3, a_4. The series is uniformly convergent within the circle about z_0 as center, which passes through the nearest of the points a_1, a_2, a_3, a_4.*

ART. 110. We may effect within this circle the same development by Taylor's Theorem in the form

$$\sqrt{R(z)} = s_0 + \frac{1}{2}\frac{R'(z_0)}{s_0}(z - z_0) + \cdots .$$

We must decide upon a definite sign of $s_0 = \sqrt{R(z_0)}$ and use this sign throughout the development. If at the beginning we decide upon the other sign, then in the series we must write $-s_0$ instead of s_0; that is, all the coefficients are given the opposite sign.

If the sign of s_0 has been chosen and if the development of s has been made, then s is defined through the above series only *within* the circle already fixed. If we consider a value of z *without* the circle of convergence, we do not know what value s will take at this point. To be more explicit we may proceed as follows:

Let z' be a point without the circle and join z' with z_0 through any path of finite length which must *not* pass indefinitely near a branch-point. Let the circle of convergence about z_0 cut this path at ξ. Then at all points of the portion of path $z_0\xi$ the corresponding values of the function are known through the series. Let z_1 be a point on this portion of path which lies sufficiently near to the periphery of the circle. We may express the value of the function at z_1, that is, $s_1 = \sqrt{R(z_1)}$ through the series

$$\begin{aligned} s_1 &= \sqrt{R(z_1)} \\ &= \left[s_0 + \frac{1}{2}\frac{R'(z_0)}{s_0}(z - z_0) + \cdots\right]_{z=z_1}. \end{aligned}$$

Fig. 29.

Thus s_1 is uniquely determined, if the sign of s_0 has been previously chosen.

We next take z_1 as the center of another circle C_1, which also must not contain a branch-point. Then precisely as we expanded s in powers of $z - z_0$ in the circle C_0 about z_0 we may now expand s within C_1 in powers of $z - z_1$ about z_1. This circle C_1 may extend up to the nearest branch-point and is not of an infinitesimally small area, since by hypothesis the path did not come indefinitely near a branch-point. The point z_1 is taken sufficiently near ξ that the circle about z_1 partly overlaps the

circle about z_0. That this may be the case z_1 must lie so close to ξ that the distance between the points is less than the radius of the circle C_1, a condition which evidently may always be satisfied. Hence the circles C_0 and C_1 have a portion of area in common. Let the power series which is convergent within C_0 be denoted by $P_0(z - z_0)$ while the one in C_1 may be represented by $P_1(z - z_1)$. As we have already seen in Chapter I the series P_1 gives for every value of z which is common to the two circles the same value as does the series P_0. But the development P_1 holds good for the entire circle C_1. We thus go in a continuous manner to values of the function which lie without the circle C_0. The series P_1 represents the *continuation* of the function s.

It is clear that this process may be repeated and that we will finally come to a circle C_m around a point z_m of the path as center within which the point z' lies. We may develop the function within C_m in positive integral powers of $z - z_m$ and may then compute $s' = \sqrt{R(z')}$ from this development. This process is called the "*Continuation of the function along a prescribed path from z_0 to z'.*" Such a continuation is possible in the entire z-plane, since z_0 may be connected by such a path with any other point z which is not a branch-point.

ART. 111. Let B and B_1 be two different paths which join z_0 and z' and suppose that neither of these points lies indefinitely near a branch-point. The question arises whether the value of the function at z' which is had through the continuation of the function along the path B_1 is the same as the one which is had through the continuation from z_0 to z' along B. It is clear that if the two values of $\sqrt{R(z')}$ thus obtained are different, they can differ only in sign.

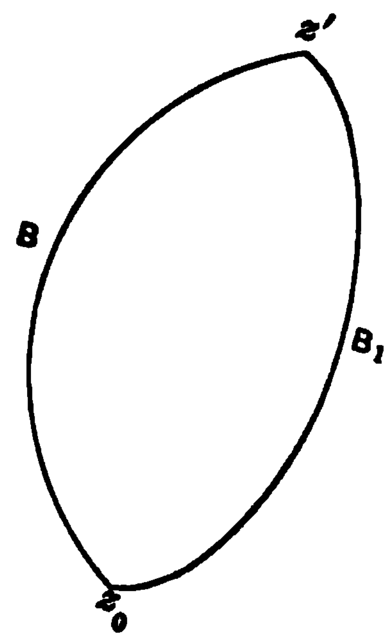


Fig. 30.

Through the circles which are necessary for the continuation of the function from z_0 to z' along B is formed a strip (see figure of preceding article) which has everywhere a finite breadth. This strip may be regarded as a "*one-value realm.*" The function s remains one-valued within this realm. First suppose that the path B_1 lies also wholly within this realm.

Since none of the circles contains a branch-point there cannot be one between B and B_1, and it is evident that we come through the continuation of the function along these curves to the point z' with the same value of the function. For let the normal at any point α_k on B cut the curve B_1 at α_k' where B_1 is taken very near to B, as shown in Fig. 31, and call α_k, α_k' a pair of *neighboring* points.

We suppose that the curves B and B_1 have been taken so near together that one of the circles employed in the continuation of the function along B contains both α_k and α_k' and that all points within this circle are ex-

pressed through $P_k(z - z_k)$; and at the same time we assume that one of the circles used in the continuation of the function along the path B_1 includes also the same points α_k, α_k' and that all points within this circle are had through the series $P_{k'}(z - z_{k'})$. Hence we must have the same value of s at the point $z = \alpha_k$ from either of the power series P_k or $P_{k'}$, provided this is true of every pair of neighboring points that preceded this pair. The same is also true of the point $z = \alpha_{k'}$. But the first pair of neighboring points was the point z_0. We therefore come to z' with the same value of s along either path B or B_1. Heffter [*Theorie der Linearen Differential-Gleichungen*, p. 72] has given a somewhat similar proof which suggested the one given here [see my *Calculus of Variations*, pp. 15, 16 and 256 *et seq.*].

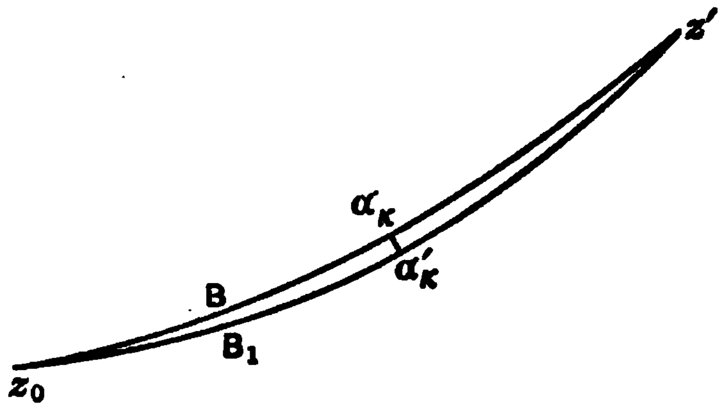


Fig. 31.

If next B and B_1 are two curves which are drawn in an arbitrary manner between z_0 and z', but which do not include a branch-point, then we may fill the surface between B and B_1 with a finite number of curves drawn from z_0 to z' which lie at a finite distance from one another and are so situated that each one lies within the one-valued realm which is formed by the circles that are necessary for the continuation of the function along a neighboring curve. Thus by means of the intermediary curves with their associated one-valued realms it is evident that we come to z' with the same value of s when we make the continuation along either of the two curves B or B_1 provided that there is no branch-point between them. It follows also that the value of the function at the point z' is independent of the form of the curve between z_0 and z'.

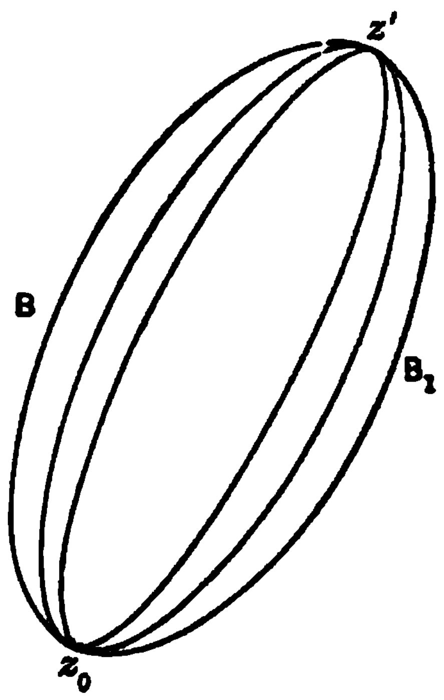


Fig. 32.

ART. 112. Let (1) and (2) be two paths between z_0 and z' which do not include a branch-point. If we go along (2) from z_0 to z' and then back again along (1) from z' to z_0, we come to the same initial value of the function. From this it follows: *If the function $s = \sqrt{R(z)}$ is continued from the point $z = z_0$ along a closed curve which does not contain a branch-point, we return after the circuit to the point z_0 with the same initial value of the function.*

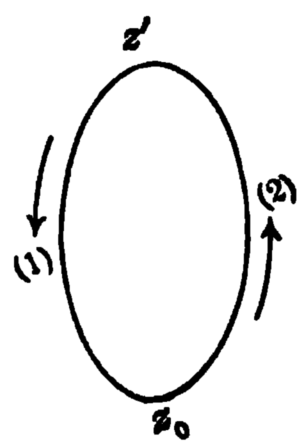


Fig. 33.

The form of the curve is arbitrary, provided only it does not inclose any branch-point. Hence instead of making a circuit around an arbitrary curve, we may choose a circle which passes through z_0.

ART. 113. Suppose next that *the closed curve includes a branch-point*, for example a_1. We again fix the sign of s_0 for $z = z_0$, and write

$$s = \sqrt{z - a_1}\sqrt{R_1(z)},$$

where

$$R_1(z) = A(z - a_2)(z - a_3)(z - a_4).$$

We may allow $\sqrt{R_1(z)}$ to have an arbitrary sign, and so choose the sign of $\sqrt{z - a_1}$ that $s = s_0$ will have the same sign for $z = z_0$ as has been previously assigned to it.

If we make a circuit about a_1, it is seen that $\sqrt{R_1(z)}$ is not affected by it, since a_1 is not a branch-point of $\sqrt{R_1(z)}$. Hence upon making a circuit about a_1 we need consider only the first factor $(z - a_1)^{\frac{1}{2}}$. We may make this circuit along a circle of radius r with a_1 as center. For the points of the periphery, it is clear that

$$| z - a_1 | = r,$$

so that

$$z - a_1 = re^{i\phi}.$$

It follows that

$$(z - a_1)^{\frac{1}{2}} = r^{\frac{1}{2}}e^{\frac{i\phi}{2}}.$$

Let the value of ϕ corresponding to $z = z_0$ be $\phi = \phi_0$, so that

$$(z_0 - a_1)^{\frac{1}{2}} = r^{\frac{1}{2}}e^{\frac{i\phi_0}{2}},$$

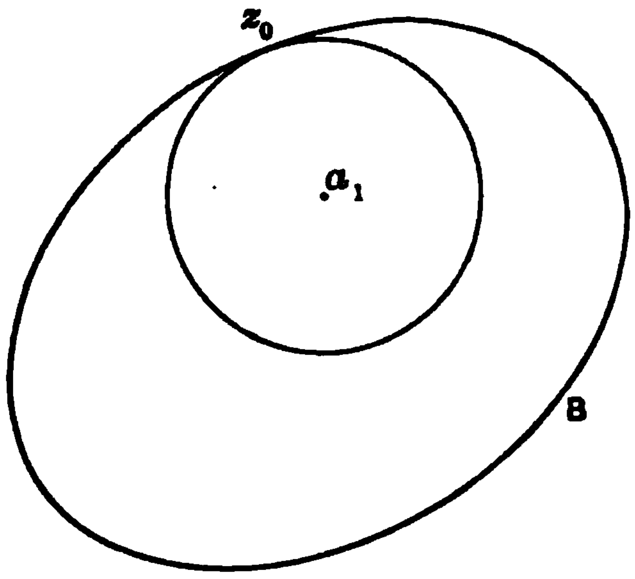


Fig. 34.

where the point z_0 of course lies upon the periphery of the circle. When a complete circuit is made about a_1, starting from z_0, it is seen that ϕ_0 is increased by 2π, and consequently after this circuit the above expression becomes

$$r^{\frac{1}{2}}e^{\frac{i(\phi_0+2\pi)}{2}} = r^{\frac{1}{2}}e^{\frac{i\phi_0}{2}}e^{i\pi} = -r^{\frac{1}{2}}e^{\frac{i\phi_0}{2}}.$$

It follows that after a circuit* about a_1 has been made, the quantity $(z - a_1)^{\frac{1}{2}}$ and consequently also $s = \sqrt{R(z)}$ changes its sign.

Further, if we make a circuit about a_1 along any arbitrary curve B which does not include any other branch-point except a_1, then s changes sign with this circuit; for this is the case when a circuit has been made about the circle around a_1, and as there is no branch-point between the circle and the path B, it follows that starting from z_0 we will again return to this point along both of the curves with the same value of the function.

ART. 114. We may next ask *what happens if the circuit includes two branch-points*. First suppose that the circuit is made along the path $z_0\alpha\beta\gamma z_0$. Let $\delta\epsilon\tau$ be a closed curve about a_1 and $\eta\theta\kappa$ a closed curve about a_2. It follows immediately from the above considerations that the

* Cf. Bobek, *Elliptische Functionen*, p. 150.

two curves between which there is no branch-point lead always to the same initial value of the function.

Hence instead of making the circuit about a_1 and a_2 along the path $z_0\alpha\beta\gamma z_0$ we may just as well make the circuit along the path $z_0\delta\epsilon\tau z_0\eta\theta\kappa z_0$, there being no branch-point between this curve and the curve $z_0\alpha\beta\gamma z_0$. After the circuit $z_0\delta\epsilon\tau z_0$ the function s changes sign as it again does after the circuit $z_0\eta\theta\kappa z_0$, so that after the two circuits around the points a_1 and a_2 we again come to the point z_0 with the initial value of s.

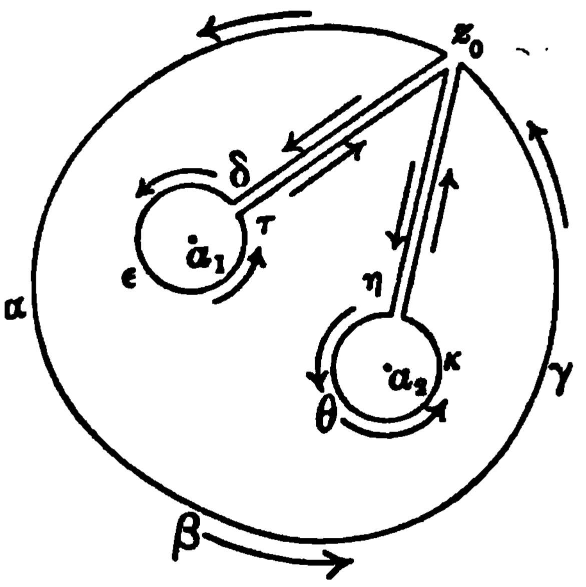


Fig. 35.

We conclude in the same way that if we make an arbitrary circuit around four branch-points we again come to the same value of the function, while if we have encircled three branch-points, we arrive at z_0 with the other value of s.

ART. 115. We may next see how the function

$$s = A^{\frac{1}{2}}\sqrt{(z-a_1)(z-a_2)\ .\ .\ .\ (z-a_n)}$$

behaves when a circuit is made around the point at infinity. When n is an even integer and when a circuit is made so as to include the n points $a_1, a_2, \ldots, a_n$, it follows from above that when z returns to its initial position, the value of s has not changed its sign. In the above expression write $z = \dfrac{1}{t}$, so that when $z = \infty$, we have $t = 0$. In the z-plane the point at infinity corresponds to the origin in the t-plane. We then have

$$s = t^{-\frac{n}{2}}A^{\frac{1}{2}}\sqrt{(1-a_1t)\,(1-a_2t)\ .\ .\ .\ (1-a_nt)}.$$

Now take a circuit about a circle with the origin as center and which does not contain one of the branch-points $a_1, a_2, \ldots, a_n$. We must therefore write

$$t = re^{i\phi},$$

and it is seen that the function s changes sign when n is an *odd* integer. In this case the origin in the t-plane is a *branch-point*, and consequently in the z-plane the point at infinity is or is not a branch-point according as n is an odd or even integer.

ART. 116. We shall draw lines connecting the points a_1 with a_2 and a_3 with a_4. The paths along which the function s is continued must never cross these lines $a_1\,a_2$ and $a_3\,a_4$. They may be called "*canals.*" The z-plane which contains these two canals may be denoted by the $\bar{z}$-plane, a dash being put over z (see Fig. 36).

If once the initial value s_0 of the function $s = \sqrt{R(z)}$ is fixed for the point z_0, then s is completely one-valued in the $\bar{z}$-plane; for in whatever manner the continuation from z_0 to z' may be made, any two different paths will always include an even number of branch-points or none, since the canals cannot be crossed. It follows that $s = \sqrt{R(z)}$ no longer depends upon the path along which this function is continued from one point to another and is consequently one-valued in the $\bar{z}$-plane. The two canals are sometimes called *branch-cuts.*

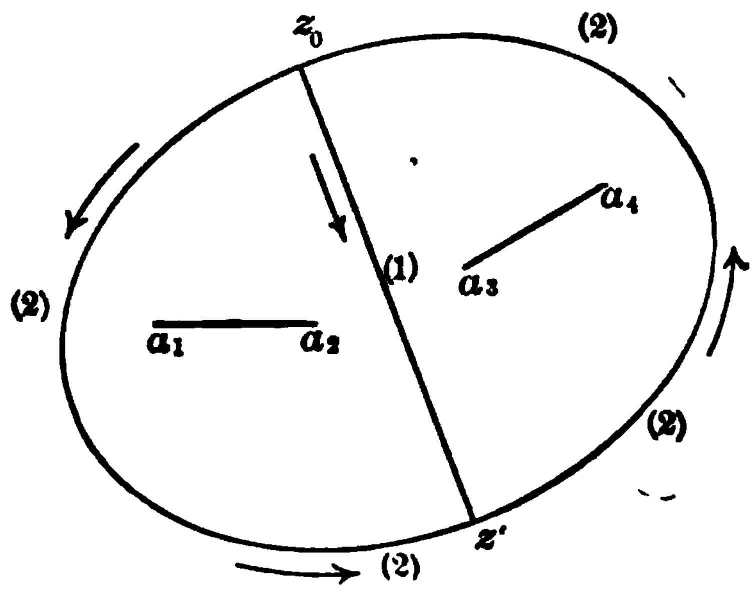


Fig. 36.

If further the sign has been ascribed to the initial value s_0 of the function s, then we may ascribe to s its proper value for every value in the $\bar{z}$-plane. These values we suppose have been written down on a *leaf,* which represents the $\bar{z}$-plane. Again starting with $-s_0$ for the initial point we consider the corresponding values of the function written down upon another plane or leaf. In this second leaf the two canals connecting a_1 with a_2 and a_3 with a_4 are also supposed to have been drawn, so that s is also one-valued on it.

We note that corresponding to the same value of z, the values of $s = \pm\sqrt{R(z)}$ in the two leaves are equal but of opposite sign. If, further, starting from a point α_1 on the upper bank of the canal we make a circuit

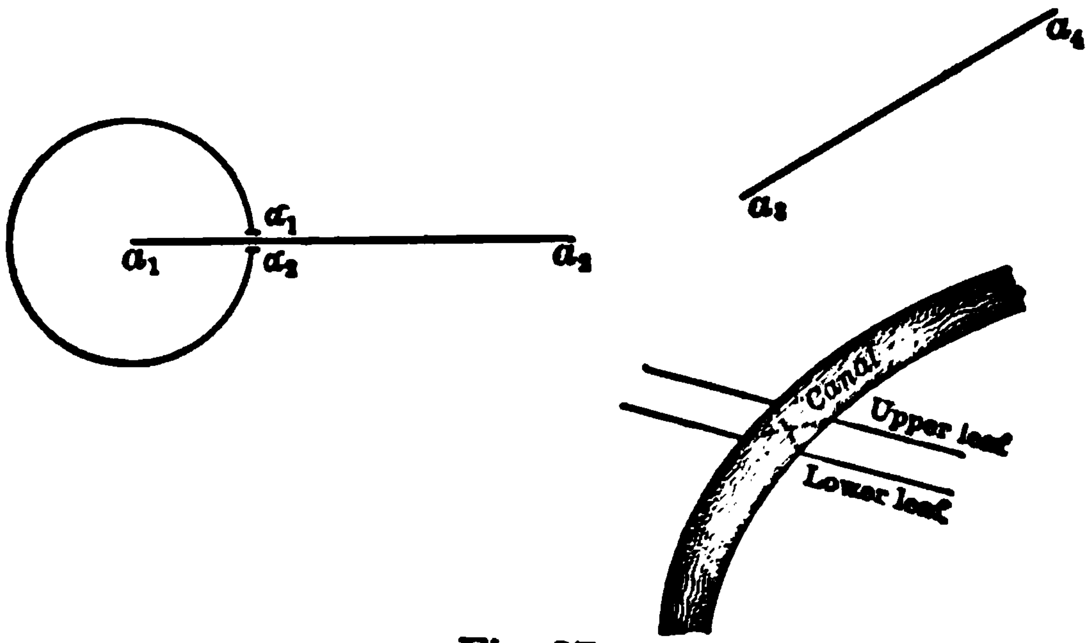


Fig. 37.

around a_1, say, and return to the point α_2 immediately opposite on the lower bank, the values of s at these two points are the same with contrary sign. The same is true for all points* opposite one another along the two canals $a_1\,a_2$ and $a_3\,a_4$.

We imagine the two leaves placed the one directly over the other, with the canals in the one leaf over those in the second leaf. The left

* Cf. Neumann, *Abel'schen Integrale,* p. 81.

bank of each canal in the upper leaf is joined with the right bank in the lower leaf and the right bank in the upper leaf with the left bank in the lower. If being in the upper leaf we cross a canal we will find ourselves in the lower leaf; and if being in the lower leaf we cross a canal we will come up in the upper leaf. Thus the values of the function s change in a continuous manner when by crossing the canals we go from one leaf into the other; and in this manner we are able to make the two-valued function s behave like a one-valued function by means of the above structure. In this structure there is no crossing from one leaf to the other except in the manner indicated by means of the canals.

The structure is called the *Riemann surface* of the function* $s = \sqrt{R(z)}$ (*cf. Grundlagen für eine allgemeine Theorie der Funktionen einer komplexen veränderlichen Grosse. Inauguraldissertation von B. Riemann. Crelle*, Bd. 54, pp. 101 *et seq.*).

If the function is continued anywhere in this Riemann surface, the function has always at any definite point a definite value, which is independent of the path along which the function has been continued. It is thus shown that *the function s is a one-valued function of position in the Riemann surface.* In this surface, if for a definite value of z the corresponding value of s is to be found, we must also indicate whether the value of z is taken in the upper or in the lower leaf.

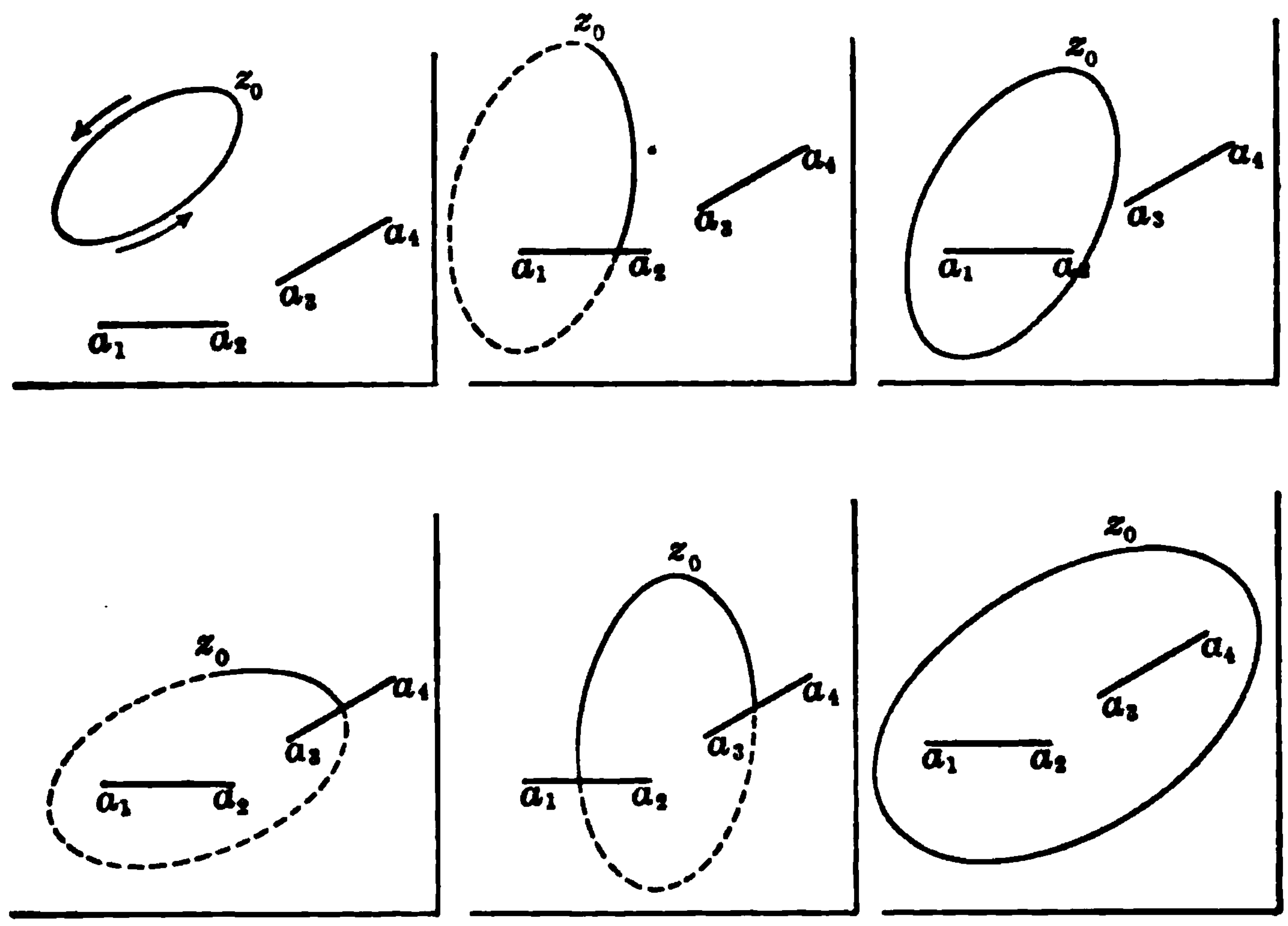


Fig. 38.

In the figures a path that is taken in the lower leaf is denoted by a broken line (----------), while a path in the upper leaf is indicated by an uninter-

* See also Neumann, *Theorie der Abel'schen Integrale*; Durège, *Elemente der Theorie der Funktionen*. For other references see Wirtinger, *Ency. der math. Wiss.*, Bd. II², Heft 1.

rupted line (—————). The fact that the function s, when a circuit is taken around *no* branch-point, or around *two* branch-points, or around *four* branch-points, retains its sign, while it changes sign if the path is around *one* or *three* such points, is brought into evidence by means of the Riemann surface. It is indicated in the figures on page 136.

We note that by a circuit around *one* or *three* branch-points we always pass from one leaf into the other, and that at two points situated the one over the other the function s has the same absolute value but different signs.

The One-Valued Functions of Position on the Riemann Surface.

ART. 117. We have defined a function as being one-valued on the Riemann surface. We may now consider more closely what is meant by such a function. When we say that a function is "one-valued on the Riemann surface," we mean something quite different from what is meant by saying a "function is one-valued." The signification of the first definition is: "If the value of the variable z is given and also the position on the Riemann surface, then the function is uniquely determined"; if, however, only z were given, the function would *not* be uniquely determined.

Let w be any function whatever of z which we suppose is one-valued on our fixed Riemann surface. In the upper leaf of this surface the function w has for a given z a definite value, say w_1, and in the lower leaf it takes another value, say w_2, for the same value of z. In the special case above, where $w = s = \pm\sqrt{R(z)}$, we have $w_1 = -w_2$. In general, however, this is not the case. But if we consider the sum $w_1 + w_2$, this sum is a one-valued function of z, for if z is given, $w_1 + w_2$ is completely determined. The same is also true of the product $w_1 \cdot w_2$.

It follows that w satisfies a quadratic equation of the form

$$w^2 - \phi(z)w + \psi(z) = 0,$$

where $\phi(z)$ and $\psi(z)$ are one-valued functions of z, such that

$$w_1 + w_2 = \phi(z) \quad \text{and} \quad w_1 \cdot w_2 = \psi(z).$$

Hence *every one-valued function of position on the Riemann surface* $s = \sqrt{R(z)}$ *is a two-valued function of z and satisfies a quadratic equation, whose coefficients are one-valued functions of z.*

In particular, we shall study those one-valued functions of position on the Riemann surface which have a *definite value* at every position on the Riemann surface. In this case $\phi(z) = w_1 + w_2$ will have a definite value for every value of z, as will also $\psi(z) = w_1 \cdot w_2$. But one-valued functions which have everywhere definite values (when therefore there is no essential singularity) are rational functions. If then w is to be a one-valued

function of position on the fixed Riemann surface and is to have everywhere on this surface a definite value, then $\phi(z)$ and $\psi(z)$ must be rational functions of z.

ART. 118. When we solve the above quadratic equation, we have

$$w = \frac{\phi(z)}{2} + \frac{1}{2}\sqrt{-4\psi(z) + \phi^2(z)},$$

where the root is to be taken positive or negative. We have thus shown that w is equal to a rational function of z, increased or diminished by the square root of a rational function.

Suppose that the radicand $-4\psi(z) + \phi^2(z) = S(z)$, say, becomes zero or infinite of the $(2n+1)$st order for $z = b$, where n is an integer.

We note that the point b cannot be a branch-point on the Riemann surface, for a_1, a_2, a_3, a_4 are the only branch-points on this surface.

We may write
$$S(z) = (z-b)^{2n+1}S_1(z),$$
where $S_1(z)$ is a rational function of z.

About b as a center describe a circle which does not inclose any other zero or infinity of $S(z)$.

We then have
$$\sqrt{S(z)} = (z-b)^{\frac{2n+1}{2}}\sqrt{S_1(z)};$$
and if z makes a circuit about the circle, the function $\sqrt{S_1(z)}$ retains its sign, while $(z-b)^{\frac{2n+1}{2}}$ changes sign. Consequently the function $\sqrt{S(z)}$ changes its sign with this circuit, so that $w = \frac{\phi(z)}{2} + \frac{\sqrt{S(z)}}{2}$ does not resume its initial value and is therefore *not* a one-valued function of position on the Riemann surface. Hence the factor $z - b$ must occur to an even power if it enters as a factor of either the numerator or the denominator of the rational function $S(z)$, so that $S(z)$ must have the form

$$S(z) = \overline{S_1(z)}^2 \left\{(z-a_1)(z-a_2)(z-a_3)(z-a_4)\right\}.$$

We may therefore write

$$\begin{aligned} w &= \tfrac{1}{2}\phi(z) + \tfrac{1}{2}S_1(z)\sqrt{(z-a_1)(z-a_2)(z-a_3)(z-a_4)} \\ &= p(z) + q(z)\sqrt{R(z)} = p + q \cdot s, \end{aligned}$$

where $p = p(z) = \frac{\phi(z)}{2}$, $q = q(z) = \frac{S_1(z)}{2}$ are rational functions of z.

It has thus been shown* that "*Every one-valued function of position, which has everywhere a definite value in our Riemann surface, is of the form*

$$w = p + qs,$$

where p and q are rational functions of z."

* Cf. Neumann, *loc. cit.*, p. 355.

Reciprocally, every function of the form $w = p + qs$ is a one-valued function of position on the Riemann surface, since p, q, s taken separately have this property. If then w has this form, it is the necessary and sufficient condition that w be a one-valued function of position on the Riemann surface.

The Zeros of the One-valued Functions of Position.

Art. 119. Let $z = \alpha$ be a position on the Riemann surface, which is different from the branch-points a_1, a_2, a_3, a_4. We may then draw a circle around α which lies entirely in one leaf of the Riemann surface.

It may happen that $w = 0$ for $z = \alpha$, while at the same time p and q are infinite for $z = \alpha$. For suppose that

$$p = \frac{e_\lambda}{(z-\alpha)^\lambda} + \frac{e_{\lambda-1}}{(z-\alpha)^{\lambda-1}} + \cdots + \frac{e_1}{z-\alpha} + c_0 + c_1(z-\alpha) + \cdots,$$

$$q = \frac{f_\mu}{(z-\alpha)^\mu} + \frac{f_{\mu-1}}{(z-\alpha)^{\mu-1}} + \cdots + \frac{f_1}{z-\alpha} + g_0 + g_1(z-\alpha) + \cdots.$$

We may also develop s for points within the circle in the form

$$s = h_0 + h_1(z-\alpha) + h_2(z-\alpha)^2 + \cdots.$$

It is evident that s is not infinite for $z = \alpha$, and it is also clear that if $\lambda \neq \mu$, then w becomes infinite for $z = \alpha$; but if $\lambda = \mu$, then we may so choose the coefficients in the development of p and q that $w = 0$ for $z = \alpha$. This will be the case if in the development of w all the negative powers and also the constant term drop out. The coefficient of $(z-\alpha)^{-\lambda}$ in this development is $e_\lambda + h_0 f_\mu$, or, since $\lambda = \mu$, we must have

$$e_\lambda + h_0 f_\lambda = 0.$$

Further, it is necessary that the coefficient of $(z-\alpha)^{-(\lambda-1)}$ be zero, that is,

$$e_{\lambda-1} + f_{\lambda-1}h_0 + f_\lambda h_1 = 0, \quad \text{etc.}$$

These conditions may all be satisfied; and consequently

$$w = k_r(z-\alpha)^r + k_{r+1}(z-\alpha)^{r+1} + \cdots,$$

where the k's are constant and where r is a positive integer greater than 0.

Finally we may write

$$w = (z-\alpha)^r[k_r + k_{r+1}(z-\alpha) + \cdots].$$

We see that w becomes zero of the rth order for $z = \alpha$. We thus experience no trouble in determining the order of zero for w at any point α, even if at this point the functions p and q become infinite. Similarly if p and q remain finite for $z = \alpha$ there is no difficulty.

ART. 120. We shall next study w in the neighborhood of one of the branch-points, a_1 say. If z makes a circuit about a_1, we return with a value of w that lies in the other leaf, and in order to reach the initial point of the circuit we must make a double circle about a_1, since by the second circuit we again come into the leaf in which the initial point is situated.

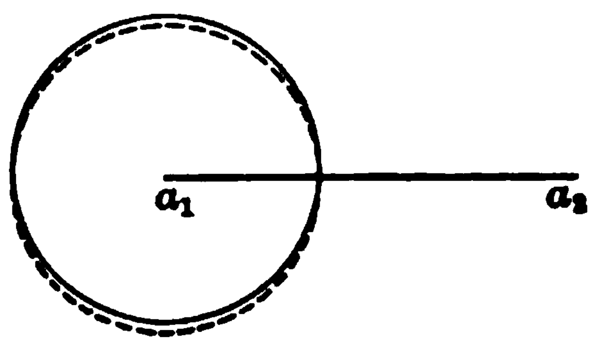


Fig. 39.

As in Art. 113, we write

$$s = \sqrt{R(z)} = (z - a_1)^{\frac{1}{2}} \sqrt{R_1(z)}.$$

Since a_1 is not a branch-point of $\sqrt{R_1(z)}$, we may expand this function in positive integral powers of $z - a_1$ and have

$$\sqrt{R(z)} = (z - a_1)^{\frac{1}{2}} [b_0 + b_1(z - a_1) + b_2(z - a_1)^2 + \cdots].$$

We put

$$(z - a_1)^{\frac{1}{2}} = t \quad \text{or} \quad t^2 = z - a_1.$$

Let a circuit be made about a_1 along a circle with radius r, so that

$$z - a_1 = t^2 = re^{i\phi},$$

or

$$t = \sqrt{r} e^{\frac{i\phi}{2}}.$$

If then z describes a circle with radius r around a_1 in the z-plane, then t describes a circle with radius $\sqrt{r}$ around the origin in the t-plane. If the circuit of z begins with the initial value $\phi = 0$, then the circuit of t begins with the value $\phi = 0$, and when ϕ increases by 2π we have $\phi/2$ increased by π. Hence to the whole circle in the z-plane there corresponds the half-circle in the t-plane, and to the double circle which z describes in the Riemann surface in order to return again to its initial point, there corresponds the simple circle in the t-plane.

Suppose that w vanishes at a branch-point, a_1 say. Further suppose by the substitution $z - a_1 = t^2$ that $p(z)$ becomes $\overline{p}(t)$ and $q(z)$ becomes $\overline{q}(t)$.

In the neighborhood of the point $t = 0$, let

$$\overline{p}(t) = \alpha_m t^m + \alpha_{m+1} t^{m+1} + \alpha_{m+2} t^{m+2} + \cdots,$$

and

$$\overline{q}(t) = \beta_n t^n + \beta_{n+1} t^{n+1} + \beta_{n+2} t^{n+2} + \cdots,$$

where m and n are integers (positive or negative including zero).

If m and n take negative or zero values, there must exist equations of condition as in the preceding article.

Since $z - a_1 = t^2$, it follows that $z - a_2 = a_1 - a_2 + t^2$, $z - a_3 = a_1 - a_3 + t^2$, $z - a_4 = a_1 - a_4 + t^2$, and consequently $R_1(z)$ becomes $V(t)$, where $V(t) = (a_1 - a_2 + t^2)(a_1 - a_3 + t^2)(a_1 - a_4 + t^2)$.

We note that this function does not vanish for $t = 0$, so that there is no branch-point of this function within the circle $t = 0$, if this circle is taken

sufficiently small. We may consequently expand $\sqrt{V(t)}$ within this circle in positive integral powers of t^2 and have

$$\sqrt{R(z)} = t[b_0 + b_1t^2 + b_2t^4 + \cdots].$$

It is further seen that if w becomes zero at the point $z - a_1 = t^2$, it may be developed in the neighborhood of $t = 0$ in positive integral powers of t in the form

$$\begin{aligned} w &= c_0t^\lambda + c_1t^{\lambda+1} + c_2t^{\lambda+2} + \cdots \\ &= c_0(z-a_1)^{\frac{\lambda}{2}} + c_1(z-a_1)^{\frac{\lambda+1}{2}} + c_2(z-a_1)^{\frac{\lambda+2}{2}} + \cdots. \end{aligned}$$

It follows also that the function w becomes zero of the λth order at the branch-point $z = a_1$. In other words, *if w becomes zero at a branch-point $z = a_k$, then* TWICE *the exponent of the lowest power of $z - a_k$ in the development of w in ascending powers of this quantity, is the* ORDER *of the zero on this position. If, however, the zero-position $z = \alpha$, say, is* NOT *a branch-point, we have the development*

$$w = c_0'(z-\alpha)^{\lambda'} + c_1'(z-\alpha)^{\lambda'+1} + \cdots,$$

and here the exponent of the LOWEST *power of $z - \alpha$ in the development in ascending powers of this quantity is the order of the zero of the function at $z = \alpha$.*

This difference respecting the order of the zeros seems at first *arbitrary*, but the significance is evidenced through the following consideration: Let α be a zero which does not coincide with one of the branch-points. We may then develop w in the form

$$w = (z-\alpha)^{\lambda'}[c_0' + c_1'(z-\alpha) + c_2'(z-\alpha)^2 + \cdots],$$

and consequently

$$\log w = \lambda' \log(z-\alpha) + \log[c_0' + c_1'(z-\alpha) + c_2'(z-\alpha)^2 + \cdots].$$

Since the expansion within the bracket does not become zero for $z = \alpha$, its logarithm is not negative infinity and the expression may be developed in integral powers of $z - \alpha$. We then have

$$\log w = \lambda' \log(z-\alpha) + e_1' + e_2'(z-\alpha) + \cdots.$$

If z makes a complete circuit about α, the power series $e_1' + e_2'(z-\alpha) + \ldots$ does not change sign; $\log(z-\alpha)$ is, however, increased by $2\pi i$ and consequently $\lambda' \log(z-\alpha)$ is increased by $2\pi i\lambda'$.

It follows that

$$\frac{1}{2\pi i}\log w$$

is increased by λ' when a circuit is made about the zero $z = \alpha$: in other words, *the order of the zero of the function w at the point $z = \alpha$ is the*

number due to the change in $\frac{1}{2\pi i}\log w$ *when* z *makes an entire circuit about* a.

This same analytic property must be retained if a is also a branch-point, say a_1.

From the development above

$$w = (z-a_1)^{\frac{\lambda}{2}}[c_0 + c_1(z-a_1)^{\frac{1}{2}} + c_2(z-a_1)^{\frac{2}{2}} + \cdots]$$

it follows that

$$\log w = \frac{\lambda}{2}\log(z-a_1) + \log[c_0 + c_1(z-a_1)^{\frac{1}{2}} + c_2(z-a_1)^{\frac{2}{2}} + \cdots],$$

or

$$\log w = \frac{\lambda}{2}\log(z-a_1) + e_0 + e_1(z-a_1)^{\frac{1}{2}} + \cdots .$$

Now to make a complete circuit around a_1 we must make a double circle. By this circuit $(z-a_1)^{\frac{1}{2}}$ does not change sign. It follows that the change experienced in $\frac{1}{2\pi i}\log w$ is λ since $\log(z-a_1)$ changes by $2\cdot 2\pi i$. But here λ is twice the exponent of the lowest power of $z-a_1$ in the above expansion of w.

The infinities of w may be treated in precisely the same way as its zeros.

Integration.

. We shall next consider the integrals taken over certain e Riemann surface. These are formed in the same manner as ·grals of functions of the complex variable in the plane.

'$(z) = p + qs$ is a function which for all points of the path of integration takes finite and continuous values, and if a definite path of integration is prescribed which is taken from the point z_0, where $\sqrt{R(z)}$ takes the value s_0, to the point z', where $\sqrt{R(z)}$ takes the value s', then the integral $\int f(z)dz$ taken over this path has a definite value. If a portion

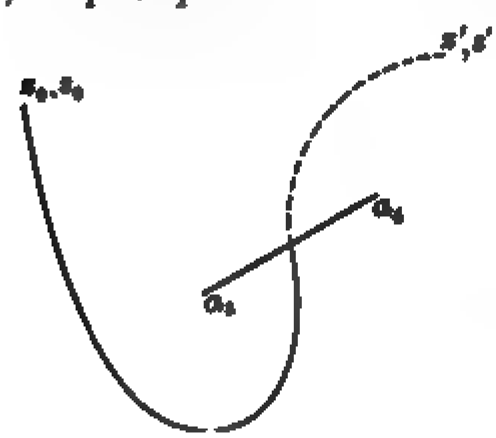


Fig. 40.

of integration lies in the lower leaf, the significance is that the .der the integral sign takes values in the lower leaf which form a connection with the values in the upper leaf.

An integral is called *closed* when the path of integration reverts to the initial point in the same sheet from which it started, as illustrated in the following figures:

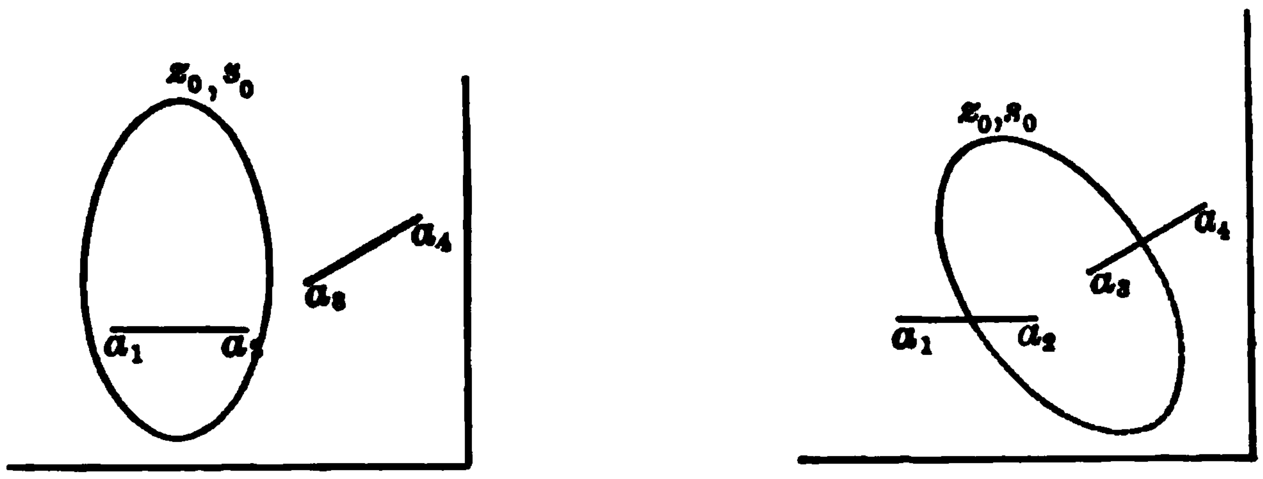


Fig. 41.

Cauchy's Theorem for the plane is also true of the Riemann surface, viz.: *If a function $f(z)$ within a portion of surface that is completely bounded, the boundaries included, is everywhere one-valued, finite and continuous, then the integral taken over the boundaries of the surface in such a way that it has the bounded surface always to the left, is zero.*

ART. 122. We must consider more closely what is meant by the *boundaries* of *a portion of surface.* The simplest case is a portion of surface as shown in the figure. We must make a distinction between an *outer* edge and an *inner* edge. If we have a point a on the inner edge and a point b on the outer edge, it is clear that we cannot go from the point a to the point b without crossing the boundary. We say in general that a portion of surface is *completely bounded* when it is impossible to go from a point on the inner edge to a point on the outer edge without crossing the boundary.

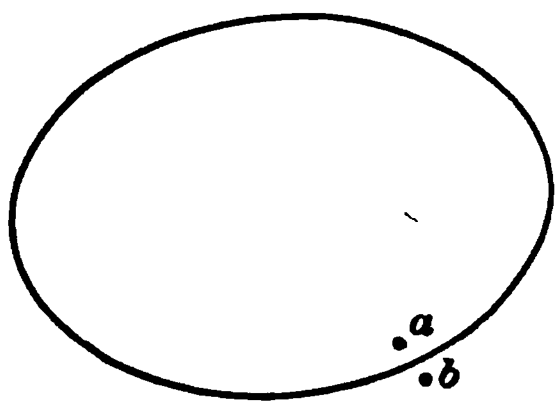


Fig. 42.

Consider next * a closed curve $\alpha\beta\gamma$ in the Riemann surface. We may go from a point a on the outer edge to a point b on the inner edge without crossing the curve $\alpha\beta\gamma$, which lies wholly in the upper leaf. Consequently the curve $\alpha\beta\gamma$ must *not* be regarded as the complete boundary of a portion of surface. But if we also draw a *congruent* curve $\alpha'\beta'\gamma'$, that is, one immediately under the first curve and in the lower leaf as shown in Fig. 44, then it is not possible to go from the point a to the point b without

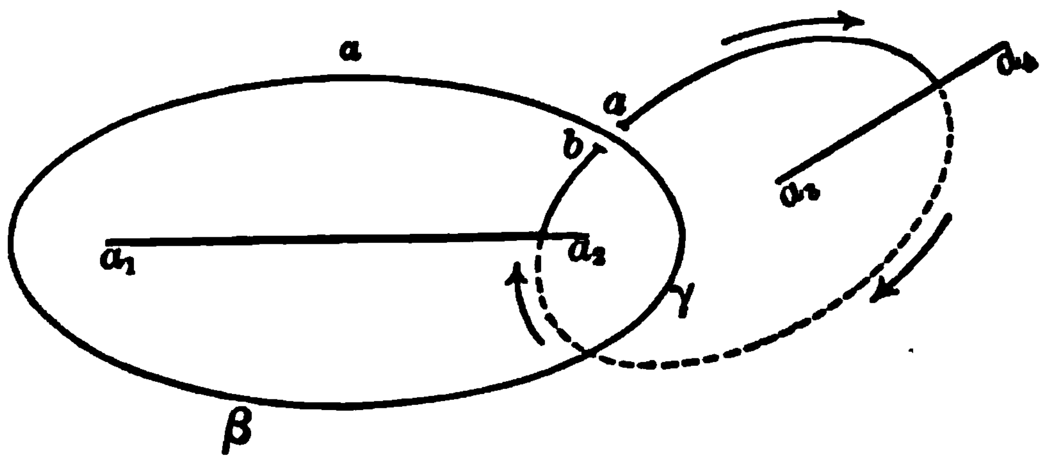


Fig. 43.

* Cf. Bobek, *loc. cit.*, p. 155.

crossing one or the other of the two curves $\alpha\beta\gamma$ or $\alpha'\beta'\gamma'$. Hence $\alpha\beta\gamma$ and $\alpha'\beta'\gamma'$ together form the complete boundary of this portion of surface of the Riemann surface. By Cauchy's Theorem the integral taken over $f(z)$, where the path of integration extends over both $\alpha\beta\gamma$ and $\alpha'\beta'\gamma'$, must be zero if the direction of integration is taken as indicated above and if $f(z)$ is one-valued, finite and continuous within and on the boundaries of this surface.

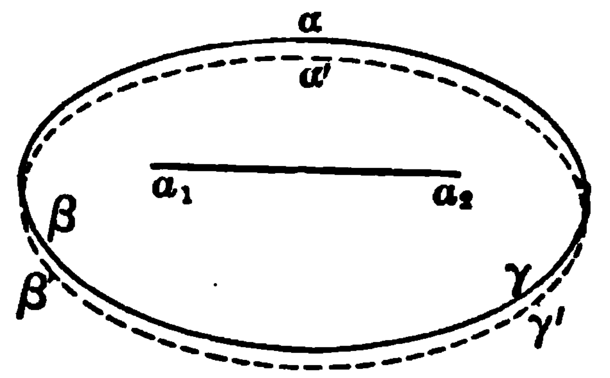


Fig. 44.

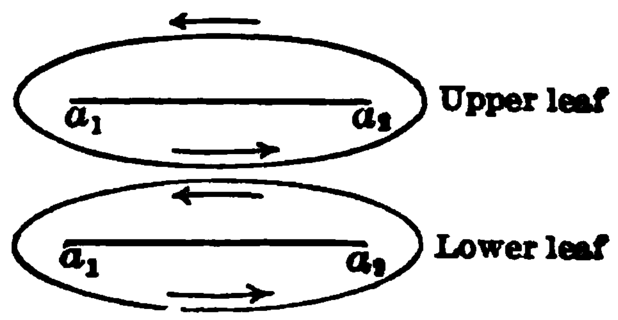


Fig. 45.

To prove this we note that instead of taking $\alpha\beta\gamma$ and $\alpha'\beta'\gamma'$ as the paths of integration we may take paths which lie indefinitely near the branch-cut a_1a_2, this one branch-cut, of course, lying in both the upper and the lower leaf. It is seen that, if the integration is taken in both the upper and the lower leaf (see Fig. 45),

$$\int wdz = \int [p + qs]dz = \int_{a_2}^{a_1} 2\, qsdz - \int_{a_2}^{a_1} 2\, qsdz = 0,$$

the elements of integration taken in the opposite directions over $\int pdz$ canceling one another.

ART. 123. If a one-valued analytic function be developed in the z-plane in the form

$$f(z) = \frac{b_\lambda}{(z-a)^\lambda} + \frac{b_{\lambda-1}}{(z-a)^{\lambda-1}} + \cdots + \frac{b_1}{z-a} + P(z-a),$$

where $P(z-a)$ denotes a power series in positive integral powers of $z-a$, then we know that the *residue* of $f(z)$ with respect to $z = a$ is

$$b_1 = \operatorname*{Res}_{z=a} f(z),$$

the quantity b_1 being the coefficient of $\dfrac{1}{z-a}$.

The same definition is given for the residue of a function of position on the Riemann surface, provided the point a does not coincide with a branch-point.

If, however, this point is a branch-point, a_1 say, and if the function becomes infinite at this point, then it follows from above that the development of $w = f(z)$ in the neighborhood of this point is

$$f(z) = \frac{b_\lambda}{(z-a_1)^{\lambda/2}} + \frac{b_{\lambda-1}}{(z-a_1)^{(\lambda-1)/2}} + \cdots + \frac{b_2}{(z-a_1)^{\frac{2}{2}}} + \frac{b_1}{(z-a_1)^{\frac{1}{2}}} + P\{(z-a_1)^{\frac{1}{2}}\}.$$

Before we define the residue here, we may consider a theorem which gives the residue in the form of an integral: If in the z-plane we draw a circle

about the infinity a of the function $f(z)$ and if $f(z)$ does not become infinite on any other point within or on the circumference of this circle, then is

$$\frac{1}{2\pi i}\int f(z)dz = \operatorname*{Res}_{z=a} f(z),$$

where the integration is taken over the circumference of the circle. We shall also retain this formula as the definition of a residue on the Riemann surface when the point a coincides with a branch-point, say a_1.

The integration is to be taken over a complete circuit about the branch-point, that is, over a double circle.

We may write under the sign of integration instead of $f(z)$ the power series by which it is represented. The general term is

$$\int (z - a_1)^{\frac{k}{2}} dz,$$

where the integration is over the double circle.

Suppose that r is the radius of the double circle, so that

$$z - a_1 = re^{i\phi},$$

and consequently also

$$\int_{\text{d'ble-circle}} (z - a_1)^{\frac{k}{2}} dz = \int_0^{4\pi} r^{\frac{k}{2}} e^{\frac{ki\phi}{2}} rie^{i\phi} d\phi = ir^{\left(1+\frac{k}{2}\right)} \int_0^{4\pi} e^{\left(1+\frac{k}{2}\right)i\phi} d\phi.$$

This integral is always zero, except when $1 + \frac{k}{2} = 0$. In this latter case

$$\int_{\text{d'ble-circle}} (z - a_1)^{\frac{k}{2}} dz = i\int_0^{4\pi} d\phi = 4\pi i.$$

It follows that

$$\operatorname*{Res}_{z=a_1} f(z) = \frac{1}{2\pi i}\int_{\text{d'ble-circle}} f(z)dz = \frac{1}{2\pi i} b_2 \cdot 4\pi i,$$

where b_2 is the coefficient of $(z - a_1)^{-1}$, since $k = -2$. We thus have finally

$$\operatorname*{Res}_{z=a_1} f(z) = 2b_2;$$

or, *the residue with respect to a branch-point is equal to* DOUBLE *the coefficient of* $(z - a_1)^{-1}$ *in the development of the function in powers of* $(z - a_1)^{\frac{1}{2}}$.

ART. 124. Suppose that a portion of surface is given which is completely bounded by certain curves. At isolated points of this surface suppose that the function becomes infinite. We draw around these points small circles, *simple* if they are not branch-points, and *double* when they are branch-points. The interior of these circles we no longer count as belonging to the surface. In this manner we derive a new portion of surface which is completely bounded on the one hand by the original curves and on the other by the small circles. The integral taken over the boundaries of this new portion of surface is zero, since the function is everywhere finite

within this surface, boundaries included. The integration is to be so taken that the interior of the portion of surface is always to the left. If the direction of integration taken over the small circles is changed so that the interiors of these circles lie to the left of the integration, then the signs of the corresponding integrals must be changed, and we have the following theorem: *The integral over the complete boundaries of the original portion of surface is equal to the sum of the integrals over the circles (or double-circles) which are drawn around the infinities (poles).*

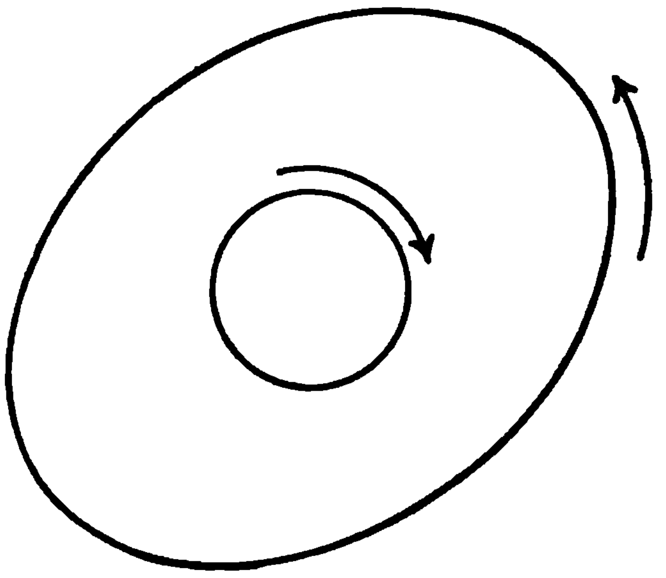


Fig. 46.

But on the other hand each of the integrals around one of the circles is equal to the residue of the function with respect to the infinities in question multiplied by $2\pi i$. We have therefore the theorem: *If a function within a completely bounded portion of surface, boundaries included, is everywhere one-valued and discontinuous only at isolated points, then the integral multiplied by $1/2\pi i$ and taken over the complete boundary of this surface is equal to the sum of the residues of the function with respect to all the points of discontinuity within the portion of surface.*

ART. 125. We saw that any one-valued function of position on the Riemann surface $s = \sqrt{R(z)}$ was of the form *

$$w = p + qs,$$

where p and q are rational functions of z and where $s = \sqrt{R(z)}$.

It follows that

$$\frac{1}{w}\frac{dw}{dz} = \frac{p' + q's + q\dfrac{R'(z)}{2s}}{p + qs} = \frac{p' + \left[q + \frac{1}{2}q\dfrac{R'(z)}{R(z)}\right]s}{p + qs}.$$

If the numerator and the denominator of the right-hand side of this expression are multiplied by $p - qs$, we have

$$\frac{1}{w}\frac{dw}{dz} = P + Qs,$$

where P and Q are rational functions of z.

It is thus seen that the logarithmic derivative of $w = p + qs$ is a rational function of z and s and indeed of the same form as is w itself.

The logarithmic derivative becomes infinite at the points where w vanishes and at the points where w becomes infinite.

* See Riemann, Werke, p. 111.

If μ is the order of the zero of the function w at the point α, then in the neighborhood of α

$$\frac{d\log w}{dz} = \frac{\mu}{z-\alpha} + P(z-\alpha) \qquad \text{[see Art. 4];}$$

and if λ is the order of infinity of the function w at the point β, then in the neighborhood of β

$$\frac{d\log w}{dz} = -\frac{\lambda}{z-\beta} + P(z-\beta).$$

It follows that

$$\operatorname*{Res}_{z=\alpha} \frac{d\log w}{dz} = \mu$$

and

$$\operatorname*{Res}_{z=\beta} \frac{d\log w}{dz} = -\lambda.$$

The above discussion is true when α and β are *not* branch-points.

If α is a branch-point, say a_1, and if w becomes zero at this point, then in the neighborhood of this point we have

$$w = (z-a_1)^{\frac{\mu}{2}}[g_0 + g_1(z-a_1)^{\frac{1}{2}} + g_2(z-a_1)^{\frac{2}{2}} + \cdots],$$

and consequently

$$\log w = \frac{\mu}{2}\log(z-a_1) + \log[g_0 + g_1(z-a_1)^{\frac{1}{2}} + \cdots].$$

It follows that

$$\frac{d\log w}{dz} = \frac{\frac{\mu}{2}}{z-a_1} + \frac{d}{dz}\log[g_0 + g_1(z-a_1)^{\frac{1}{2}} + \cdots].$$

Since the logarithmic expression does not become infinite for $z = a_1$, it may be developed in the form

$$\log[g_0 + g_1(z-a_1)^{\frac{1}{2}} + \cdots] = h_0 + h_1(z-a_1)^{\frac{1}{2}} + \cdots,$$

and consequently

$$\frac{d\log w}{dz} = \frac{\frac{\mu}{2}}{z-a_1} + \frac{h_1}{2}(z-a_1)^{-\frac{1}{2}} + \cdots.$$

We therefore have (cf. Art. 120)

$$\operatorname*{Res}_{z=a_1} \frac{d\log w}{dz} = 2\cdot\frac{\mu}{2} = \mu.$$

If on the other hand a_1 is an infinity of the λth order of w, then is

$$\operatorname*{Res}_{z=a_1} \frac{d\log w}{dz} = -\lambda.$$

ART. 126. We shall now apply Cauchy's Theorem to the function

$$\frac{d \log w}{dz} = P + Qs.$$

As the portion of surface over whose boundaries the integration is to be taken we shall choose a region which contains all the infinities of the function $P + Qs$.

In order to have such a surface, we construct in the Riemann surface a very small circle which does not contain any of the infinities of $P+Qs$. The rest of the Riemann surface, that is, the entire Riemann surface excepting the small circle, will then contain all the infinities of $P+Qs$. The point at infinity may be one of these infinities. In the latter case we make the substitution $z = \frac{1}{t}$. The function $P + Qs$ becomes by this substitution, say

$$\begin{aligned} P + Qs &= P_1(t) + Q_0(t) \sqrt{\left(\frac{1}{t} - a_1\right)\left(\frac{1}{t} - a_2\right)\left(\frac{1}{t} - a_3\right)\left(\frac{1}{t} - a_4\right)} \\ &= P_1(t) + Q_1(t) \sqrt{(1 - a_1 t)(1 - a_2 t)(1 - a_3 t)(1 - a_4 t)}, \end{aligned}$$

where
$$Q_1(t) = \frac{Q_0(t)}{t^2}.$$

The functions $P_1(t)$ and $Q_1(t)$ are rational functions of t; and in the t-plane the origin is now an infinity. The other infinities in the old Riemann surface remain at finite distances from the origin on the new Riemann surface, whose branch-points are the reciprocal of those in the old Riemann surface.*

We thus have no trouble in computing the order of the infinity at the point infinity.

The boundary of the region is evidently that of the small circle, and the integration is to be taken so that the region without the circle lies to the left.

After the theorem of Art. 92, when we remain on the original Riemann surface

$$\frac{1}{2\pi i}\int \frac{d \log w}{dz} dz = \sum \text{Res} \frac{d \log w}{dz} = \sum \text{Res } (P + Qs),$$

where the integration is taken so that the bounded region is on the left, that is, so that the interior of the small circle is on the right.

Noting that the integral taken over the boundary of this small circle, within which there is no infinity of the function, is zero, it is seen that

$$\Sigma \text{ Res } (P + Qs) = 0,$$

where the summation extends over all the infinities of $P + Qs$.

* Cf. Neumann, *loc. cit.*, p. 111.

These residues fall into two groups: those of the one group have reference to the infinities of the function $\frac{d \log w}{dz}$, which exist through the vanishing of w, while those of the other group refer to the infinities of $\frac{d \log w}{dz}$, which are also the infinities of w.

If by $\Sigma\mu$ we denote the sum of the orders of the zeros and by $\Sigma\lambda$ the sum of the orders of the infinities of w, then for $P + Qs$ the sum of the residues of the first group is $\Sigma\mu$, while $-\Sigma\lambda$ is the sum of the residues of the second group.

It follows at once that

$$\Sigma \text{ Res } (P + Qs) = \Sigma\mu - \Sigma\lambda = 0,$$

or

$$\Sigma\lambda = \Sigma\mu.$$

It has thus been shown that *the sum of the orders of the zeros of w is equal to the sum of the orders of its infinities;* or, in other words, *the function w becomes as often zero as it does infinity in the Riemann surface, if a zero of the μth order is counted μ-ply and an infinity of the λth order is counted λ-ply.*

ART. 127. Suppose that k is an arbitrary constant and write

$$p + qs = k.$$

The function $p + qs - k$ is a rational function in z and s. It becomes infinite as often as $p + qs$ is infinite, and since the relation

$$\Sigma\lambda = \Sigma\mu$$

is true also here, it becomes zero as often as $p + qs$ becomes zero.

We thus have the following theorem: *The equation*

$$p + qs = k$$

has in the Riemann surface as many solutions as $p + qs$ has infinities. Hence also the function $p + qs$ takes every value in the Riemann surface an equal number of times.

ART. 128. We have often employed the term "*complete boundary*" and have in particular considered this expression in Art. 122. We shall again emphasize the fact that it is of extreme importance to understand the full significance of this term. If from a portion of surface A a piece is cut out, for example a circle around a point of discontinuity, then in this new portion of surface every closed curve no longer forms a complete boundary. If P is the small

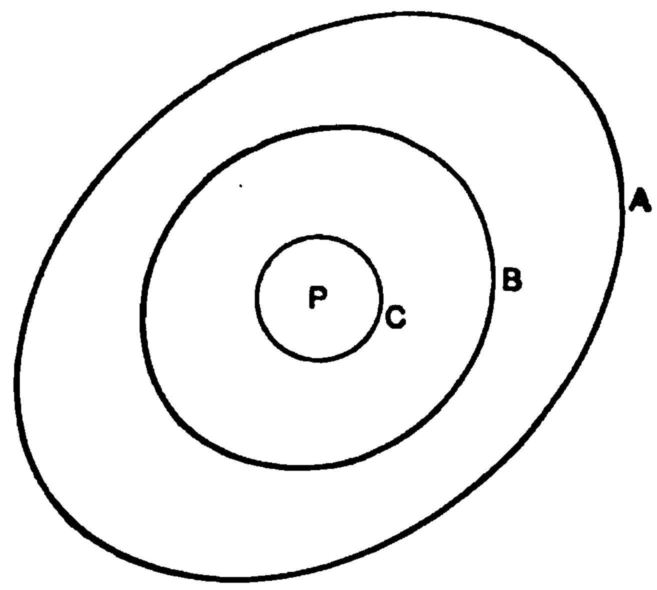


Fig. 47.

circle that has been cut out of A, then the closed curve B no longer forms a complete boundary, since B and C together constitute this com-

plete boundary. If from any portion of the surface A we cut out a circle and join this circle with the original boundary by means of a *cross-cut*, it is then impossible to draw a closed curve in A which does not form the complete boundary of a portion of surface, so long as we do not cross the cross-cut.

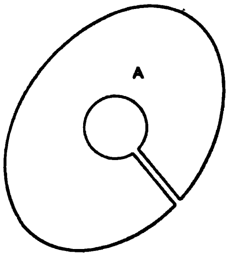


Fig. 48.

Every surface which has the property that every closed curve drawn in it is the complete boundary of a portion of surface, is called a *simply connected surface.**

The Riemann surface on which the function $w = p + qs$ is represented is not a simply connected one. We may, however, as shown in the figure, easily transform it into a simply connected surface by drawing the two canals a and b. We note that one-half of the canal b lies in the lower leaf. These canals *cannot* be crossed by going from one leaf into the other as is the case with the canals a_1a_2 and a_3a_4.

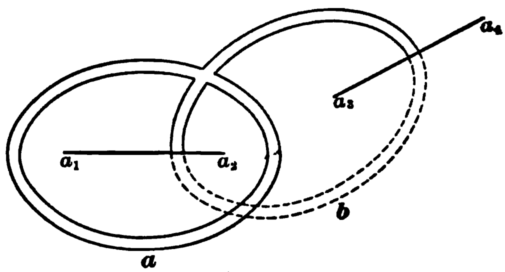


Fig. 49.

The Riemann surface containing the two canals a and b we denote by T'. The surface which does not have these canals is denoted by T. The surface T' is said to be of *order* † unity. We note that two canals or cross-cuts were necessary to make it simply connected. One may easily be convinced by trial that every closed curve in T' forms the complete boundary of a portion of surface, so long as the curve does not cross the canals a and b.

ART. 129. Anticipating some of the more complicated results of the next Chapter, we may consider here the simpler case of the function

$$s^2 = r(z),$$

where $r(z) = A(z - a_1)(z - a_2)$. The associated Riemann surface consists of two leaves connected along the canal a_1a_2.

The integrals

$$P = \int \frac{dz}{\sqrt{r(z)}},$$

* Cf. Neumann, *loc. cit.*, p. 146.

† In general, if N denotes the number of branch-points belonging to any function, n the number of leaves in the associated Riemann surface, and p the class or order of the Riemann surface, then (see Forsyth, *Theory of Functions*, p. 356) $N = 2p + 2n - 2$. (Cf. Riemann, Werke, p. 114.) The name *deficiency* was introduced by Cayley, *On the Transformation of Plane Curves.* 1865. The deficiency of a curve is the *class* or *order* of the Riemann surface associated with its equation; that is, $y^2 = R(x)$ is a curve of *deficiency* unity, if $s^2 = R(z)$ is a Riemann surface of *order* unity.

where the paths of integration are taken over the two curves (1) and (2), are equal since the function $\frac{1}{\sqrt{r(z)}}$ is one-valued finite and continuous for all points of the surface between these two curves.

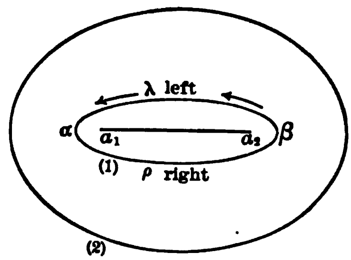


Fig. 50.

If we let the path of integration (1) approach indefinitely near the canal a_1a_2, then, since the values of $\sqrt{r(z)}$ on the right and left banks of this canal have contrary signs, we have

$$P = \int_{\beta\lambda\alpha} \frac{dz}{\sqrt{r(z)}} + \int_{\alpha\rho\beta} \frac{dz}{\sqrt{r(z)}} = 2\int_{a_2}^{a_1} \frac{dz}{\sqrt{r(z)}},$$

where in the last integral the integration is taken along the upper leaf and the left bank.

It follows that P is different from zero and consequently also the integral taken over a curve such as (2) is *not* zero.

This two-leaved Riemann surface T we next cut by a canal so that the integral

$$u = \int \frac{dz}{\sqrt{r(z)}}$$

will be a one-valued function of position in the surface where the cut has been made. This integral will then be independent of the path of integration, which we have just shown by going around the canal a_1a_2 is not the case in the Riemann surface before the cut has been made.

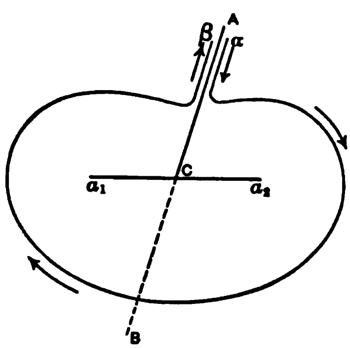


Fig. 51.

From a point C on the upper bank of the canal we draw a line CA which goes off towards infinity and this line is indefinitely continued from C in the other direction in the lower leaf. We thus form a cut or canal AB which is not to be crossed. The surface with this new canal we call T'.

From the figure it is seen that we may go from any point α on the bank of the canal AB to a point β immediately opposite on the other bank without crossing either the canal AB or the canal a_1a_2, but it is impossible to make a circuit around the canal a_1a_2 or around either of the branch-points a_1 or a_2 without crossing one of these canals. It follows that the above integral in T' is one-valued.

ART. 130. Next let

$$\bar{u}(z, s) = \int_{z_0, s_0}^{z, s} \frac{dz}{\sqrt{r(z)}}, \quad \text{where the path of integration is in } T';$$

and let

$$u(z, s) = \int_{z_0, s_0}^{z, s} \frac{dz}{\sqrt{r(z)}}, \quad \text{where the path is in } T.$$

The integration in both cases is always counted from a fixed point z_0, s_0, which as a rule may be arbitrarily taken, but when once taken must be retained as the lower limit for all the integrals that come under the discussion.

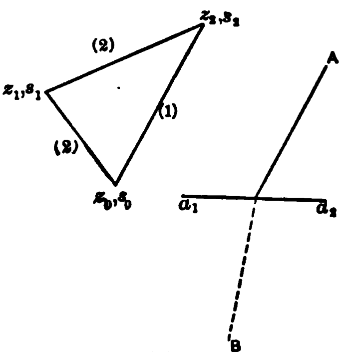


Fig. 52.

We know that if the function $\sqrt{r(z)}$ is one-valued, finite and continuous within the area situated within the two curves (1) and (2) of the figure,

$$\int_{\substack{z_0, s_0 \\ (1)}}^{z_2, s_2} = \int_{\substack{z_0, s_0 \\ (2)}}^{z_2, s_2} = \int_{z_0, s_0}^{z_1, s_1} + \int_{z_1, s_1}^{z_2, s_2},$$

where the integrand $\dfrac{dz}{\sqrt{r(z)}}$ is to be understood with every integral.

It follows that in T'

$$\int_{z_1, s_1}^{z_2, s_2} = \int_{z_0, s_0}^{z_2, s_2} - \int_{z_0, s_0}^{z_1, s_1} = \bar{u}(z_2, s_2) - \bar{u}(z_1, s_1).$$

Next take the integral from z_0, s_0 to z, s in T where there being no canal AB we go by the way of the two points λ and ρ.

We have

$$u(z, s) = \int_{z_0, s_0}^{\rho} + \int_{\rho}^{\lambda} + \int_{\lambda}^{z, s},$$

where the distance between λ and ρ being indefinitely small the middle integral on the right may be neglected. But from above

$$\int_{z_0, s_0}^{\rho} = \bar{u}(\rho)$$

and

$$\int_{\lambda}^{z, s} = \bar{u}(z, s) - \bar{u}(\lambda),$$

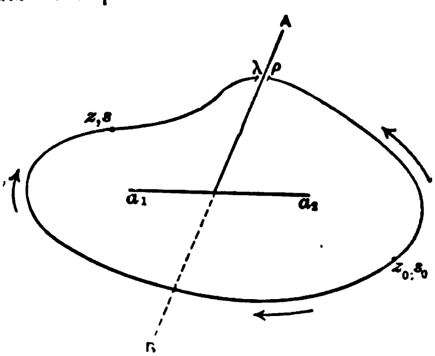


Fig. 53.

where both of these integrals are in the T' surface. From this it is seen that

$$u(z, s) = \bar{u}(z, s) + \bar{u}(\rho) - \bar{u}(\lambda),$$

where $\bar{u}(\rho)$ and $\bar{u}(\lambda)$ are the integrals from z_0, s_0 to ρ and from z_0, s_0 to λ in the T' surface, the path of integration being taken in any manner so long as neither of the canals a_1a_2 and AB is crossed.

On the other hand,

$$u(\rho) - \bar{u}(\lambda) = \int_\lambda^\rho \frac{dz}{\sqrt{r(z)}} \text{ in } T';$$

or, from the figure,

$$\int_\rho^\lambda = \int_\rho^\alpha + \int_{\alpha\gamma\beta} + \int_\beta^\lambda,$$

and since

$$\int_\rho^\alpha = -\int_\beta^\lambda,$$

we have

$$\int_\rho^\lambda = \int_{\alpha\gamma\beta} = 2\int_{a_2}^{a_1} = -P,$$

where the integration of the last integral is taken in the upper leaf and the *lower* bank of the canal a_1a_2.

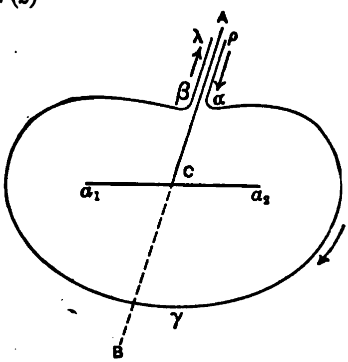


Fig. 54.

We have finally

$$u(z, s) = \bar{u}(z, s) + P,$$

where P is a quantity which does not depend upon the path z_0, s_0 to z_1, s_1.

The quantity P is called the *modulus of periodicity.*

If the path of integration is taken so that we pass from the right to the left bank of the canal AB, then is

$$u(z, s) = \bar{u}(z, s) - P.$$

The integral in T differs from the integral in T' only by a positive or negative multiple of P, this multiple depending upon the number of times and the direction the canal AB has been crossed [see Durège, *Elliptische Functionen* (2d ed.), p. 370].

EXAMPLE

Show that $P = 2\pi$ in the case of $u = \int \frac{dz}{\sqrt{1 - z^2}}$.

REALMS OF RATIONALITY.

ART. 131. Let z be a complex variable which may take all real or complex, finite and infinite values. Consider the collectivity of all rational functions of z with arbitrary constant real or complex coefficients. These functions form a closed realm, the individual functions of which repeat themselves through the processes of addition, subtraction, multiplication and division, since clearly the sum, the difference, the product, and the quotient of two or more rational functions is a rational function and consequently an individual of the realm.

This realm of rationality we shall denote by (z). Consider next the one-valued functions on the fixed Riemann surface. If we denote any such function by $w_1 = p_1 + q_1 s$ and any other such function by $w_2 = p_2 + q_2 s$, then the sum, difference, product and quotient of the two functions w_1 and w_2 are functions of the form

$$w = p + qs.$$

It is evident that if we add (or *adjoin*) the algebraic quantity s to the realm (z), we will have another realm (z, s), the individual functions or *elements* of which repeat themselves through the processes of addition, subtraction, multiplication and division. This realm we shall call the *elliptic* realm. It includes the former realm. We note that every element of this realm is a one-valued function of position on the fixed Riemann surface. In the present Chapter we have proved that every element of the realm (z, s) takes every arbitrary value that it can take an equal number of times. It also follows that within this elliptic realm there does not exist an element that becomes infinite of the first order at only one point of the Riemann surface. This latter statement is left as an exercise (see Thomae, *Functionen einer complexen Veränderlichen*, p. 94).

CHAPTER VII

THE PROBLEM OF INVERSION

ARTICLE 132. We have seen (Chapter V) that every one-valued doubly periodic function of the second order which has no essential singularity in the finite portion of the plane, or Riemann surface, satisfies a certain differential equation in which the independent variable does not explicitly appear. This equation may be written

$$\frac{dz}{du} = p(z) + \sqrt{R(z)},$$

where $p(z)$ is an integral function of at most the second degree and $R(z)$ is an integral function of the fourth degree. We saw in the preceding Chapter that $p(z) + \sqrt{R(z)}$ is a one-valued function of position on the fixed Riemann surface. We are thus led to the study of the integral

$$u = \int \frac{dz}{p(z) + \sqrt{R(z)}}.$$

As the lower limit of this integral we take any point z_0 of the Riemann surface, at which s has the value $s_0 = +\sqrt{R(z_0)}$. Throughout the whole discussion this point z_0, s_0 will be taken as the initial point. The integral is taken along any path of integration to the point z, s.

It follows then that

$$u = \int_{z_0,\, s_0}^{z,\, s} \frac{dz}{p(z) + \sqrt{R(z)}}$$

is a definite function of the upper limit, a function which is *dependent upon* the path of integration.

We may also consider the upper limit z as a function of u; and we shall now raise the question: *Under what conditions is the upper limit z a one-valued function of u?*

It is possible that the point z, s lies in the neighborhood of a branch-point a_1, say.

We then have the following development:

$$p(z) = p(a_1) + \frac{p'(a_1)}{1!}(z - a_1) + \frac{p''(a_1)}{2!}(z - a_1)^2 + \cdots,$$

$$\sqrt{R(z)} = b_1(z - a_1)^{\frac{1}{2}} + b_2(z - a_1)^{\frac{3}{2}} + \cdots;$$

and consequently

$$p(z) + \sqrt{R(z)} = p(a_1) + b_1(z - a_1)^{\frac{1}{2}} + \frac{p'(a_1)}{1}(z - a_1)^{\frac{2}{2}} + b_2(z - a_1)^{\frac{3}{2}} + \cdots.$$

We thus have a series which proceeds in ascending powers of $(z - a_1)$.

ART. 133. *Suppose that $p(a_1)$ does not vanish.*

We may then develop $\dfrac{1}{p(z)+\sqrt{R(z)}}$ in integral powers of $(z-a_1)^{\frac{1}{2}}$ in the form

$$\frac{1}{p(z)+\sqrt{R(z)}} = \frac{1}{p(a_1)} + c_1(z-a_1)^{\frac{1}{2}} + c_2(z-a_1)^{\frac{2}{2}} + \cdots.$$

If we put

$$\int_{z_0, s_0}^{a_1} \frac{dz}{p(z)+\sqrt{R(z)}} = \alpha,$$

it is seen that

$$u = \int_{z_0, s_0}^{z, s} \frac{dz}{p(z)+\sqrt{R(z)}} = \int_{z_0, s_0}^{a_1} \frac{dz}{p(z)+\sqrt{R(z)}} + \int_{a_1}^{z, s} \frac{dz}{p(z)+\sqrt{R(z)}}$$
$$= \alpha + \int_{a_1}^{z, s} \frac{dz}{p(z)+\sqrt{R(z)}}.$$

We have here assumed that the point z, s has been so chosen that there is no point of discontinuity of the integrand within the triangle $a_1 z z_0$.

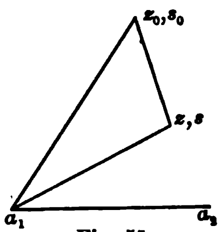


Fig. 55.

It follows that

$$u - \alpha = \int_{a_1}^{z, s} \frac{dz}{p(z)+\sqrt{R(z)}}.$$

By hypothesis the point z, s lies in the neighborhood of a_1, that is, on the inside of a circle within which the series developed above is convergent. We may therefore integrate this series and have

$$u - \alpha = \int_{a_1}^{z, s} \left\{ \frac{1}{p(a_1)} + c_1(z-a_1)^{\frac{1}{2}} + c_2(z-a_1)^{\frac{2}{2}} + \cdots \right\} d(z-a_1)$$
$$= \frac{z-a_1}{p(a_1)} + \frac{2}{3}c_1(z-a_1)^{\frac{3}{2}} + \cdots.$$

If we put $z - a_1 = t^2$, we have

$$u - \alpha = \frac{t^2}{p(a_1)} + \frac{2}{3}c_1 t^3 + \cdots$$
$$= t^2\left[\frac{1}{p(a_1)} + \frac{2}{3}c_1 t + \cdots\right]$$

It follows that

$$(u-\alpha)^{\frac{1}{2}} = t\left[\frac{1}{p(a_1)^{\frac{1}{2}}} + f_2 t + \cdots\right];$$

or

$$(u-\alpha)^{\frac{1}{2}} = \frac{t}{p(a_1)^{\frac{1}{2}}} + f_2 t^2 + \cdots,$$

where of course the quantities $c_1, c_2, \ldots, f_2$, etc., are constants. By the reversion of this series we have

$$t = g_1(u-\alpha)^{\frac{1}{2}} + g_2(u-\alpha)^{\frac{2}{2}} + \cdots.$$

But since $t^2 = z - a_1$, it is seen that z is *two-valued* and not *one-valued* in the neighborhood of $u = \alpha$.

ART. 134. If $p(a_1) = 0$, the above development becomes

$$\frac{1}{p(z) + \sqrt{R(z)}} = e_{-1}(z - a_1)^{-\frac{1}{2}} + e_0 + e_1(z - a_1)^{\frac{1}{2}} + \cdots .$$

We then have

$$\begin{aligned} u - \alpha &= \int [e_{-1}(z - a_1)^{-\frac{1}{2}} + e_0 + e_1(z - a_1)^{\frac{1}{2}} + \cdots] d(z - a_1) \\ &= 2 e_{-1}(z - a_1)^{\frac{1}{2}} + e_0(z - a_1)^{\frac{2}{2}} + \cdots \\ &= 2 e_{-1} t + e_0 t^2 + \cdots . \end{aligned}$$

From this we conclude that t is developable in positive integral powers of $u - \alpha$ and consequently is one-valued in the neighborhood of $u = \alpha$. It follows also that z is one-valued in the neighborhood of this point.

Hence in order that z be a one-valued function of u, it is necessary that $p(a_1) = 0$. In the same way it may be shown that $p(a_2) = 0 = p(a_3) = p(a_4)$.

On the other hand, $p(z)$ is an integral algebraic function of at most the second degree in z. Such a function cannot vanish at more than two points without being identically zero. It follows that $p(z) = 0$. We therefore have the theorem: *In order that z be a one-valued function of u, it is necessary that* $p(z) = 0$ *and consequently also that*

$$\frac{dz}{du} = \sqrt{R(z)}.$$

ART. 135. The last investigation would be true even if

$$\alpha = \int_{z_0, s_0}^{a} \frac{dz}{p(z) + \sqrt{R(z)}}$$

were infinite. We may prove, however, as follows that this integral is never infinite.

We saw above that $\dfrac{1}{p(z) + \sqrt{R(z)}}$ is developable in a power series which is convergent within a certain circle. Let this circle cut the path of integration at the point z', s'. We then have

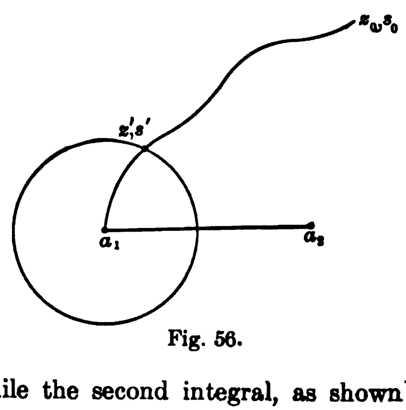


Fig. 56.

$$\int_{z_0, s_0}^{a_1} \frac{dz}{p(z) + \sqrt{R(z)}} = \int_{z_0, s_0}^{z', s'} \frac{dz}{p(z) + \sqrt{R(z)}} + \int_{z', s'}^{a_1} \frac{dz}{p(z) + \sqrt{R(z)}} .$$

The first integral on the right is finite, since it does not become infinite for any value between z_0, s_0 and z', s'; while the second integral, as shown above, may be expressed through the series

$$\left[2 e_{-1}(z - a_1)^{\frac{1}{2}} + e_0(z - a_1)^{\frac{2}{2}} + \cdots \right]_{z', s'}^{a_1} .$$

This series is finite for the values z', s' and a_1. It follows therefore that

$$\alpha = \int_{z_0, s_0}^{a_1} \frac{dz}{p(z) + \sqrt{R(z)}}$$

has a finite value even when $p(a_1) = 0$, and at the same time it has been shown that the integral

$$\int_{z_0, s_0} \frac{dz}{p(z) + \sqrt{R(z)}}$$

is finite when the upper limit is a branch-point.

ART. 136. We may now confine ourselves to the consideration of the integral

$$u = \int_{z_0, s_0}^{z, s} \frac{dz}{\sqrt{R(z)}}.$$

This integral is called an *elliptic integral of the first kind.* We have seen that the integral u remains finite when the upper limit coincides with a branch-point. We shall next see that this integral remains finite when the path of integration goes into infinity.

In one of the leaves of the Riemann surface, for example the upper, draw a circle with the origin as center which includes all the branch-points. On the outside of this circle the quantity $\sqrt{R(z)}$ and consequently also $\frac{1}{\sqrt{R(z)}}$ is one-valued; for if we make a closed circuit without this circle it includes either none or all the branch-points and consequently $\frac{1}{\sqrt{R(z)}}$ does not change its value.

We have

$$\frac{1}{\sqrt{R(z)}} = \frac{1}{z^2}\left[1 - \frac{a_1}{z}\right]^{-\frac{1}{2}}\left[1 - \frac{a_2}{z}\right]^{-\frac{1}{2}}\left[1 - \frac{a_3}{z}\right]^{-\frac{1}{2}}\left[1 - \frac{a_4}{z}\right]^{-\frac{1}{2}}.$$

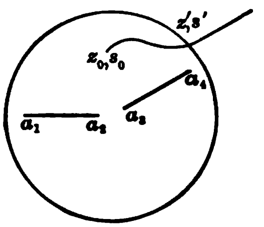


Fig. 57.

Since $\frac{a_1}{z}$, $\frac{a_2}{z}$, $\frac{a_3}{z}$, $\frac{a_4}{z}$ are proper fractions for all values of z without this circle, each of the above factors is developable in positive integral powers of $\frac{1}{z}$, so that

$$\frac{1}{\sqrt{R(z)}} = \frac{1}{z^2}\left\{g_0 + g_1\frac{1}{z} + g_2\frac{1}{z^2} + \cdots\right\},$$

which series is convergent for all values of z without the circle.

Let z', s' be the point where the path of integration starting from the point z_0, s_0 and leading to infinity, cuts the circle.

We have

$$\int_{z_0, s_0}^{\infty} \frac{dz}{\sqrt{R(z)}} = \int_{z_0, s_0}^{z', s'} \frac{dz}{\sqrt{R(z)}} + \int_{z', s'}^{\infty} \frac{dz}{\sqrt{R(z)}}.$$

We have seen that the first integral on the right is always finite, whether the path of integration goes through a branch-point or not. For the second integral we have

$$\int_{z',s'}^{\infty}\frac{dz}{\sqrt{R(z)}}=\int_{z',s'}^{\infty}\left[\frac{g_0}{z^2}+\frac{g_1}{z^3}+\cdots\right]dz$$
$$=\left[-\frac{g_0}{z}-\frac{1}{2}\frac{g_1}{z^2}-\cdots\right]_{z',s'}^{\infty},$$

an expression which is finite for both the upper and the lower limit. We have thus shown that the *integral*

$$\int_{z_0,s_0}^{z,s}\frac{dz}{\sqrt{R(z)}}$$

is finite everywhere, even when the upper limit is indefinitely large or if it coincides with one of the branch-points.

ART. 137. We represent by T' the Riemann surface of Art. 128 in which the canals a and b have been drawn. We noted that any closed curve on this surface formed the complete boundary of a portion of surface. If on this surface the curve C includes one or several branch-points, for example a_1, we isolate them by means of small double circles. If K denotes the double circle about a_1, and if the curve C includes only one such branch-point, then by Cauchy's Theorem we have

$$\int_C\frac{dz}{s}+\int_K\frac{dz}{s}=0,\ \text{ where } s=\sqrt{R(z)}.$$

Note that in this second integral the integration is over two circles lying directly the one over the other in the two leaves of the Riemann surface. In these two leaves the quantity s has opposite signs, while at points the one over the other the absolute values of s and z are equal. It follows that in the integral $\int_K\frac{dz}{s}$ the elements of integration cancel in pairs, so that this integral is zero. We have thus shown that the integral $\int_C\frac{dz}{s}$ taken over any closed curve in T' is zero.

If in T' we draw any two curves (1) and (2) between the points z_0, s_0 and z, s, without crossing either of the canals a *or* b, the two curves will form a closed curve, and from what we have just seen

$$\int_{\substack{z_0,s_0\\(1)}}^{z,s}\frac{dz}{s}+\int_{\substack{z,s\\(2)}}^{z_0,s_0}\frac{dz}{s}=0,$$

or

$$\int_{\substack{z_0,s_0\\(1)}}^{z,s}\frac{dz}{s}=\int_{\substack{z_0,s_0\\(2)}}^{z,s}\frac{dz}{s},$$

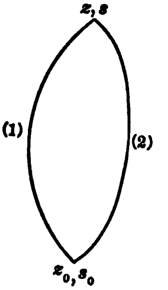


Fig. 58.

the numbers in parentheses under the integral signs denoting the paths along which the integration has been taken.

Hence if we write

$$\bar{u}(z, s) = \int_{z_0, s_0}^{z, s} \frac{dz}{s},$$

where the dash over u signifies that the integration is to be taken in the Riemann surface T', in which the canals a and b are *not* to be crossed, it follows from above that $\bar{u}(z, s)$ *is entirely independent of the path of integration. It follows also that the integral* $\bar{u}(z, s)$ *is a one-valued definite function of the upper limit.*

ART. 138. We shall consider next the integral

$$u(z, s) = \int_{z_0, s_0}^{z, s} \frac{dz}{s},$$

where the path of integration is taken in the Riemann surface T, which does *not* contain the canals a and b. We shall show that here the integral $u(z, s)$ is *not* a one-valued function of the upper limit z, s, but depends upon the path of integration.

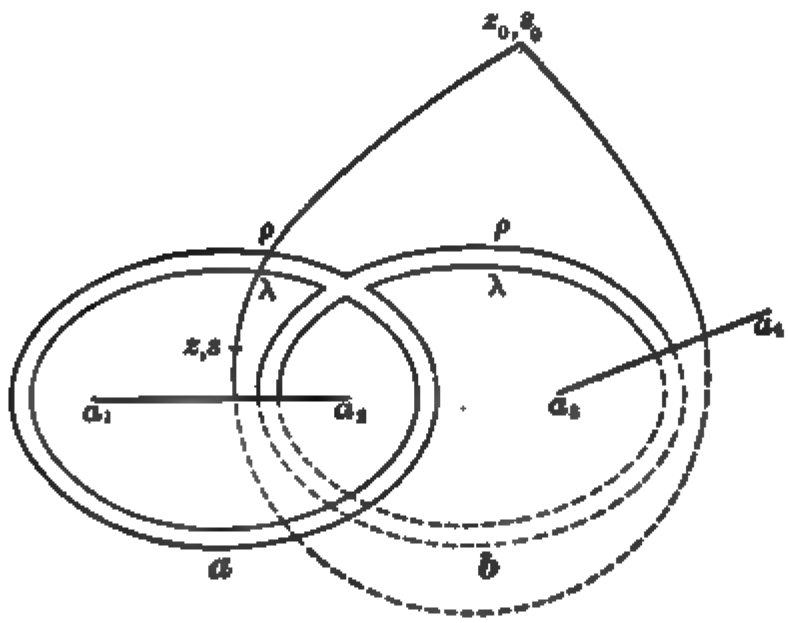


Fig. 59.

In the T-surface the integral corresponding to $\bar{u}(z, s)$ is

$$u(z, s) = \int_{z_0, s_0}^{\rho} \frac{dz}{s} + \int_{\rho}^{\lambda} \frac{dz}{s} + \int_{\lambda}^{z, s} \frac{dz}{s}.$$

The points ρ and λ are supposed to lie indefinitely near each other, so
sequently have

(A)

We have seen th
dent of the path (

$$\int_{z_1, s_1}^{z_2, s_2}$$

Returning to the e
is crossed between

Further, there is no canal between λ and z, s. It follows from what we have just shown that

$$\int_{\lambda}^{z,s} \frac{dz}{s} = \overline{u}(z, s) - \overline{u}(\lambda) \text{ in } T',$$

where we go in T' from z_0, s_0 to z, s by crossing the canals $a_3 a_4$ and $a_1 a_2$ as shown in the figure. We have to make the same crossings to go from z_0, s_0 to λ. We therefore have from the equation (A)

$$u(z, s) = \overline{u}(z, s) + \overline{u}(\rho) - \overline{u}(\lambda).$$

If the canal $\boldsymbol{a}$ had been crossed at any other point ρ_1, λ_1 instead of at ρ, λ, we would have had

$$u(z, s) = \overline{u}(z, s) + \overline{u}(\rho_1) - \overline{u}(\lambda_1).$$

Consider the difference

$$\left\{\overline{u}(\rho) - \overline{u}(\lambda)\right\} - \left\{\overline{u}(\rho_1) - \overline{u}(\lambda_1)\right\},$$

or

$$\left\{\overline{u}(\rho) - \overline{u}(\rho_1)\right\} - \left\{\overline{u}(\lambda) - \overline{u}(\lambda_1)\right\}.$$

The points ρ and ρ_1 are both on the same side of the canal $\boldsymbol{a}$, while the point λ and λ_1 are both on the opposite bank.

It is seen that

$$\overline{u}(\rho) - \overline{u}(\rho_1) = \int_{\rho_1}^{\rho} \frac{dz}{s} \text{ in } T',$$

and

$$\overline{u}(\lambda) - \overline{u}(\lambda_1) = \int_{\lambda_1}^{\lambda} \frac{dz}{s} \text{ in } T',$$

where the path of integration in T' may be quite arbitrary, provided only it does not cross the canals $\boldsymbol{a}$ and $\boldsymbol{b}$. We may therefore take the path of integration from ρ to ρ_1 indefinitely near the right bank of the canal, while the path from λ to λ_1 is taken indefinitely near the left bank. Since these two paths differ from each other by an infinitesimal quantity, the integrals over them are equal. It follows then that

$$\left\{\overline{u}(\rho) - \overline{u}(\lambda)\right\} - \left\{\overline{u}(\rho_1) - \overline{u}(\lambda_1)\right\} = 0,$$

and consequently $\overline{u}(\rho) - \overline{u}(\lambda)$ has the same value at whatever point the crossing has taken place.

ART. 139. If we cross the canal $\boldsymbol{a}$ from z_0, s_0 to z, s in the opposite direction from that gone over in the previous case, we have

$$\begin{aligned}\int_{z_0, s_0}^{z, s} \frac{dz}{s} &= \int_{z_0, s_0}^{\lambda} \frac{dz}{s} + \int_{\rho}^{z, s} \frac{dz}{s} \text{ (in } T) \\ &= \overline{u}(\lambda) + \overline{u}(z, s) - \overline{u}(\rho) \text{ (in } T'), \\ &= \overline{u}(z, s) + \overline{u}(\lambda) - \overline{u}(\rho).\end{aligned}$$

We note that in T' we must go from z_0, s_0 to the canal joining a_1 and a_2 and after crossing this canal into the lower leaf come out again into the upper leaf by crossing the canal a_3a_4 and then proceed to z, s. We thus see that when we cross the canal a in the opposite direction to that followed in the previous article we have to subtract the quantity $\bar{u}(\rho)-\bar{u}(\lambda)$ from $\bar{u}(z, s)$.

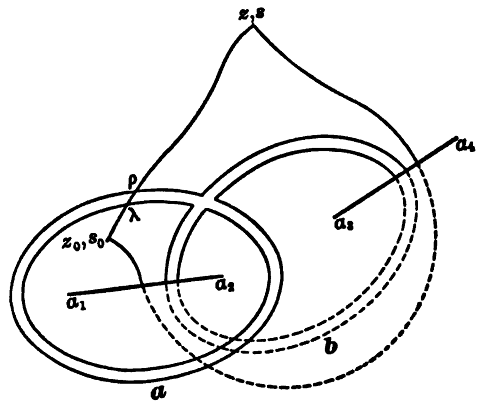


Fig. 60.

If the canal a is crossed μ times in the first direction and ν times in the second direction, we will have

$$u(z, s) = \bar{u}(z, s) + (\mu - \nu)\left\{\bar{u}(\rho) - \bar{u}(\lambda)\right\}.$$

We have precisely the same result if we cross the canal b. Of course, the constant $\bar{u}(\rho) - \bar{u}(\lambda)$ is different here from what it was in the previous case when we crossed the canal a.

We shall write.

$$\textbf{for the canal } a : \bar{u}(\lambda) - \bar{u}(\rho) = A,$$
$$\textbf{for the canal } b : \bar{u}(\rho) - \bar{u}(\lambda) = B.$$

We therefore have in general

$$u(z, s) = \bar{u}(z, s) + mA + nB,$$

where m and n are positive or negative integers and where $u(z, s)$ is the integral in which the path of integration is free, $\bar{u}(z, s)$ being the integral in the Riemann surface T', in which the canals a and b cannot be crossed. The quantities A and B are called the *Moduli of Periodicity*.

ART. 140. We have seen that if a and b are two quantities whose quotient is not real and if the coefficient of i in the complex quantity $\frac{b}{a}$ is *positive*, we may determine a function $\Phi(u)$ which satisfies the two functional equations

$$\Phi(u + a) = \Phi(u),$$
$$\Phi(u + b) = e^{-\frac{\pi i k}{a}(2u+b)}\Phi(u).$$

This function is (cf. Art. 86)

$$\Phi(u) = \sum_{m=-\infty}^{m=+\infty} B_m Q^{\frac{m^2}{k}} e^{\frac{2\pi i}{a}mu},$$

where $Q = e^{\pi i \frac{b}{a}}$ and $B_{m+k} = B_m$.

If the two moduli of periodicity A and B have the property that the coefficient of i in $\frac{A}{B}$ is *positive*, then we may write $a = B$ and $b = A$ and form a function $\Phi(u)$ so that

$$\Phi(u) = \sum_{m=-\infty}^{m=+\infty} B_m Q_0^{\frac{m^2}{k}} e^{\frac{2\pi i}{B} mu},$$

where $Q_0 = e^{\pi i \frac{A}{B}}$ and $B_{m+k} = B_m$.

We then have

$$\Phi(u + B) = \Phi(u),$$
$$\Phi(u + A) = e^{-\frac{\pi i k}{B}(2u+A)} \Phi(u).$$

Instead of the variable u we may introduce any variable quantity, say

$$u(z, s) = \int_{z_0, s_0}^{z, s} \frac{dz}{s}.$$

We then have

$$\Phi(u) = \Phi[u(z, s)] = \Psi(z, s), \text{ say}.$$

It is seen that $\Psi(z, s)$ is a function of position in the Riemann surface and is *not* a one-valued function; that is, when z, s are given, $\Psi(z, s)$ does not take *one* definite value. For $u(z, s)$ depends upon the path of integration, so that (cf. Art. 139)

$$u(z, s) = \bar{u}(z, s) + mA + nB.$$

Hence the complex of values $\Psi(z, s)$ which belong to one position z, s is expressed through

$$\Psi(z, s) = \Phi[\bar{u}(z, s) + mA + nB],$$

where m and n are integers.

Since $\Phi(u)$ has the period B, the above complex of values reduces to

$$\Phi[\bar{u}(z, s) + mA].$$

We saw in Art. 91 that the following relation existed for the general Φ-function:

$$\Phi(u + mb) = e^{-\frac{\pi i k}{a}(2mu+m^2 b)} \Phi(u).$$

Consequently the complex of values above becomes

$$\Phi[u(z, s) + mA] = e^{-\frac{\pi i k}{B}[2m\bar{u}(z,s)+m^2 A]} \Phi[\bar{u}(z, s)].$$

It is evident that $\Phi[\bar{u}(z, s)] = \overline{\Psi}(z, s)$ is a one-valued function of position on the Riemann surface T'. It also follows that between $\Psi(z, s)$ and $\overline{\Psi}(z, s)$ there exists the relation

$$\Psi(z, s) = e^{-\frac{\pi i k}{B}[2m\bar{u}(z,s)+m^2 A]} \overline{\Psi}(z, s).$$

The integer m is positive or negative depending upon the number of times the path of integration has crossed the canal $\boldsymbol{a}$ and upon the direction at the crossing.

ART. 141. We saw in Art. 94 that

$$\Phi(u) = B_0\Phi_0(u) + B_1\Phi_1(u) + \cdots + B_{k-1}\Phi_{k-1}(u).$$

Let the corresponding Ψ-functions be denoted by

$$\Psi_0, \Psi_1, \Psi_2, \ldots, \Psi_{k-1}.$$

We then have, for example,

$$\Psi_1(z, s) = e^{-\frac{\pi i k}{B}[2m\bar{u}(z,s)+m^2A]}\overline{\Psi}_1(z, s),$$

$$\Psi_2(z, s) = e^{-\frac{\pi i k}{B}[2m\bar{u}(z,s)+m^2A]}\overline{\Psi}_2(z, s).$$

It follows that

$$\frac{\Psi_1(z, s)}{\Psi_2(z, s)} = \frac{\overline{\Psi}_1(z, s)}{\overline{\Psi}_2(z, s)};$$

and since $\overline{\Psi}_1(z, s)$, $\overline{\Psi}_2(z, s)$ are both one-valued functions of position on the Riemann surface, it is also seen that $\dfrac{\Psi_1(z, s)}{\Psi_2(z, s)}$ is a one-valued function of position on the Riemann surface.

The functions $\Psi(z, s) = \Phi[u(z, s)]$ are infinite series which are convergent for all values of the argument $u(z, s)$ which are not infinitely large (Art. 86). We have proved, however, that

$$u(z, s) = \int_{z_0, s_0}^{z, s}\frac{dz}{s}$$

is infinite for no point of the Riemann surface, including the point at infinity. It follows that $\Psi_1(z, s)$, $\Psi_2(z, s)$ are everywhere convergent and consequently the quotient $\dfrac{\Psi_1(z, s)}{\Psi_2(z, s)}$ has definite values everywhere on the Riemann surface. But a one-valued function of z, s which has everywhere a definite value is a rational function of z, s. It follows then that

$$\frac{\Psi_1(z, s)}{\Psi_2(z, s)} = R(z, s),$$

where R denotes a rational function.

ART. 142. Let us next study more closely some of the subjects which we have passed over rather rapidly.

We had on the canal $\boldsymbol{a}$: $\bar{u}(\lambda) - \bar{u}(\rho) = A$,

on the canal $\boldsymbol{b}$: $\bar{u}(\rho) - \bar{u}(\lambda) = B$.

It made no difference where the point λ, ρ was situated on the canal. We may therefore take the point α, α' where the canal $\boldsymbol{b}$ cuts the canal $\boldsymbol{a}$ and have accordingly

$$\bar{u}(\alpha') - \bar{u}(\alpha) = A,$$

or

$$A = \int_\alpha^{\alpha'}\frac{dz}{s} \text{ in } T', \text{ (cf. Neumann, } loc.\ cit., \text{ p. 248]},$$

the integration being in the negative direction. In the T'-surface we may, starting with α, follow the canal b around to the point α', and consequently have

$$A = \int_b \frac{dz}{s} \text{ in } T',$$

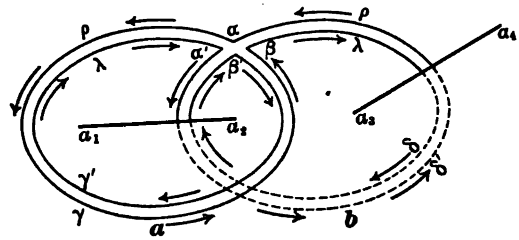


Fig. 61.

the integration being in the negative direction; *i.e.*, the quantity A *is the closed integral around the canal* b.

In the same way

$$B = \overline{u}(\rho) - \overline{u}(\lambda)$$
$$= \overline{u}(\alpha) - \overline{u}(\beta) = \int_\beta^\alpha \frac{dz}{s} = \int_a \frac{dz}{s} \text{ in } T',$$

the integration being in the negative direction. We have thus shown that B *is the closed integral over the canal* a.

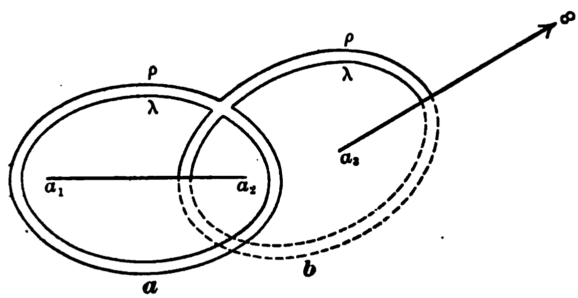


Fig. 62.

ART. 143. In the previous discussions we have assumed that $R(z)$ is of the fourth degree in z. When $R(z)$ is of the third degree, we have only three finite branch-points, a_1, a_2, a_3, say. But here the point at infinity is also a branch-point (Art. 115). We may therefore connect a_1 and a_2 by a canal and a_3 with the point at infinity. The Riemann surface may then be represented as in the former case (see figure).

ART. 144. In the derivation of the function $\Phi(u)$ the ratio $\frac{A}{B}$ cannot be real. Following the methods of Riemann* we shall show that this ratio is imaginary and that the coefficient of i must be positive, a result which was also necessary in the previous discussion.

We saw that $\overline{u}(z, s)$ was a one-valued function of position on the Riemann surface T'. All functions of the complex variable are in general also complex, and we may consequently write

$$\overline{u}(z, s) = p + iq.$$

* Riemann, *Theorie der Abel'schen Functionen, Crelle*, Bd. 54, p. 145; see also Koenigsberger, *Elliptische Functionen*, pp. 368, 369; Fuchs, *Crelle*, Bd. 83, pp. 13 *et seq.*

The quantity $\bar{u}(z, s)$ is everywhere finite in T', and from the developments by which it was shown always to be finite, it is readily proved to be also continuous.

If we write
$$z = \mathrm{x} + i\mathrm{y},$$
then p and q are everywhere one-valued, finite and continuous functions of x, y.

Noting that $\dfrac{\partial \bar{u}(z, s)}{\partial \mathrm{x}} = \dfrac{\partial \bar{u}(z, s)}{\partial z}\dfrac{\partial z}{\partial \mathrm{x}} = \dfrac{\partial \bar{u}(z, s)}{\partial z} \cdot 1 = \dfrac{\partial \bar{u}}{\partial z} = \dfrac{1}{\sqrt{R(z)}}$,

it is seen that $\dfrac{\partial \bar{u}}{\partial \mathrm{x}}$ is infinite for $z = a_1, a_2, a_3$, or a_4. On the other hand,
$$\frac{\partial \bar{u}}{\partial \mathrm{x}} = \frac{\partial p}{\partial \mathrm{x}} + i\frac{\partial q}{\partial \mathrm{x}},$$
and consequently either $\dfrac{\partial p}{\partial \mathrm{x}}$ or $\dfrac{\partial q}{\partial \mathrm{x}}$ or both of these derivatives are infinite for $z = a_1, a_2, a_3$, or a_4.

Form next the integral
$$\int p dq = \int p\left[\frac{\partial q}{\partial \mathrm{x}} d\mathrm{x} + \frac{\partial q}{\partial \mathrm{y}} d\mathrm{y}\right] = \int p\frac{\partial q}{\partial \mathrm{x}} d\mathrm{x} + \int p\frac{\partial q}{\partial \mathrm{y}} d\mathrm{y},$$
where the integration is to be taken over the whole boundary of the Riemann surface T'. This surface, see figure in the preceding article, is bounded by the two banks λ and ρ of the two canals $\boldsymbol{a}$ and $\boldsymbol{b}$. It is seen that we may go over both the banks λ and ρ of $\boldsymbol{a}$ and $\boldsymbol{b}$ with a single trace. The integral $\int p dq$ taken over this trace may be divided into several integrals as follows:
$$\int p dq = \int_{\alpha\gamma\beta \text{ on} + \boldsymbol{a}}^{(\rho)} p dq + \int_{\beta\delta\beta' \text{ on} - \boldsymbol{b}}^{(\lambda)} p dq + \int_{\beta'\gamma'\alpha' \text{ on} - \boldsymbol{a}}^{(\lambda)} p dq + \int_{\alpha'\delta'\alpha \text{ on} + \boldsymbol{b}}^{(\rho)} p dq,$$
where (ρ) as an upper index means that we are on the right bank, $\alpha\gamma\beta$ means the portion of curve gone over, and $+\,\boldsymbol{a}$ means on the canal $\boldsymbol{a}$ in the positive direction.

ART. 145. We saw above that
$$d\bar{u} = \frac{dz}{\sqrt{R(z)}} = dp + idq,$$
or
$$\frac{dz}{\sqrt{R(z)}} = \frac{d\mathrm{x} + id\mathrm{y}}{\sqrt{R(z)}} = dp + idq.$$

If we write
$$\frac{1}{\sqrt{R(z)}} = \phi(\mathrm{x}, \mathrm{y}) + i\psi(\mathrm{x}, \mathrm{y}),$$
then is
$$\phi(\mathrm{x}, \mathrm{y})d\mathrm{x} - \psi(\mathrm{x}, \mathrm{y})d\mathrm{y} + i\{\psi(\mathrm{x}, \mathrm{y})d\mathrm{x} + \phi(\mathrm{x}, \mathrm{y})d\mathrm{y}\} = dp + idq.$$

It follows that
$$\phi(\mathrm{x}, \mathrm{y})d\mathrm{x} - \psi(\mathrm{x}, \mathrm{y})d\mathrm{y} = dp,$$
$$\psi(\mathrm{x}, \mathrm{y})d\mathrm{x} + \phi(\mathrm{x}, \mathrm{y})d\mathrm{y} = dq.$$

The function $\psi(\mathsf{x}, \mathsf{y})$, which is the coefficient of i in $\dfrac{1}{\sqrt{R(z)}}$, will have at two opposite points on the left and right banks of the canals values which are different only by an infinitesimal small quantity, since the canals $\boldsymbol{a}$ and $\boldsymbol{b}$ are indefinitely narrow. The same is true of the function $\phi(\mathsf{x}, \mathsf{y})$. It follows that dq will have at two points opposite each other on the canal $\boldsymbol{a}$ the same values, but the signs will be different, since the integration at these points has been taken in the opposite direction.

We may therefore write the above integral in the form

$$\int pdq = \int_{+\boldsymbol{a}}\{\overset{(\rho)}{p} - \overset{(\lambda)}{p}\}dq + \int_{+\boldsymbol{b}}\{\overset{(\rho)}{p} - \overset{(\lambda)}{p}\}dq.$$

In Art. 139 we put $A = \bar{u}(\lambda) - \bar{u}(\rho)$ on the canal $\boldsymbol{a}$;

or
$$\begin{aligned} A &= \overset{(\lambda)}{p} + i\overset{(\lambda)}{q} - \{\overset{(\rho)}{p} + i\overset{(\rho)}{q}\} \\ &= \overset{(\lambda)}{p} - \overset{(\rho)}{p} + i\{\overset{(\lambda)}{q} - \overset{(\rho)}{q}\}. \end{aligned}$$

If further we write $A = \alpha + i\beta$, then is

$$\alpha = \overset{(\lambda)}{p} - \overset{(\rho)}{p} \textit{ on the canal } \boldsymbol{a}.$$

We also had

$$\begin{aligned} B = \bar{u}(\rho) - \bar{u}(\lambda) &= \overset{(\rho)}{p} + i\overset{(\rho)}{q} - \{\overset{(\lambda)}{p} + i\overset{(\lambda)}{q}\} \\ &= \overset{(\rho)}{p} - \overset{(\lambda)}{p} + i\{\overset{(\rho)}{q} - \overset{(\lambda)}{q}\}, \end{aligned}$$

and writing
$$B = \gamma + i\delta,$$

it follows that $\gamma = \overset{(\rho)}{p} - \overset{(\lambda)}{p}$ *on the canal* $\boldsymbol{b}$. It is seen at once that the above integral may be written

$$\int pdq = -\alpha\int_{+\boldsymbol{a}} dq + \gamma\int_{+\boldsymbol{b}} dq.$$

Since $d\bar{u} = \dfrac{dz}{\sqrt{R(z)}}$, it is clear that

$$\begin{aligned} A &= \int_{\boldsymbol{b}} \frac{dz}{s} = \int_{\boldsymbol{b}} (dp + idq) \\ &= \int_{\boldsymbol{b}} dp + i\int_{\boldsymbol{b}} dq. \end{aligned}$$

Further, since $A = \alpha + i\beta$, we have

$$\beta = \int_{\boldsymbol{b}} dq; \text{ and similarly}$$

$$\delta = \int_{\boldsymbol{a}} dq.$$

The integral above is finally

$$\int pdq = \gamma\beta - \alpha\delta.$$

ART. 146. We shall calculate the same integral in another manner. *Suppose that P and Q are real functions of the real variables* $\mathbf{x}$ *and* $\mathbf{y}$; *then the curvilinear integral*

$$\int (P d\mathbf{x} + Q d\mathbf{y}),$$

where the integration is taken over the complete boundary of a region within and on the boundary of which P and Q together with their partial derivatives of the first and second order are one-valued, finite and continuous, is equal to the surface integral

$$\text{(A)} \qquad \int\int \left(\frac{\partial Q}{\partial \mathbf{x}} - \frac{\partial P}{\partial \mathbf{y}}\right) d\mathbf{x} d\mathbf{y},$$

*taken over the same region.**

Consider the curvilinear integral

$$\int p dq = \int \left(p \frac{\partial q}{\partial \mathbf{x}} d\mathbf{x} + p \frac{\partial q}{\partial \mathbf{y}} d\mathbf{y}\right),$$

where as above the integration is to be taken over both banks of the two canals $\mathfrak{a}$ and $\mathfrak{b}$ in the Riemann surface T'. We have seen that p is one-valued, finite and continuous within this surface, since it is the real part of $u(z, s)$. But (see Art. 144) $\frac{\partial q}{\partial \mathbf{x}}$ and $\frac{\partial q}{\partial \mathbf{y}}$ become infinite at the points a_1, a_2, a_3 and a_4.

Hence to apply the theorem just stated, we must cut these points out of the surface by means of very small double circles. The resulting Riemann surface call T''. In this surface the conditions required are satisfied. The curvilinear integral must now also be taken over the double circles. But as shown in Art. 137 the integrals over these double circles are zero.

If then we write in the formula

$$\int (P d\mathbf{x} + Q d\mathbf{y}) = \int\int \left(\frac{\partial Q}{\partial \mathbf{x}} - \frac{\partial P}{\partial \mathbf{y}}\right) d\mathbf{x} d\mathbf{y}$$

instead of $P d\mathbf{x} + Q d\mathbf{y}$ the quantity $p \frac{\partial q}{\partial \mathbf{x}} d\mathbf{x} + p \frac{\partial q}{\partial \mathbf{y}} d\mathbf{y}$, we will have to substitute

for $\frac{\partial Q}{\partial \mathbf{x}}$ the expression $\frac{\partial p}{\partial \mathbf{x}} \frac{\partial q}{\partial \mathbf{y}} + p \frac{\partial^2 q}{\partial \mathbf{x} \partial \mathbf{y}}$

and for $\frac{\partial P}{\partial \mathbf{y}}$ the expression $\frac{\partial p}{\partial \mathbf{y}} \frac{\partial q}{\partial \mathbf{x}} + p \frac{\partial^2 q}{\partial \mathbf{x} \partial \mathbf{y}}$;

and consequently

$$\int p dq = \int\int \left[\frac{\partial p}{\partial \mathbf{x}} \frac{\partial q}{\partial \mathbf{y}} - \frac{\partial p}{\partial \mathbf{y}} \frac{\partial q}{\partial \mathbf{x}}\right] \partial \mathbf{x} \partial \mathbf{y}.$$

* Forsyth, p. 23; see also Casorati, *Teorica delle funzioni di variabili complesse*, pp. 64–69; Neumann, *Abel'sche Integrale*, 2d ed., p. 390. Schwarz, *Ges. Werke*, Bd. II, has shown that there are certain limitations of this theorem; and Picard, *Traité d'Analyse*, t. 2, pp. 38 *et seq.*

But since $\quad \dfrac{\partial p}{\partial x} = \dfrac{\partial q}{\partial y}$ and $\dfrac{\partial p}{\partial y} = -\dfrac{\partial q}{\partial x}$

(being the conditions that $\bar{u}(z, s) = p + iq$ have a definite derivative), and since $\int pdq = \beta\gamma - \alpha\delta$, it follows that

$$\beta\gamma - \alpha\delta = \int\int\left[\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2\right]dxdy.$$

As the elements under the sign of integration are essentially positive, it is seen that $\beta\gamma - \alpha\delta$ is a positive quantity.

But we have

$$\frac{B}{A} = \frac{\gamma + i\delta}{\alpha + i\beta} = \frac{(\gamma + i\delta)(\alpha - i\beta)}{\alpha^2 + \beta^2} = \frac{\alpha\gamma + \beta\delta}{\alpha^2 + \beta^2} + i\frac{\alpha\delta - \beta\gamma}{\alpha^2 + \beta^2}.$$

Since $\alpha\delta - \beta\gamma$ is different from zero, *the ratio* $\dfrac{B}{A}$ *is not real,** and the coefficient of i in $\dfrac{B}{A}$ is negative; *hence the coefficient of i in* $\dfrac{A}{B}$ *is positive.*

We may therefore (see Art. 86) form functions $\Phi(u)$ such that

$$\Phi(u + B) = \Phi(u),$$

$$\Phi(u + A) = e^{-\frac{\pi ik}{B}(2u+A)}\Phi(u).$$

ART. 147. In the expression

$$\Psi(z, s) = e^{-\frac{\pi ik}{B}[2m\bar{u}(z,s)+m^2A]}\overline{\Psi}(z, s),$$

since $\bar{u}(z, s)$ is always finite, the exponential factor is always finite so long as m is finite. Further, since Φ is only infinite for infinite values of its argument, it follows that

$$\overline{\Psi}(z, s) = \Phi[\bar{u}(z, s)]$$

is never infinite. Hence also $\Psi(z, s)$ is only infinite when m is infinite.

It is also evident that $\Psi(z, s)$ can only be zero when $\overline{\Psi}(z, s) = 0$.

We shall now see how often the function $\overline{\Psi}(z, s)$ becomes zero on the Riemann surface T''.

In Art. 92 we saw that if a function $f(z, s)$ is discontinuous at isolated positions within a portion of surface, but otherwise is one-valued and finite, then

$$\frac{1}{2\pi i}\int\frac{d\log f(z, s)}{dz}dz,$$

where the integration is taken over the complete boundary of the portion of surface, is equal to the sum of the orders of the zeros of the function

* Cf. Thomae, *Abriss einer Theorie der Functionen*, etc., p. 102; Falk, *Acta Math.*, Bd. 70; Pringsheim, *Math. Ann.*, Bd. 27.

diminished by the sum of the orders of its infinities within the portion of surface in question; *i.e.*,

$$\frac{1}{2\pi i}\int\frac{d\log f(z, s)}{dz}\,dz = \Sigma\mu - \Sigma\lambda.$$

As the portion of surface we shall take the surface T' which is bounded by the canals a and b, and for $f(z, s)$ we have here $\overline{\Psi}(z, s)$. There being no infinities, $\Sigma\lambda = 0$, and consequently

$$\frac{1}{2\pi i}\int\frac{d\log\overline{\Psi}(z, s)}{dz}\,dz,$$

where the integration taken over both banks of the canals a and b is equal to the sum of the orders of the zeros in T'.

Now *on the canal* a we have $\overline{u}(\lambda) - \overline{u}(\rho) = A$,

or $$\overline{u}(\lambda) = \overline{u}(\rho) + A.$$

It follows that

$$\Phi[\overline{u}(\lambda)] = \Phi[\overline{u}(\rho) + A] = e^{-\frac{\pi ik}{B}\{2u(\rho)+A)\}}\Phi[\overline{u}(\rho)],$$

and consequently that

$$\text{(M)}\qquad \frac{\Phi[\overline{u}(\lambda)]}{\Phi[\overline{u}(\rho)]} = \frac{\overline{\Psi}(\overset{(\lambda)}{z, s})}{\overline{\Psi}(\overset{(\rho)}{z, s})} = e^{-\frac{\pi ik}{B}[2\overline{u}(\rho)+A]}.$$

On the canal b we have $\overline{u}(\rho) - \overline{u}(\lambda) = B$,

or $$\overline{u}(\lambda) = \overline{u}(\rho) - B.$$

It follows that

$$\Phi[\overline{u}(\lambda)] = \Phi[\overline{u}(\rho) - B] = \Phi[\overline{u}(\rho)],$$

or

$$\text{(N)}\qquad \frac{\Phi[\overline{u}(\lambda)]}{\Phi[\overline{u}(\rho)]} = \frac{\overline{\Psi}(\overset{(\lambda)}{z, s})}{\overline{\Psi}(\overset{(\rho)}{z, s})} = 1.$$

From the figure in Art. 142 it is seen that

$$\begin{aligned}\int_{T'}\frac{d\log\overline{\Psi}(z, s)}{dz}\,dz &= \int_{\alpha\gamma\beta \text{ on } +a}^{(\rho)}\frac{d\log\overline{\Psi}(z, s)}{dz}\,dz + \int_{\beta\delta\beta' \text{ on } -b}^{(\lambda)}\frac{d\log\overline{\Psi}(z, s)}{dz}\,dz\\ &+ \int_{\beta'\gamma'\alpha' \text{ on } -a}^{(\lambda)}\frac{d\log\overline{\Psi}(z, s)}{dz}\,dz + \int_{\alpha'\delta'\alpha \text{ on } +b}^{(\rho)}\frac{d\log\overline{\Psi}(z, s)}{dz}\,dz\\ &= \int_{+a}\frac{d}{dz}\left[\log\frac{\overline{\Psi}(\overset{(\rho)}{z, s})}{\overline{\Psi}(\overset{(\lambda)}{z, s})}\right]dz + \int_{+b}\frac{d}{dz}\left[\log\frac{\overline{\Psi}(\overset{(\rho)}{z, s})}{\overline{\Psi}(\overset{(\lambda)}{z, s})}\,dz\right],\end{aligned}$$

which owing to (M) and (N)

$$\begin{aligned}&= \int_{+a}\frac{d}{dz}\left\{\frac{\pi ik}{B}[2\,\overline{u}(\rho) + A]\right\}dz = \frac{2\pi ik}{B}\int_{+a}\frac{d}{dz}[\overline{u}(\rho)]dz\\ &= \frac{2\pi ik}{B}\int_{+a}\frac{dz}{\sqrt{R(z)}}.\end{aligned}$$

But from Art. 139 it is seen that

$$\int_{a} \frac{dz}{\sqrt{R(z)}} = B.$$

We therefore have finally

$$\frac{1}{2\pi i}\int \frac{d \log \overline{\Psi}(z, s)}{dz}\, dz = \Sigma\mu = k.$$

It is thus seen that *the intermediary function* $\overline{\Psi}(z, s)$ has k zeros on the surface T'; and since $\Psi(z, s)$ vanishes on the same points as $\overline{\Psi}(z, s)$, it follows that $\Psi(z, s)$ has k zeros on the Riemann surface T.

ART. 148. We saw (Art. 87) that when $k = 2$

$$\Phi(u + a) = \Phi(u),$$

$$\Phi(u + b) = e^{-\frac{2\pi i}{a}(2u+b)}\Phi(u).$$

Further, write $Q = q^{\frac{1}{4}}$, and it follows that

$$\Phi_0(u) = \Theta_1(u) = \sum_{\mu=-\infty}^{\mu=+\infty} q^{\mu^2} e^{\frac{4\pi i}{a}\mu u},$$

$$\Phi_1(u) = \mathrm{H}_1(u) = \sum_{\mu=-\infty}^{\mu=+\infty} q^{\left(\frac{2\mu+1}{2}\right)^2} e^{\frac{2\pi i}{a}(2\mu+1)u}.$$

If in $\Theta_1(u)$ we write $-\mu$ in the place of μ, the summation is not thereby changed, and we have

$$\Theta_1(u) = \sum_{\mu=-\infty}^{\mu=+\infty} q^{\mu^2} e^{-\frac{4\pi i}{a}\mu u}.$$

From this it is seen that $\Theta_1(u) = \Theta_1(-u)$, or $\Theta_1(u)$ *is an even function.* Similarly writing $-\mu-1$ for μ in the formula for $\mathrm{H}_1(u)$ we have

$$\mathrm{H}_1(u) = \sum_{\mu=-\infty}^{\mu=+\infty} q^{\left(\frac{2\mu+1}{2}\right)^2} e^{-\frac{2\pi i}{a}(2\mu+1)u},$$

or $\mathrm{H}_1(u) = \mathrm{H}_1(-u)$, *so that this function is also even.*

ART. 149. If in $\Theta_1(u)$ we write $u(z, s)$ instead of u, then $\Theta_1(u)$ becomes $\Psi_0(z, s) = \overline{\Psi}_0(z, s) \cdot e^{-\frac{2\pi i}{B}[2m\bar{u}(z,s)+m^2 A]}$ (cf. Art. 140).

Suppose that, starting from a point z_0, s_0 in the upper leaf of the Riemann surface T', a path of integration is taken to the point z, s, which may cross the canals $\boldsymbol{a}$ and $\boldsymbol{b}$ as often as we choose. The point z, s may lie in either the upper or the lower leaf. Next starting from the point z_0, $-s_0$, which lies immediately under the point z_0, s_0, let us construct a second path, which is everywhere congruent to the first path, that is, which lies in the under

leaf when the first path is in the upper, and is in the upper leaf when the first path is in the under. If further we form the integral of the first kind $u(z, s)$ for each of these two paths, and add the two integrals, it is seen that the elements of integration cancel in pairs, so that

$$\underset{(\mathrm{I})}{\int_{z_0, s_0}^{z, s}} \frac{dz}{s} + \underset{(\mathrm{II})}{\int_{z_0, -s_0}^{z, -s}} \frac{dz}{s} = 0,$$

where (I) and (II) are used to denote the paths of integration. Suppose that z_0, s_0 coincides with one of the branch-points, for example with a_1, then z_0, s_0 and z_0, $-s_0$ coincide, and we have

$$\underset{(\mathrm{I})}{\int_{a_1}^{z, s}} \frac{dz}{s} + \underset{(\mathrm{II})}{\int_{a_1}^{z, -s}} \frac{dz}{s} = 0,$$

or
$$\bar{u}(z, s) + gA + hB + u(z, -s) + g'A + h'B = 0,$$

where g, g', h, h' denote integers.

It follows that
$$\bar{u}(z, s) + \bar{u}(z, -s) = \gamma A + \delta B,$$

where γ and δ are integers.

If then we take a branch-point as the initial point of the path of integration, the function $\bar{u}(z, s)$ *has at two points situated the one over the other in the Riemann surface* T', *values whose sum is equal to integral multiples of* A *and* B.

ART. 150. If we write $\bar{u}(z, s)$ for u in $\Theta_1(u)$, we have the function $\bar{\Psi}_0(z, s)$; similarly let $\bar{\Psi}_1(z, s)$ denote the result of substituting $\bar{u}(z, s)$ for u in $\mathrm{H}_1(u)$. Then noting the relations existing between Ψ_0, $\bar{\Psi}_0$ and between Ψ_1 and $\bar{\Psi}_1$, it is seen (cf. Art. 141) that

$$\frac{\Psi_0(z, s)}{\Psi_1(z, s)} = \frac{\bar{\Psi}_0(z, s)}{\bar{\Psi}_1(z, s)} = R(z, s),$$

where $R(z, s)$ denotes a rational function of its arguments.

It will be shown in the following Chapter that

$$R(z, s) = g(z) + s \cdot h(z),$$

where $g(z)$ and $h(z)$ are rational functions of z alone.

We form next
$$R(z, -s) = \frac{\Psi_0(z, -s)}{\Psi_1(z, -s)} = \frac{\Theta_1[\bar{u}(z, -s)]}{\mathrm{H}_1[\bar{u}(z, -s)]}$$
$$= \frac{\Theta_1[-\bar{u}(z, s) + \gamma A + \delta B]}{\mathrm{H}_1[-\bar{u}(z, s) + \gamma A + \delta B]} = \frac{\Theta_1[-\bar{u}(z, s)]}{\mathrm{H}_1[-\bar{u}(z, s)]},$$

as is seen from the functional equations which Θ_1 and H_1 satisfy. Since Θ_1 and H_1 are even functions, it follows that

$$R(z, -s) = \frac{\Theta_1[\bar{u}(z, s)]}{\mathrm{H}_1[\bar{u}(z, s)]} = R(z, s).$$

We therefore have

$$g(z) - s \cdot h(z) = g(z) + s \cdot h(z),$$

and consequently

$$s \cdot h(z) = 0.$$

Since s is not identically zero, we must have

$$h(z) = 0;$$

and finally

$$R(z, s) = g(z),$$

or $R(z, s)$ is a rational function of z alone.

ART. 151. Since $k = 2$, it follows that H_1 and Θ_1 have two zeros of the first order on the Riemann surface; and since the quotient of these two functions is a rational function of z, it is evident that

$$\text{(M)} \qquad \frac{\mathrm{H}_1(u)}{\Theta_1(u)} = \frac{A_1 z + A_2}{A_3 z + A_4},$$

where the A's are constants. This function has the two zeros of the first order

$$z = -\frac{A_2}{A_1}, \quad s = \pm\sqrt{R\left[-\frac{A_2}{A_1}\right]},$$

and the two infinities

$$z = -\frac{A_4}{A_3}, \quad s = \pm\sqrt{R\left[-\frac{A_4}{A_3}\right]}.$$

Remark.—If the zero $z = -\dfrac{A_2}{A_1}$ is a branch-point, say a_1, then (see Art. 120) *twice* the exponent of the lowest power of $z - a_1 = z + \dfrac{A_2}{A_1}$ in the development in ascending powers of $z - a_1$ is the order of the zero. But as the development of the numerator of the above expression is simply $A_1\left[z + \dfrac{A_2}{A_1}\right]$, it is seen that 2 is the order of the zero for $z = -\dfrac{A_2}{A_1}$. Such a zero is therefore to be counted as two zeros of the first order. The case where $-\dfrac{A_4}{A_3}$ is a branch-point may be treated in an analogous manner.

ART. 152. It follows directly from equation (M) above that

$$z = \frac{A_2\Theta_1(u) - A_4\mathrm{H}_1(u)}{A_3\mathrm{H}_1(u) - A_1\Theta_1(u)},$$

from which it is seen that z is a *one-valued doubly periodic function of u with periods A and B.* We call *z the inverse of the elliptic integral u, where*

$$u = \int^z \frac{dz}{\sqrt{A(z - a_1)(z - a_2)(z - a_3)(z - a_4)}}.$$

Although u is not a one-valued function of z (Art. 139), the *inverse function z is one-valued in u.* The constant A under the radical is of course not the same constant as the period A.

We may also note that

$$s = \frac{dz}{du}$$

is a one-valued function of u; for the derivative of a one-valued doubly periodic function is one-valued and doubly periodic.

ART. 153. The following remarks of Lejeune Dirichlet (*Gedächtnissrede auf Jacobi;* Jacobi's Werke, Bd. I, pp. 9 and 10) are instructive and historical:

"Es ist Legendres unvergänglicher Ruhm in den eben erwähnten Entdeckungen die Keime eines wichtigen Zweiges der Analysis erkannt und durch die Arbeit eines halben Lebens auf diesen Grundlagen eine selbständige Theorie errichtet zu haben, welche alle Integrale umfasst, in denen keine andere Irrationalität enthalten ist als eine Quadratwurzel, unter welcher die Veränderliche den 4ten Grad nicht übersteigt. Schon Euler hatte bemerkt, mit welchen Modificationen sein Satz auf solche Integrale ausgedehnt werden kann; Legendre, indem er von dem glücklichen Gedanken ausging, alle diese Integrale auf feste canonische Formen zurückzuführen, gelangte zu der für die Ausbildung der Theorie so wichtig gewordenen Erkenntniss, dass sie in drei wesentlich verschiedene Gattungen zerfallen. Indem er dann jede Gattung einer sorgfältigen Untersuchung unterwarf, entdeckte er viele ihrer wichtigsten Eigenschaften, von welchen namentlich die, welche der dritten Gattung zukommen, sehr verborgen und umgemein schwer zugänglich waren. Nur durch die ausdaurerndste Beharrlichkeit, die den grossen Mathematiker immer von neuem auf den Gegenstand zurückkommen liess, gelang es ihm hier Schwierigkeiten zu besiegen, welche mit den Hülfsmitteln, die ihm zu Gebote standen, kaum überwindlich sheinen mussten. . . .

"Während die früheren Bearbeiter dieses Gegenstandes das elliptische Integral der ersten Gattung als eine Function seiner Grenze ansahen, erkannten Abel und Jacobi unabhängig von einander, wenn auch der erstere einige Monate früher, die Nothwendigkeit die Betrachtungsweise umzukehren und die Grenze nebst zwei einfachen von ihr abhängigen Grössen, die so unzertrennlich mit ihr verbunden sind wie der Sinus zum Cosinus gehört, als Functionen des Integrals zu behandeln, gerade wie man schon früher zur Erkenntniss der wichtigsten Eigenschaften der vom Kreise abhängigen Transcendenten gelangt war, indem man den Sinus und Cosinus als Functionen des Bogens und nicht diesen als eine Function von jenen betrachtete.

"Ein zweiter Abel und Jacobi gemeinsamer Gedanke, der Gedanke das Imaginäre in diese Theorie einzuführen, war von noch grösserer Bedeutung und Jacobi hat es später oft wiederholt, dass die Einführung des Imaginären allein alle Räthsel der früheren Theorie gelöst habe."

ART. 154. If we had not wished to study the one-valued functions of position on the Riemann surface $s = \sqrt{R(z)}$, we might have shown immediately that

$$\left(\frac{dz}{du}\right)^2 = R(z).$$

For in the differential equation (cf. Art. 106)

$$\left(\frac{dz}{du}\right)^2 + A_1(z)\frac{dz}{du} + A_0(z) = 0,$$

or

$$A_0(z)\left(\frac{du}{dz}\right)^2 + A_1(z)\frac{du}{dz} + 1 = 0,$$

when a definite value is given to z, say z_0, then the sum of the two roots of the equation is

$$\left(\frac{du}{dz}\right)_1 + \left(\frac{du}{dz}\right)_2 = -\frac{A_1(z_0)}{A_0(z_0)}. \tag{i}$$

On the other hand, corresponding to the value z_0 there are within the initial period-parallelogram two values of u say u_1 and u_2. Also, since $u_1 + u_2 =$ Constant, it follows that

$$\frac{du_1}{dz} + \frac{du_2}{dz} = 0. \tag{ii}$$

But the left-hand side of (i) is the same as the left-hand side of (ii), and consequently* $A_1(z) = 0$.

ART. 155. *A Theorem due to Liouville.* Suppose that $w = F(u)$ is a doubly periodic function of the kth order with periods a and b; also let $z = f(u)$ be a doubly periodic function of the second order with the same periods. There exists then (see Art. 104) an integral algebraic equation of the form

$$G(w, z) = 0,$$

which is of the second degree in w and of the kth degree in z.

This equation may be written

$$Lw^2 + 2Pw + Q = 0,$$

L, P and Q being integral functions of degree not greater than k in z.

It follows that

$$w = \frac{-P \pm\sqrt{P^2 - LQ}}{L} = \frac{-P + \sigma}{L},$$

where

$$\sigma = \pm\sqrt{P^2 - LQ}.$$

We therefore have

$$\sigma = Lw + P,$$

so that σ is a one-valued function of w.

We saw above that

$$\frac{dz}{du} = \pm\sqrt{R(z)}.$$

* Cf. Harkness and Morley, *Theory of Functions*, p. 293, where numerous other references are given.

It is also seen that corresponding to one value of z there are two values of σ differing only in sign, and corresponding to this same value of z there are two values of $\frac{dz}{du}$ which differ only in sign.

Hence $T(z) = \sigma \div (dz/du)$ is a *one-valued* function of z with periods a and b. It follows also (see Art. 104) that an algebraic equation exists between $\sigma \div (dz/du)$ and z; and consequently $\sigma \div (dz/du)$ is indeterminate for no value of u. But a one-valued function which has no essential singularity is a rational function (Chapter I). Hence $T(z)$ is a rational function of z.

It is also seen that

$$w = \frac{-P + T(z)\dfrac{dz}{du}}{L} = p + qs,$$

p and q being rational functions of z.

We have thus shown that w may be expressed rationally in z and $s = \frac{dz}{du}$;* or $w = R(z, s)$, *which theorem is due to Liouville.*

ART. 156. *A Theorem of Briot and Bouquet* (*Fonctions Elliptiques*, p. 278). Suppose that $w = F(u)$ is a doubly periodic function of the kth order with primitive periods a and b and let $t = f_1(u)$ denote any other doubly periodic function with the same periods. We shall show that t is a rational function of w and $\frac{dw}{du}$.

There exists (Art. 104) between w and $w' = \frac{dw}{du}$ an integral algebraic equation

(I) $$G(w, w') = 0,$$

which is of the kth degree in w'.

Hence corresponding to one value of w there correspond in general k values of w' in a period-parallelogram. Suppose that for the value w_0 there correspond the k values

$$w_1', w_2', \ldots, w_k'. \tag{1}$$

Further, since w is of the kth order, there correspond k values of u to w_0 in the period-parallelogram, say

$$u_1, u_2, \ldots, u_k. \tag{2}$$

We also know that between the functions t and w there is an algebraic equation

(II) $$G_1(w, t) = 0$$

of the kth degree in t, so that corresponding to the value w_0 there are k values of t, say

$$t_1, t_2, \ldots, t_k. \tag{3}$$

* Liouville, *Crelle*, Bd. 88, p. 277, and *Comptes Rendus*, t. 32, p. 450.

We note that the system of values (3) correspond to the system of values (1) in such a way that to every system of values (w, w') there corresponds one definite value of t and only one.

The functions

$$tw', tw'^2, \ldots, tw'^{k-1}$$

enjoy the same property.

It follows that the sums

$$\begin{array}{l}
t_1 + t_2 + t_3 + \cdots + t_k = P_0, \\
t_1w_1' + t_2w_2' + t_3w_3' + \cdots + t_kw_k' = P_1, \\
t_1w_1'^2 + t_2w_2'^2 + t_3w_3'^2 + \cdots + t_kw_k'^2 = P_2, \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
t_1w_1'^{k-1} + t_2w_2'^{k-1} + t_3w_3'^{k-1} + \cdots + t_kw_k'^{k-1} = P_{k-1}
\end{array}$$

are one-valued functions of w, and have *definite values* for all values of w on the Riemann surface. They are therefore rational functions of w.

ART. 157. If we multiply the above equations respectively by

$$A_{k-1}, A_{k-2}, \ldots A_1, 1,$$

add the results and equate to zero the coefficients of

$$t_2, t_3, \ldots, t_k,$$

we will have the system of k equations:

$$\left.\begin{array}{l}
w_2'^{k-1} + A_1w_2'^{k-2} + A_2w_2'^{k-3} + \cdots + A_{k-2}w_2' + A_{k-1} = 0, \\
w_3'^{k-1} + A_1w_3'^{k-2} + A_2w_3'^{k-3} + \cdots + A_{k-2}w_3' + A_{k-1} = 0, \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
w_k'^{k-1} + A_1w_k'^{k-2} + A_2w_k'^{k-3} + \cdots + A_{k-2}w_k' + A_{k-1} = 0;
\end{array}\right\} \quad (4)$$

and the additional equation

$$\begin{aligned}
t_1[w_1'^{k-1} + A_1w_1'^{k-2} + A_2w_1'^{k-3} + \cdots + A_{k-2}w_1' + A_{k-1}] \\
= P_{k-1} + A_1P_{k-2} + \cdots + A_{k-2}P_1 + A_{k-1}P_0. \qquad (5)
\end{aligned}$$

The equations (4) show that the quantities $A_1, A_2, \ldots, A_{k-1}$ are the coefficients of an algebraic equation of the $k-1$st degree whose roots are $w_2', w_3', \ldots, w_k'$.

We obtain this equation by dividing (I) arranged in decreasing powers of w' by $w' - w_1'$. The coefficients of the quotient, which are integral functions of w_0 and w_1', will give the quantities $A_1, A_2, \ldots, A_{k-1}$.

From equation (5) *we have t expressed as a rational function of w and w'.*

This theorem is a generalization of Liouville's Theorem above. In Chapter XX we shall again prove indirectly both theorems.

ART. 158. We shall prove in Chapter XVI that the doubly periodic function of the second order $z = \phi(u)$ is such that $\phi(u + v)$ may be expressed rationally in terms of $\phi(u)$, $\phi'(u)$, $\phi(v)$, $\phi'(v)$, say

$$\phi(u + v) = R_1[\phi(u), \phi'(u), \phi(v), \phi'(v)], \tag{1}$$

where R with a suffix denotes a rational function, and consequently also

$$\phi'(u + v) = R_2[\phi(u), \phi'(u), \phi(v), \phi'(v)]. \tag{2}$$

For the present admit the above statements.

By Liouville's Theorem it follows that $w = F(u)$ is a rational function of $\phi(u)$ and $\phi'(u)$, or $\quad F(u) = R_3[\phi(u), \phi'(u)].$

We consequently have

$$\begin{aligned} F(u + v) &= R_3[\phi(u + v), \phi'(u + v)] \\ &= R_4[\phi(u), \phi'(u), \phi(v), \phi'(v)]. \end{aligned} \tag{3}$$

Also from Briot and Bouquet's Theorem

$$\phi(u) = R_5[F(u), F'(u)]$$

and

$$\phi'(u) = R_6[F(u), F'(u)].$$

Hence from (3) we see that

$$F(u + v) = R_7[F(u), F'(u), F(v), F'(v)].$$

It has therefore been proved, since w satisfies the latent test expressed by the eliminant equation, that this function has an algebraic addition-theorem, and *in fact is such** *that* $F(u + v)$ *may be expressed rationally in terms of* $F(u)$, $F'(u)$, $F(v)$, $F'(v)$.

This property, see Chapter II, also belongs to the rational functions and to the simply periodic functions.

It has thus been demonstrated that *to any one-valued function* $\phi(u)$ *which has everywhere in the finite portion of the plane the character of an integral or (fractional) rational function, belongs the property that* $\phi(u + v)$ *is rationally expressible through* $\phi(u)$, $\phi'(u)$, $\phi(v)$, $\phi'(v)$. As it was shown in Art. 74 that a one-valued analytic function cannot have more than two periods, it follows (cf. also Art. 41) that a one-valued analytic function which has an algebraic addition-theorem is either

I, a rational function of u,

II, a rational function of $e^{\frac{\pi i u}{\omega}}$,

III, a rational function of z and $\dfrac{dz}{du}$.

The first two cases (Art. 41) are limiting cases of the third. Every *transcendental* one-valued analytic function which has an algebraic addition-theorem is necessarily a simply or a doubly periodic function.

* See Schwarz, *Ges. Math. Abhandl.*, Vol. II, p. 265.

ART. 159. We have seen that any rational function of z and s is a one-valued function of position on the Riemann surface s. Hence the function w of the preceding article, which is the most general one-valued doubly periodic function, is a one-valued function of position on the Riemann surface.* The quantity s is the root of the algebraic equation

$$s^2 - R(z) = 0,$$

and by *adjoining* this algebraic quantity to the realm of rational quantities (z) we have the more extended realm (z, s) composed of all rational functions of both z and s. This latter realm includes the former. Since all functions of the realm (z, s) are one-valued functions of position on the Riemann surface T and since this surface is of *deficiency* or *order* unity, we may say the realm (s, z), the *elliptic realm*, is of the *first* order, the realm of rational functions (z) being of the *zero* order.

We thus see that the study of functions belonging to the realm of order unity is coincident with the study of the doubly periodic functions and in fact the study of one necessitates the study of the other.

The elliptic or doubly periodic realm (s, z), where

$$s = \sqrt{A\,(z - a_1)(z - a_2)(z - a_3)(z - a_4)} = \frac{dz}{du},$$

degenerates into the simply periodic realm when any pair of branch-points are equal and into the realm of rational functions (z) when two pairs of branch-points are equal (including of course the case where all the branch-points are equal).

Thus the elliptic realm (z, s) includes the three classes of one-valued functions:

First, the rational functions,
Second, the simply periodic functions,
Third, the doubly periodic functions.

All these functions, and only these, have algebraic addition-theorems. In other words, *all functions of the realm (z, s) have algebraic addition-theorems, and no one-valued function that does not belong to this realm has an algebraic addition-theorem.* We have thus proved that *the one-valued functions of position on the Riemann surface*

$$s^2 = R(z),$$

R denoting an integral function of the third or fourth degree in z, belong to the closed realm (z, s) of order unity, and all elements of this realm and no others have algebraic addition-theorems.

* Cf. Klein, *Theorie der elliptischen Modulfunctionen*, Bd. I, pp. 147 and 539.

CHAPTER VIII

ELLIPTIC INTEGRALS IN GENERAL

The three kinds of elliptic integrals. Normal forms.

ARTICLE 160. At the end of the last Chapter we saw that the most general elliptic function could be expressed as a rational function of z, s. We shall now consider the integral of such an expression.*

Let $R_1(z, s)$ denote a rational function of z, s. This function may be written in the form

$$R_1(z, s) = \frac{A_0 + A_1 s + A_2 s^2 + \cdots + A_k s^k}{B_0 + B_1 s + B_2 s^2 + \cdots + B_l s^l},$$

where the A's and B's are integral functions of z. Owing to the relation $s^2 = A(z - a_1)(z - a_2)(z - a_3)(z - a_4)$, it is seen that the even powers of s are integral functions of z, while the odd powers of s are equal to an integral function of z multiplied by s, so that

$$R_1(z, s) = \frac{A_0' + A_1' s}{B_0' + B_1' s} = \frac{(A_0' + A_1' s)(B_0' - B_1' s)}{B_0'^2 - B_1'^2 s^2}$$
$$= \frac{C + Ds}{E},$$

where C, D, and E are integral functions of z as are A_0', A_1', B_0' and B_1'.

Writing
$$\frac{C}{E} = p(z) \text{ and } \frac{D}{E} = q(z),$$
it is seen that
$$R_1(z, s) = p(z) + q(z) \cdot s = p(z) + \frac{Q(z)}{s},$$

where $q(z) \cdot s^2 = Q(z)$ and where $p(z)$, $q(z)$, and $Q(z)$ are rational functions of z. (See also Arts. 125 *et seq.*)

Consider next the integral

$$\int R_1(z, s) dz = \int p(z) dz + \int \frac{Q(z)}{s} dz.$$

The first integral on the right may be reduced at once to elementary integrals, so that we may confine our attention to the integral

$$\int \frac{Q(z)}{s} dz, \text{ which may be written} \int \frac{f(z)}{\sqrt{R(z)}} dz,$$

$f(z)$ denoting a rational function of z, and $s = \sqrt{R(z)}$.

* Legendre, *Mémoire sur les transcendantes elliptiques*, 1794. See also Legendre, *Fonctions Elliptiques*, t. I, Chap. I.

ART. 161. Suppose * in general that

$$R(z) = C_0 z^n + C_1 z^{n-1} + \cdots + C_n,$$

where the C's are constants. When n is greater than 4, the integral

$$\int \frac{f(z)}{\sqrt{R(z)}}\, dz$$

is no longer an *elliptic* but a *hyperelliptic* integral; when $n = 3$ or 4 we have the elliptic integrals, and when $n = 2$ we have the integrals that are connected with the circular functions.

The rational function $f(z)$ may be written

$$f(z) = \frac{g_1(z)}{g(z)} = G(z) + \frac{G_1(z)}{g(z)},$$

the g's and G's denoting integral functions, and say

$$g(z) = B(z - b_1)^{\lambda_1}(z - b_2)^{\lambda_2}(z - b_3)^{\lambda_3} \cdots .$$

Hence when resolved into partial fractions

$$f(z) = G(z) + \sum_i \frac{A_{\lambda_i}}{(z - b_i)^{\lambda_i}}, \quad (A_{\lambda_i} \text{ constants}),$$

and also

$$\int \frac{f(z)dz}{\sqrt{R(z)}} = \int \frac{G(z)}{\sqrt{R(z)}}\, dz + \sum_i A_{\lambda_i} \int \frac{dz}{(z - b_i)^{\lambda_i}\sqrt{R(z)}}.$$

Since $G(z)$ is an integral function, the first integral on the right-hand side may be resolved into a number of integrals of the form

$$\int \frac{z^k}{\sqrt{R(z)}}\, dz.$$

We thus have two general types of integrals to consider,

$$I_k = \int \frac{z^k}{\sqrt{R(z)}}\, dz$$

and

$$H_k = \int \frac{dz}{(z-b)^k \sqrt{R(z)}}.$$

ART. 162. Form the expression

$$\frac{d}{dz}\left[z^k \sqrt{R(z)}\right] = kz^{k-1}\sqrt{R(z)} + \frac{1}{2}\frac{R'(z)}{\sqrt{R(z)}} z^k = \frac{z^{k-1}}{2\sqrt{R(z)}}[2kR(z) + zR'(z)]$$

$$= \frac{z^{k-1}}{2\sqrt{R(z)}}[(2k+n)C_0 z^n + (2k+n-1)C_1 z^{n-1} + (2k+n-2)C_2 z^{n-2} + \cdots$$
$$+ (2k+1)C_{n-1} z + 2kC_n].$$

* Briot et Bouquet, *Fonctions Elliptiques*, p. 436; see also Koenigsberger, *Elliptische Functionen*, p. 260; Appell et Lacour, *Fonctions Elliptiques*, p. 235.

It follows through integration that

$$2z^k\sqrt{R(z)} = (2k+n)C_0I_{k+n-1} + (2k+n-1)C_1I_{k+n-2} + \cdots$$
$$+ (2k+1)C_{n-1}I_k + 2kC_nI_{k-1}.$$

If in this expression we put $k = 0$, it is seen that I_{n-1} may be expressed through $I_{n-2}, I_{n-3}, \ldots, I_0, I_{-1}$ and through the function $\sqrt{R(z)}$; when k is put $= 1$, we may express I_n through $I_{n-1}, I_{n-2}, \ldots, I_0$ and through the function $z\sqrt{R(z)}$. If further we write for I_{n-1} its value, we may express I_n through $I_{n-2}, I_{n-3}, \ldots, I_0, I_{-1}$ and an algebraic function. This algebraic function is an integral function of the first degree in z multiplied by $\sqrt{R(z)}$.

Continuing in the same manner, we may express $I_{n+\lambda}$ through $I_{n-2}, I_{n-3}, \ldots, I_0, I_{-1}$ and an algebraic function which is an integral function of the $\lambda + 1$ degree in z multiplied by $\sqrt{R(z)}$.

ART. 163. We consider next the integrals of the type H_k. Form the expression

$$\frac{d}{dz}\left[\frac{\sqrt{R(z)}}{(z-b)^k}\right] = -\frac{k}{(z-b)^{k+1}}\sqrt{R(z)} + \frac{1}{2}\frac{R'(z)}{(z-b)^k\sqrt{R(z)}}$$
$$= \frac{1}{2\sqrt{R(z)}(z-b)^{k+1}}[-2kR(z) + R'(z)(z-b)].$$

If we write $$-2kR(z) + R'(z)(z-b) = \phi(z),$$
then is
$$\frac{d^\nu\phi(z)}{dz^\nu} = -2kR^{(\nu)}(z) + R^{(\nu+1)}(z)(z-b) + \nu R^{(\nu)}(z),$$
or
$$\phi^{(\nu)}(z) = (\nu - 2k)R^{(\nu)}(z) + (z-b)R^{(\nu+1)}(z).$$

It follows, since
$$\phi(z) = \phi(b) + \frac{z-b}{1!}\phi'(b) + \frac{(z-b)^2}{2!}\phi''(b) + \cdots,$$
that
$$\phi(z) = -2kR(b) + \frac{1-2k}{1!}R'(b)(z-b) + \frac{2-2k}{2!}R''(b)(z-b)^2$$
$$+ \frac{3-2k}{3!}R^{(3)}(b)(z-b)^3 + \cdots + \frac{n-1-2k}{(n-1)!}R^{(n-1)}(b)(z-b)^{n-1}$$
$$+ \frac{n-2k}{n!}R^{(n)}(b)(z-b)^n.$$

We therefore have

$$\frac{d}{dz}\left[\frac{\sqrt{R(z)}}{(z-b)^k}\right] = \frac{1}{2\sqrt{R(z)}(z-b)^{k+1}}\left\{-2kR(b) + \frac{1-2k}{1!}R'(b)(z-b) + \cdots\right.$$
$$\left. + \frac{n-1-2k}{(n-1)!}R^{(n-1)}(b)(z-b)^{n-1} + \frac{n-2k}{n!}R^{(n)}(b)(z-b)^n\right\}.$$

Integrating it is seen that

$$\frac{2\sqrt{R(z)}}{(z-b)^k} = -2kR(b)H_{k+1} + \frac{1-2k}{1!}R'(b)H_k + \frac{2-2k}{2!}R''(b)H_{k-1} + \cdots$$
$$+ \frac{n-1-2k}{(n-1)!}R^{(n-1)}(b)\,H_{k-n+2} + \frac{n-2k}{n!}R^{(n)}(b)\,H_{k-n+1}.$$

If we put $k = 1$, we see that H_2 may be expressed through

$$H_1,\ H_0,\ H_{-1},\ H_{-2},\ \ldots,\ H_{-(n-2)},\ \text{and}\ \frac{\sqrt{R(z)}}{z-b}.$$

This is correct only if $R(b) \neq 0$; *i.e.*, if b is *not* a root of the equation $R(z) = 0$. This case is for the moment excluded. We note that

$$H_0 = \int\frac{dz}{\sqrt{R(z)}} = I_0; \qquad H_{-1} = \int\frac{(z-b)\,dz}{\sqrt{R(z)}} = I_1 - bI_0;\ \cdots;$$
$$H_{-(n-2)} = \int\frac{(z-b)^{n-2}\,dz}{\sqrt{R(z)}}.$$

From this it is seen that the integrals $H_0, H_{-1}, H_{-2}, \ldots, H_{-(n-2)}$ may be expressed through integrals of the type I_k. Hence the integral H_1 alone offers something new.

We note that H_2 may be expressed through $H_1, I_0, I_1, \ldots, I_{n-2}$ and through an algebraic function of z. If we put $k = 2$, we may express H_3 through $H_2, H_1, \ldots, H_{-(n-3)}$ and through $\frac{\sqrt{R(z)}}{(z-b)^2}$; or, if for H_2 we write its value just found, H_3 may be expressed through $H_1, I_0, I_1, \ldots, I_{n-2}$ and an algebraic function of z. In general, we may express H_m through $H_1, I_0, I_1, \ldots, I_{n-2}$ and an algebraic function of z. We thus have to consider only the integrals $I_0, I_1, \ldots, I_{n-2}$ and $H_1 = I_{-1}$, since I_{-1} is a special case of H_1, viz., when $b = 0$.

If b is a root of the equation $R(z) = 0$, then the term with H_{k+1} drops out. Since $R(z)$ cannot have a double root, as otherwise it could be taken from under the root sign in $\sqrt{R(z)}$, we may in this case express H_1 through the integrals $H_0, H_{-1}, \ldots, H_{-(n-2)}, \frac{\sqrt{R(z)}}{z-b}$; and consequently through integrals of the type I_k alone.

ART. 164. We have therefore to consider the integrals

$$I_k = \int\frac{z^k dz}{\sqrt{R(z)}},$$

where $k = 0, 1, \ldots, n-2$, where n is the degree of the integral function $R(z)$, and in addition the integral

$$H_1 = \int\frac{dz}{(z-b)\sqrt{R(z)}},$$

where b is a root of the equation $g(z) = 0$. We note that there are as many integrals of the type H_1 as there are distinct roots of the equation $g(z) = 0$. The quantity b is called the *parameter* (Legendre, *Fonctions Elliptiques*, t. I, p. 18) of the integral H_1.

ART. 165. For the elliptic integrals, if $n = 4$, we have the integrals I_0, I_1, I_2, H_1; if $n = 3$, there are the integrals I_0, I_1, H_1. In the first of these cases we shall see that I_1 reduces to elementary integrals; and with Legendre we call

$$I_0 = \int \frac{dz}{\sqrt{R(z)}} \text{ an elliptic integral of the } \textit{first kind},$$

$$I_2 = \int \frac{z^2 dz}{\sqrt{R(z)}} \text{ an elliptic integral of the } \textit{second kind},$$

and

$$H_1 = \int \frac{dz}{(z-b)\sqrt{R(z)}} \text{ an elliptic integral of the } \textit{third kind}.$$

LEGENDRE'S NORMAL FORMS.

ART. 166. In the expression

$$\frac{dz}{\sqrt{R(z)}} = \frac{dz}{\sqrt{A(z-a_1)(z-a_2)(z-a_3)(z-a_4)}},$$

let us make the homographic transformation

$$z = \frac{at+b}{ct+d}.$$

It follows that

$$z - a_k = \frac{(a-ca_k)t + b - da_k}{ct+d} \qquad (k = 1, 2, 3, 4)$$

and

$$dz = \frac{ad-bc}{(ct+d)^2}.$$

We then have

$$\frac{dz}{\sqrt{R(z)}} = \frac{(ad-bc)dt}{\sqrt{A[(a-ca_1)t+b-da_1][(a-ca_2)t+b-da_2][(a-ca_3)t+b-da_3][(a-ca_4)t+b-da_4]}}.$$

We note that the expression under the root sign is not essentially changed, since we still have an integral function of the fourth degree, the branch-points, however, being different.

Legendre * conceived the idea of so determining the constants a, b, c, d that only the even powers of t remain under the root sign. If we neglect the constant A, the radicand may be written

$$[g_0t^2 + g_1t + g_2][h_0t^2 + h_1t + h_2],$$

* Legendre, *loc. cit.*, Chap. II.

where
$$g_0 = (a - ca_1)(a - ca_2),$$
$$g_1 = (a - ca_1)(b - da_2) + (a - ca_2)(b - da_1),$$
$$g_2 = (b - da_1)(b - da_2),$$
and where h_0, h_1, h_2 are had when we interchange a_1 with a_3 and a_2 with a_4 in the expression for the g's.

That the coefficients of t^3 and t disappear, we must have
$$h_0 g_1 + g_0 h_1 = 0,$$
$$g_1 h_2 + h_1 g_2 = 0.$$
These two equations are satisfied if we put
$$g_1 = 0 \text{ and } h_1 = 0.$$
From the expression $g_1 = 0$ it follows that
$$2\,ab - (ad + bc)(a_1 + a_2) + 2\,cda_1a_2 = 0;$$
and from $h_1 = 0$ we have
$$2\,ab - (ad + bc)(a_3 + a_4) + 2\,cda_3a_4 = 0.$$
These two equations may be written
$$2 - \left\{\frac{d}{b} + \frac{c}{a}\right\}(a_1 + a_2) + 2\,\frac{d}{b}\,\frac{c}{a}\,a_1a_2 = 0,$$
$$2 - \left\{\frac{d}{b} + \frac{c}{a}\right\}(a_3 + a_4) + 2\,\frac{d}{b}\,\frac{c}{a}\,a_3a_4 = 0.$$
From them we may determine $\frac{d}{b} + \frac{c}{a}$ and $\frac{d}{b} \cdot \frac{c}{a}$ considered as unknown quantities.

If $a_3 + a_4 = a_1 + a_2$ and $a_3 \cdot a_4 = a_1 \cdot a_2$, the two equations reduce to one and then we need only determine the quantities $\frac{d}{b} + \frac{c}{a}$ and $\frac{d}{b} \cdot \frac{c}{a}$ so that they satisfy the one equation. When these two quantities have been determined, the quantities $\frac{d}{b}$ and $\frac{c}{a}$ may be found from a quadratic equation.

When these conditions have all been satisfied, then in the expression
$$[g_0t^2 + g_1t + g_2][h_0t^2 + h_1t + h_2]$$
the coefficients of t in both factors drop out.

We have finally
$$\frac{dz}{\sqrt{R(z)}} = \frac{(ad - bc)dt}{\sqrt{A(g_0t^2 + g_2)(h_0t^2 + h_2)}}.$$

Legendre further wrote $\frac{g_0}{g_2} = -p^2$, $\frac{h_0}{h_2} = -q^2$

so that
$$\frac{dz}{\sqrt{R(z)}} = \frac{(ad - bc)dt}{\sqrt{Ag_2h_2(1 - p^2t^2)(1 - q^2t^2)}}.$$

If finally we write $t = \frac{\mathfrak{z}}{p}$ (the Gothic $\mathfrak{z}$ being a different variable from the italic z), we have

$$\frac{dz}{\sqrt{R(z)}} = \frac{\frac{1}{p}(ad - bc)d\mathfrak{z}}{\sqrt{Ag_2h_2\{1 - \mathfrak{z}^2\}\left\{1 - \frac{q^2}{p^2}\mathfrak{z}^2\right\}}}.$$

If we put
$$\frac{q^2}{p^2} = k^2 \quad \text{and} \quad C = \frac{ad - bc}{p\sqrt{Ag_2h_2}},$$
the above expression is

$$\frac{dz}{\sqrt{R(z)}} = \frac{Cd\mathfrak{z}}{\sqrt{(1 - \mathfrak{z}^2)(1 - k^2\mathfrak{z}^2)}}.$$

The quantity k is called the *modulus* (Legendre, *loc. cit.*, p. 14). In theoretical investigations it may take any value whatever, real or imaginary; but in the applications to geometry, physics, and mechanics we shall see in the Second Volume that it is necessary to make this modulus real and less than unity.

ART. 167. If we make the above substitutions the general integral of Art. 160

$$\int\frac{Q(z)dz}{\sqrt{R(z)}} \quad \text{becomes} \quad \int\frac{f(\mathfrak{z})d\mathfrak{z}}{\sqrt{(1 - \mathfrak{z}^2)(1 - k^2\mathfrak{z}^2)}},$$

where $f(\mathfrak{z})$ denotes a rational function of $\mathfrak{z}$. We may write this function in the form

$$f(\mathfrak{z}) = \frac{\phi(\mathfrak{z}^2) + \mathfrak{z}\phi_1(\mathfrak{z}^2)}{\psi(\mathfrak{z}^2) + \mathfrak{z}\psi_1(\mathfrak{z}^2)},$$

where ϕ, ϕ_1, ψ, ψ_1 denote integral functions. If we multiply the numerator and the denominator of this last expression by $\psi(\mathfrak{z}^2) - \mathfrak{z}\psi_1(\mathfrak{z}^2)$, it is seen that $f(\mathfrak{z}) = f_0(\mathfrak{z}^2) + \mathfrak{z}f_1(\mathfrak{z}^2)$, where f_0 and f_1 are rational functions of $\mathfrak{z}$.

The above integral correspondingly becomes

$$\int\frac{f(\mathfrak{z})d\mathfrak{z}}{\sqrt{(1-\mathfrak{z}^2)(1-k^2\mathfrak{z}^2)}} = \int\frac{f_0(\mathfrak{z}^2)d\mathfrak{z}}{\sqrt{(1-\mathfrak{z}^2)(1-k^2\mathfrak{z}^2)}} + \int\frac{\mathfrak{z}f_1(\mathfrak{z}^2)d\mathfrak{z}}{\sqrt{(1-\mathfrak{z}^2)(1-k^2\mathfrak{z}^2)}}.$$

The second integral on the right-hand side may be reduced to elementary integrals by the substitution $\mathfrak{z}^2 = \zeta$.

Proceeding as in the general case above and noting that

$$\frac{d}{d\mathfrak{z}}\left[\mathfrak{z}^{2k+1}\sqrt{(1-\mathfrak{z}^2)(1-k^2\mathfrak{z}^2)}\right] = \frac{c_0\mathfrak{z}^{2k} + c_1\mathfrak{z}^{2k+2} + c_2\mathfrak{z}^{2k+4}}{\sqrt{(1-\mathfrak{z}^2)(1-k^2\mathfrak{z}^2)}}$$

and

$$\frac{d}{d\mathfrak{z}}\left[\frac{\mathfrak{z}\sqrt{(1-\mathfrak{z}^2)(1-k^2\mathfrak{z}^2)}}{(\mathfrak{z}^2 - b)^k}\right] = \frac{a_0 + a_1(\mathfrak{z}^2 - b) + a_2(\mathfrak{z}^2 - b)^2 + a_3(\mathfrak{z}^2 - b)^3}{(\mathfrak{z}^2 - b)^{k+1}\sqrt{(1-\mathfrak{z}^2)(1-k^2\mathfrak{z}^2)}},$$

it may be shown that the integral

$$\int \frac{f_0(\mathbf{z}^2)d\mathbf{z}}{\sqrt{(1-\mathbf{z}^2)(1-k^2\mathbf{z}^2)}}$$

is dependent upon the evaluation of the integrals

$$\int \frac{d\mathbf{z}}{\sqrt{(1-\mathbf{z}^2)(1-k^2\mathbf{z}^2)}}, \quad \int \frac{\mathbf{z}^2 d\mathbf{z}}{\sqrt{(1-\mathbf{z}^2)(1-k^2\mathbf{z}^2)}},$$
$$\int \frac{d\mathbf{z}}{(\mathbf{z}^2-b)\sqrt{(1-\mathbf{z}^2)(1-k^2\mathbf{z}^2)}}.$$

These integrals are known as Legendre's *normal integrals of the first, second, and third kinds* respectively.

ART. 168. The name "*elliptic integral*" is due to the fact that such an integral appears in the rectification of an ellipse. Writing the equation of the ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the length of arc is determined through

$$s = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^x \sqrt{\frac{a^4 - (a^2 - b^2)x^2}{a^2(a^2 - x^2)}}\, dx.$$

If the numerical eccentricity is introduced:

$$s = \int_0^x \sqrt{\frac{a^2 - e^2x^2}{a^2 - x^2}}\, dx = \int_0^x \frac{a^2 - e^2x^2}{\sqrt{(a^2 - x^2)(a^2 - e^2x^2)}}\, dx.$$

If further we put $x = a \sin \phi$, it is seen that

$$s = \int_0^\phi \sqrt{1 - e^2 \sin^2 \phi}\, d\phi.$$

This is also taken as a type of normal elliptic integral of the second kind,* being in fact composed of the normal forms of the first and second kinds as above defined. Regarding the forms of the integral of the second kind see Chapter XIII.

ART. 169. If the integral which we have to consider is of the form

$$\int \frac{f(z)dz}{\sqrt{az^3 + 3bz^2 + 3cz + d}},$$

where $f(z)$ again denotes a rational function of z, we may by writing

$$z = mt + n$$

make
$$az^3 + 3bz^2 + 3cz + d = 4t^3 - g_2t - g_3,$$

where g_2 and g_3 are constants.

This is effected by writing $n = -\frac{b}{a}$, $am^3 = 4$.

* The elliptic integral of the second kind was considered by the Italian mathematician Fagnano (1700–1766) and was later recognized as a peculiar transcendent by Euler (in 1761).

The above integral then becomes

$$\int \frac{F(t)dt}{\sqrt{4\,t^3 - g_2 t - g_3}},$$

where $F(t)$ is a rational function of t. The evaluation of this integral (cf. Art. 165) depends upon that of the three typical integrals

$$\int \frac{dt}{\sqrt{4\,t^3 - g_2 t - g_3}}, \quad \int \frac{t\,dt}{\sqrt{4\,t^3 - g_2 t - g_3}}, \quad \int \frac{dt}{(t - b)\sqrt{4\,t^3 - g_2 t - g_3}},$$

which correspond to the normal forms employed by Weierstrass.

ART. 170. In the expression

$$(1) \qquad R(z) = A(z - a_1)(z - a_2)(z - a_3)(z - a_4),$$

make the homographic transformation

$$(2) \qquad z = \frac{\alpha + \beta \mathbf{z}}{1 - \mu \mathbf{z}},$$

and so determine the coefficients * that to

$$z = a_1, \quad z = a_2, \quad z = a_3, \quad z = a_4$$

correspond

$$\mathbf{z} = -\frac{1}{k}, \quad \mathbf{z} = -1, \quad \mathbf{z} = +1, \quad \mathbf{z} = +\frac{1}{k}.$$

It follows immediately from (2) that

$$(3) \qquad z - a_1 = \frac{p(1 + k\mathbf{z})}{1 - \mu \mathbf{z}}, \qquad (4) \qquad z - a_2 = \frac{q(1 + \mathbf{z})}{1 - \mu \mathbf{z}},$$

$$(5) \qquad z - a_3 = \frac{r(1 - \mathbf{z})}{1 - \mu \mathbf{z}}, \qquad (6) \qquad z - a_4 = \frac{s(1 - k\mathbf{z})}{1 - \mu \mathbf{z}},$$

where p, q, r, s are constants which may be determined as follows: In (4) write $z = a_3$, $\mathbf{z} = 1$, and in (5) put $z = a_2$, $\mathbf{z} = -1$. We thus have

$$a_3 - a_2 = \frac{2q}{1 - \mu}, \quad a_2 - a_3 = \frac{2r}{1 + \mu}.$$

Equations (4) and (5) thereby become

$$(7) \qquad \frac{z - a_2}{a_3 - a_2} = \frac{1 - \mu}{2} \cdot \frac{1 + \mathbf{z}}{1 - \mu \mathbf{z}}, \qquad (8) \qquad \frac{z - a_3}{a_2 - a_3} = \frac{1 + \mu}{2} \cdot \frac{1 - \mathbf{z}}{1 - \mu \mathbf{z}}.$$

In the same manner we derive from equations (3) and (6) the following:

$$(9) \qquad \frac{z - a_1}{a_4 - a_1} = \frac{1 - \frac{\mu}{k}}{2} \cdot \frac{1 + k\mathbf{z}}{1 - \mu \mathbf{z}} \qquad (10) \qquad \frac{z - a_4}{a_1 - a_4} = \frac{1 + \frac{\mu}{k}}{2} \cdot \frac{1 - k\mathbf{z}}{1 - \mu \mathbf{z}}.$$

* Koenigsberger, *Elliptische Functionen*, p. 271.

Equations (7) and (8) become through division

$$\frac{z - a_2}{z - a_3} = \frac{\mu - 1}{\mu + 1} \cdot \frac{1 + \mathbf{z}}{1 - \mathbf{z}}.$$

Writing in this equation $z = a_4$, $\mathbf{z} = \frac{1}{k}$, we have

$$\frac{a_4 - a_2}{a_4 - a_3} = \frac{\mu - 1}{\mu + 1} \cdot \frac{k + 1}{k - 1};$$

and similarly for the values $z = a_1$, $\mathbf{z} = -\frac{1}{k}$, the same equation gives

$$\frac{a_1 - a_2}{a_1 - a_3} = \frac{\mu - 1}{\mu + 1} \cdot \frac{k - 1}{k + 1}.$$

The quantities k and μ may be determined from the last two expressions in the form

$$(11) \qquad \left(\frac{1 - k}{1 + k}\right)^2 = \frac{a_1 - a_2}{a_1 - a_3} \cdot \frac{a_4 - a_3}{a_4 - a_2},$$

$$(12) \qquad \frac{1 + \mu}{1 - \mu} = \frac{a_1 - a_3}{a_1 - a_2} \cdot \frac{1 - k}{1 + k}.$$

From the equations (9) and (7) we have

$$\frac{dz}{d\mathbf{z}} = \frac{\left(1 - \frac{\mu^2}{k^2}\right) k (a_4 - a_1)}{2(1 - \mu\mathbf{z})^2}, \qquad \frac{dz}{d\mathbf{z}} = \frac{(1 - \mu^2)(a_3 - a_2)}{2(1 - \mu\mathbf{z})^2},$$

and consequently

$$(13) \qquad \frac{dz}{d\mathbf{z}} = \frac{\sqrt{(1 - \mu^2)\left(1 - \frac{\mu^2}{k^2}\right)} \sqrt{k(a_4 - a_1)(a_3 - a_2)}}{2(1 - \mu\mathbf{z})^2}.$$

Through the multiplication of (7), (8), (9), (10) and (13), it follows at once that

$$\int \frac{dz}{\sqrt{R(z)}} = \frac{1}{M} \int \frac{d\mathbf{z}}{\sqrt{(1 - \mathbf{z}^2)(1 - k^2\mathbf{z}^2)}},$$

where

$$M = \frac{1}{2} \sqrt{\frac{A(a_3 - a_2)(a_4 - a_1)}{k}},$$

and where z and $\mathbf{z}$ as determined from (7) and (8) are connected by the relation

$$z = \frac{a_3 + a_2}{2} + \frac{a_3 - a_2}{2} \cdot \frac{\mathbf{z} - \mu}{1 - \mu\mathbf{z}},$$

the quantities μ and k being determined from equations (12) and (13).

ART. 171. If in the equation

$$\left(\frac{1-k}{1+k}\right)^2 = \frac{a_1-a_2}{a_1-a_3}\cdot\frac{a_4-a_3}{a_4-a_2},$$

we put the right-hand side $= \tau$, then the six different anharmonic ratios which may be had by the interchange of the a's are denoted by

$$\tau, \frac{1}{\tau};\ 1-\tau, \frac{1}{1-\tau};\ \frac{\tau}{\tau-1}, \frac{\tau-1}{\tau};$$

and corresponding to each of these values there are two values of k, in all twelve values of k.

Denoting any one of these values by k, it is seen that all twelve may be expressed in the form

$$\pm k,\quad \pm\frac{1}{k},\quad \pm\left(\frac{1-\sqrt{k}}{1+\sqrt{k}}\right)^2,\quad \pm\left(\frac{1+\sqrt{k}}{1-\sqrt{k}}\right)^2,\quad \pm\left(\frac{1-i\sqrt{k}}{1+i\sqrt{k}}\right)^2,\quad \pm\left(\frac{1+i\sqrt{k}}{1-i\sqrt{k}}\right)^2.$$

(Cf. Abel, Œuvres, T. I, pp. 408, 458, 568, 603; Cayley's *Elliptic Functions*, p. 372.)

Remark. — We may make use of the above results to transform the expression

$$\frac{dz}{\sqrt{A(z-a_1)(z-a_2)(z-a_3)}}$$

into Legendre's normal form.

Noting that

$$A(z-a_1)(z-a_2)(z-a_3)$$
$$= \underset{a_4=\infty}{\text{Limit}}\left[-\frac{A}{a_4}(z-a_1)(z-a_2)(z-a_3)(z-a_4)\right],$$

we have to write in the formulas above $-\dfrac{A}{a_4}$ in the place of A, and let a_4 become infinite.

We then have

$$\int\frac{dz}{\sqrt{A(z-a_1)(z-a_2)(z-a_3)}} = \frac{1}{M}\int\frac{d\mathbf{z}}{\sqrt{(1-\mathbf{z}^2)(1-k^2\mathbf{z}^2)}},$$

where

$$M = \frac{1}{2}\sqrt{\frac{A(a_2-a_3)}{k}},$$

$$z = \frac{a_3+a_2}{2} + \frac{a_3-a_2}{2}\cdot\frac{\mathbf{z}-k}{1-k\mathbf{z}},$$

and

$$\left(\frac{1-k}{1+k}\right)^2 = \frac{a_1-a_2}{a_1-a_3},\qquad \mu = k.$$

ART. 172. In the expression

$$\sqrt{R(z)} = \sqrt{A(z-a_1)(z-a_2)(z-a_3)(z-a_4)}$$

$$= (z-a_1)^2\sqrt{A\frac{z-a_2}{z-a_1}\cdot\frac{z-a_3}{z-a_1}\cdot\frac{z-a_4}{z-a_1}},$$

write $$\frac{z-a_4}{z-a_1} = t-e, \quad \text{or} \quad z = \frac{a_1 t - a_1 e - a_4}{t-e-1},$$

$$dt = \frac{a_4 - a_1}{(z-a_1)^2}dz.$$

If we put $\sqrt{(a_1-a_2)(a_1-a_3)A} = 2M$,

we have

$$\sqrt{R(z)} = 2M\frac{(z-a_1)^2}{a_1-a_4}\sqrt{(t-e)\left\{t-\left(e-\frac{a_2-a_4}{a_1-a_2}\right)\right\}\left\{t-\left(e-\frac{a_3-a_4}{a_1-a_3}\right)\right\}}.$$

Choose e so that $$3e - \frac{a_2-a_4}{a_1-a_2} - \frac{a_3-a_4}{a_1-a_2} = 0.$$

Let e_1 be the value of e that satisfies this equation, and write

$$e_2 = e_1 - \frac{a_2-a_4}{a_1-a_2} \quad \text{and} \quad e_3 = e_1 - \frac{a_3-a_4}{a_1-a_3}.$$

We finally have

$$\frac{dz}{\sqrt{R(z)}} = \frac{1}{2M}\frac{dt}{\sqrt{(t-e_1)(t-e_2)(t-e_3)}}$$

$$= \frac{1}{M}\frac{dt}{\sqrt{4t^3-g_2t-g_3}},$$

where $$e_1e_2 + e_2e_3 + e_3e_1 = -\tfrac{1}{4}g_2,$$

$$e_1e_2e_3 = \tfrac{1}{4}g_3.$$

It also follows that

$$\int\frac{P(z)dz}{\sqrt{R(z)}} = \int\frac{p(t)dt}{\sqrt{4t^3-g_2t-g_3}},$$

where P and p denote rational functions.

The quantities g_2 and g_3 which occur in Weierstrass's normal form are called *invariants*, their invariantive character being especially evidenced in the *Theory of Transformation*. We may now consider more carefully their meaning.

ART. 173. Write $$u = \int\frac{dz}{\sqrt{R(z)}},$$

where the function $R(z)$ may be written

$$R(z) = a_0z^4 + 4a_1z^3 + 6a_2z^2 + 4a_3z + a_4.$$

Write
$$z = \frac{x_1}{x_2},$$
$$dz = \frac{x_2 dx_1 - x_1 dx_2}{x_2^2},$$
where the variables x_1, x_2 individually are not determined, but only their quotient.

We then have
$$R(z) = \frac{1}{x_2^4}\{a_0 x_1^4 + 4a_1 x_1^3 x_2 + 6a_2 x_1^2 x_2^2 + 4a_3 x_1 x_2^3 + a_4 x_2^4\}$$
$$= \frac{1}{x_2^4} f(x_1, x_2).$$

It is seen that $f(x_1, x_2)$ is a *binary form** of the fourth degree. We have at once
$$\frac{dz}{\sqrt{R(z)}} = \frac{x_2 dx_1 - x_1 dx_2}{\sqrt{f(x_1, x_2)}}.$$

If next we write
$$x_1 = ay_1 + by_2,$$
$$x_2 = cy_1 + dy_2,$$
$$z = \frac{ay_1 + by_2}{cy_1 + dy_2},$$
it is seen that $f(x_1, x_2)$ becomes another binary form $\phi(y_1, y_2)$ of the fourth degree.

ART. 174. In general make the above substitutions in the binary form of the nth degree
$$f(x_1, x_2) = a_0 x_1^n + n_1 a_1 x_1^{n-1} x_2 + n_2 a_2 x_1^{n-2} x_2^2 + \cdots + a_n x_2^n,$$
where n_1, n_2, . . . are the binomial coefficients.

We thus derive another binary form of the nth degree $\phi(y_1, y_2)$. It is seen at once that
$$\frac{1}{x_2^n} f(x_1, x_2) = a_0 z^n + n_1 a_1 z^{n-1} + n_2 a_2 z^{n-2} + \cdots + a_n$$
$$= a_0(z - \alpha_1)(z - \alpha_2) \ldots (z - \alpha_n), \quad \text{say}.$$

It follows that

$$= a_0(x_1 - \alpha_1 x_2)(x_1 - \alpha_2 x_2) \ldots (x_1 - \alpha_n x_2),$$

ly

$$= a_0'(y_1 - \beta_1 y_2)(y_1 - \beta_2 y_2) \ldots (y_1 - \beta_n y_2).$$

$$- \alpha_1 x_2 = ay_1 + by_2 - \alpha_1(cy_1 + dy_2)$$
$$= (a - \alpha_1 c)\left\{y_1 - \frac{\alpha_1 d - b}{a - \alpha_1 c} y_2\right\},$$

Bocher, *Introduction to Higher Algebra*, p. 260.

it is evident that one of the β's, say

$$\beta_1 = \frac{\alpha_1 d - b}{a - \alpha_1 c},$$

and similarly

$$\beta_2 = \frac{\alpha_2 d - b}{a - \alpha_2 c}, \text{ etc.}$$

From this it is clear that, if some of the α's are equal, some of the β's are also equal, and that there are just as many equal roots in the equation $\phi(y_1, y_2) = 0$ as there are in $f(x_1, x_2) = 0$.

ART. 175. The above correspondence gives rise to the following consideration: Suppose we have given the quadratic form

$$a_0 z^2 + 2 a_1 z + a_2.$$

The roots of the quadratic equation

$$a_0 z^2 + 2 a_1 z + a_2 = 0$$

are

$$z = -\frac{a_1}{a_0} \pm \frac{1}{a_0}\sqrt{a_1^2 - a_0 a_2}.$$

If we write $a_1^2 - a_0 a_2 = D(a_0, a_1, a_2)$, we know that the two roots of the quadratic equation are equal if D is equal to zero. The quantity D after Gauss is called the *discriminant* of the quadratic equation.

Also for forms of higher order we may derive such discriminants, whose vanishing is the condition that the associated equation have equal roots.*

The quantity $D(a_0, a_1, a_2, \ldots, a_n)$ is an integral rational function of $a_0, a_1, \ldots, a_n$ and is homogeneous with respect to these quantities.

If next we form the discriminant $D(a_0', a_1', a_2', \ldots, a_n')$ of the form

$$\phi(y_1, y_2) = a_0' y_1^n + n_1 a_1' y_1^{n-1} y_2 + n_2 a_2' y_1^{n-2} y_2^2 + \cdots + a_n' y_2^n,$$

then the vanishing of this discriminant is the condition that $\phi(y_1, y_2)$ have equal roots β. But we saw that $\phi(y_1, y_2)$ had equal roots when the roots of $f(x_1, x_2)$ are equal. It follows that

$$D(a_0', a_1', \ldots, a_n') = C D(a_0, a_1, \ldots, a_n),$$

where C is a constant factor. This constant factor * is $\{ad - bc\}^{n(n-1)}$.

* Cf. Salmon, *Modern Higher Algebra*, p. 98; Burnside and Panton, *Theory of Equations* (3d ed.), p. 357; etc.

* Cf. Salmon, *loc. cit.*, p. 108; Bôcher, *loc. cit.*, p. 238.

ART. 176. If the function $f(x_1,x_2)=a_0x_1^n+n_1a_1x_1^{n-1}x_2+\cdots+a_nx_2^n$ becomes through the substitutions

$$\begin{array}{c||l} x_1 & ay_1+by_2, \\ x_2 & cy_1+dy_2, \end{array}$$

$$\phi(y_1,y_2)=a_0'y_1^n+n_1a_1'y_1^{n-1}y_2+\cdots+a_n'y_2^n,$$

and if I is a function of the coefficients such that

$$I(a_0',a_1',\ldots,a_n')=(ad-bc)^\mu I(a_0,a_1,\ldots,a_n),$$

where μ is an integer, then I is called an *invariant* of the form $f(x_1,x_2)$.

It may be shown * that, if I is an invariant, μ must be equal to $\frac{1}{2}n\rho$, where ρ is the degree of I with respect to the coefficients $a_0, a_1, \ldots, a_n$. The quantity μ is sometimes called the *index* of the invariant.

The following theorem is also true:† *All the invariants of a binary form $f(x_1,x_2)$ may be expressed rationally through a certain number of them which are called the fundamental invariants.*

For the form of the fourth degree,

$$f(x_1,x_2)=a_0x_1^4+4a_1x_1^3x_2+6a_2x_1^2x_2^2+4a_3x_1x_2^3+a_4x_2^4,$$

there are only two fundamental invariants (cf. Sylvester, *Phil. Mag.*, April, 1853).

The one of these is ‡

$$I_2=a_0a_4-4a_1a_3+3a_2.$$

If by the given transformations we bring $f(x_1,x_2)$ to the form

$$\phi(y_1,y_2)=a_0'y_1^4+4a_1'y_1^3y_2+\cdots+a_4'y_2^4,$$

then it is easy to show that

$$a_0'a_4'-4a_1'a_3'+3a_2'^2=(a_0a_4-4a_1a_3+3a_2^2)(ad-bc)^4.$$

In this case $\rho=2$, $n=4$, $\mu=\frac{1}{2}n\rho=4$.

We thus have

$$I_2'=I_2(ad-bc)^4.$$

The other fundamental invariant § is

$$I_3=a_0a_2a_4+2a_1a_2a_3-a_2^3-a_0a_3^2-a_4a_1^2.$$

It is seen at once that

$$I_3'=I_3(ad-bc)^6.$$

* Cf. Salmon, *loc. cit.*, p. 130; Burnside and Panton, *loc. cit.*, p. 376.

† Cf. Salmon, pp. 111, 132, 175; Bôcher, *loc. cit.*, Chap. XVII, and Burnside and Panton, p. 405.

‡ Salmon, *loc. cit.*, p. 112. Cayley, *Cambridge Math. Journ.* (1845), Vol. IV, p. 193, introduced this invariant.

§ To Boole, *Cambridge Math. Journ.* (1841), Vol. III, pp. 1–106, is due the discovery of this invariant; see also *Cambridge Math. Journ.*, Vol. IV, p. 209; *Cambridge and Dublin Math. Journ.*, Vol. I, p. 104; *Crelle*, Bd. 30, etc.; and Eisenstein, *Crelle*, Bd. 27, p. 81; Aronhold, *Crelle*, Bd. 39, p. 140.

ART. 177. The discriminant D of the binary form $f(x_1, x_2)$ may be rationally expressed (cf. Salmon, *loc. cit.*, p. 112) in terms of I_2 and I_3 in the form

$$D = I_2^3 - 27 I_3^2.$$

It is evident that

$$D' = I_2'^3 - 27 I_3'^2 = D(ad - bc)^{12}.$$

ART. 178. The functional-determinant or *Jacobian* of the two forms $\psi_1(x_1, x_2)$, $\psi_2(x_1, x_2)$ may be written

$$F = \begin{vmatrix} \dfrac{\partial \psi_1}{\partial x_1}, & \dfrac{\partial \psi_1}{\partial x_2} \\ \dfrac{\partial \psi_2}{\partial x_1}, & \dfrac{\partial \psi_2}{\partial x_2} \end{vmatrix}.$$

If we make the substitution

$$x_1 = \lambda_1(y_1, y_2), \; x_2 = \lambda_2(y_1, y_2),$$

λ_1 and λ_2 being functional signs, then $\psi_1(x_1, x_2)$ becomes a function of y_1, y_2, which may be symbolically denoted by $[\psi_1(x_1, x_2)]_s$ and $\psi_2(x_1, x_2)$ becomes by the same substitution $[\psi_2(x_1, x_2)]_s$.

We form the functional determinant of these two forms

$$\Phi = \begin{vmatrix} \dfrac{\partial[\psi_1(x_1, x_2)]_s}{\partial y_1}, & \dfrac{\partial[\psi_1(x_1, x_2)]_s}{\partial y_2} \\ \dfrac{\partial[\psi_2(x_1, x_2)]_s}{\partial y_1}, & \dfrac{\partial[\psi_2(x_1, x_2)]_s}{\partial y_2} \end{vmatrix},$$

and we shall study the relation between F and Φ.

It is evident, since

$$\frac{\partial \psi}{\partial y_1} = \frac{\partial \psi}{\partial x_1}\frac{\partial x_1}{\partial y_1} + \frac{\partial \psi}{\partial x_2}\frac{\partial x_2}{\partial y_1},$$

that

$$\Phi = \begin{vmatrix} \left[\dfrac{\partial \psi_1(x_1,x_2)}{\partial x_1}\right]_s \dfrac{\partial \lambda_1}{\partial y_1} + \left[\dfrac{\partial \psi_1(x_1,x_2)}{\partial x_2}\right]_s \dfrac{\partial \lambda_2}{\partial y_1}, & \left[\dfrac{\partial \psi_1(x_1,x_2)}{\partial x_1}\right]_s \dfrac{\partial \lambda_1}{\partial y_2} + \left[\dfrac{\partial \psi_1(x_1,x_2)}{\partial x_2}\right]_s \dfrac{\partial \lambda_2}{\partial y_2} \\ \left[\dfrac{\partial \psi_2(x_1,x_2)}{\partial x_1}\right]_s \dfrac{\partial \lambda_1}{\partial y_1} + \left[\dfrac{\partial \psi_2(x_1,x_2)}{\partial x_2}\right]_s \dfrac{\partial \lambda_2}{\partial y_1}, & \left[\dfrac{\partial \psi_2(x_1,x_2)}{\partial x_1}\right]_s \dfrac{\partial \lambda_1}{\partial y_2} + \left[\dfrac{\partial \psi_2(x_1,x_2)}{\partial x_2}\right]_s \dfrac{\partial \lambda_2}{\partial y_2} \end{vmatrix}$$

$$= \begin{vmatrix} \left[\dfrac{\partial \psi_1}{\partial x_1}\right]_s, & \left[\dfrac{\partial \psi_1}{\partial x_2}\right]_s \\ \left[\dfrac{\partial \psi_2}{\partial x_1}\right]_s, & \left[\dfrac{\partial \psi_2}{\partial x_2}\right]_s \end{vmatrix} \begin{vmatrix} \dfrac{\partial \lambda_1}{\partial y_1}, & \dfrac{\partial \lambda_1}{\partial y_2} \\ \dfrac{\partial \lambda_2}{\partial y_1}, & \dfrac{\partial \lambda_2}{\partial y_2} \end{vmatrix} = [F]_s \begin{vmatrix} \dfrac{\partial \lambda_1}{\partial y_1}, & \dfrac{\partial \lambda_1}{\partial y_2} \\ \dfrac{\partial \lambda_2}{\partial y_1}, & \dfrac{\partial \lambda_2}{\partial y_2} \end{vmatrix}.$$

Suppose next that

$$x_1 = \lambda_1(y_1, y_2) = ay_1 + by_2,$$
$$x_2 = \lambda_2(y_1, y_2) = cy_1 + dy_2.$$

We then have

$$\Phi = [F]_s \begin{vmatrix} a, & b \\ c, & d \end{vmatrix} = [F]_s(ad - bc).$$

ART. 179. Let $f(x_1, x_2)$ be a binary form of the nth degree. It is seen that $\frac{\partial f(x_1, x_2)}{\partial x_1}$ and $\frac{\partial f(x_1, x_2)}{\partial x_2}$ are binary forms of the degree $n - 1$.

The functional-determinant F of these two functions

$$\begin{vmatrix} \frac{\partial^2 f}{\partial x_1^2}, & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}, & \frac{\partial^2 f}{\partial x_2^2} \end{vmatrix} = H(f), \text{ say,}$$

is called the *Hessian covariant* * of the form f. Suppose that by the substitution

$$x_1 = ay_1 + by_2,$$
$$x_2 = cy_1 + dy_2,$$

the function $f(x_1, x_2)$ becomes $\phi(y_1, y_2)$ and form the Hessian covariant for this latter function, viz.,

$$H(\phi) = \begin{vmatrix} \frac{\partial}{\partial y_1}\left(\frac{\partial \phi}{\partial y_1}\right), & \frac{\partial}{\partial y_2}\left(\frac{\partial \phi}{\partial y_1}\right) \\ \frac{\partial}{\partial y_1}\left(\frac{\partial \phi}{\partial y_2}\right), & \frac{\partial}{\partial y_2}\left(\frac{\partial \phi}{\partial y_2}\right) \end{vmatrix}.$$

We have

$$\frac{\partial \phi}{\partial y_1} = \left[\frac{\partial f}{\partial x_1}\right]_s \frac{\partial x_1}{\partial y_1} + \left[\frac{\partial f}{\partial x_2}\right]_s \frac{\partial x_2}{\partial y_1},$$

or

$$\frac{\partial \phi}{\partial y_1} = \left[\frac{\partial f}{\partial x_1}\right]_s a + \left[\frac{\partial f}{\partial x_2}\right] c;$$

and similarly

$$\frac{\partial \phi}{\partial y_2} = \left[\frac{\partial f}{\partial x_1}\right]_s b + \left[\frac{\partial f}{\partial x_2}\right] d.$$

When these values are substituted in the above determinant, it follows that

$$H(\phi) = \begin{vmatrix} a\frac{\partial}{\partial y_1}\left[\frac{\partial f}{\partial x_1}\right]_s + c\frac{\partial}{\partial y_1}\left[\frac{\partial f}{\partial x_2}\right]_s, & a\frac{\partial}{\partial y_2}\left[\frac{\partial f}{\partial x_1}\right]_s + c\frac{\partial}{\partial y_2}\left[\frac{\partial f}{\partial x_2}\right]_s \\ b\frac{\partial}{\partial y_1}\left[\frac{\partial f}{\partial x_1}\right]_s + d\frac{\partial}{\partial y_1}\left[\frac{\partial f}{\partial x_2}\right]_s, & b\frac{\partial}{\partial y_2}\left[\frac{\partial f}{\partial x_1}\right]_s + d\frac{\partial}{\partial y_2}\left[\frac{\partial f}{\partial x_2}\right]_s \end{vmatrix}$$

$$= (ad - bc)\begin{vmatrix} \frac{\partial}{\partial y_1}\left[\frac{\partial f}{\partial x_1}\right]_s, & \frac{\partial}{\partial y_2}\left[\frac{\partial f}{\partial x_1}\right]_s \\ \frac{\partial}{\partial y_1}\left[\frac{\partial f}{\partial x_2}\right]_s, & \frac{\partial}{\partial y_2}\left[\frac{\partial f}{\partial x_2}\right]_s \end{vmatrix}.$$

Further, since $\frac{\partial}{\partial y_1}\left[\frac{\partial f}{\partial x_1}\right]_s = \left[\frac{\partial^2 f}{\partial x_1^2}\right]_s a + \left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]_s c$, etc., we have

$$H(\phi) = (ad - bc)\begin{vmatrix} a\left[\frac{\partial^2 f}{\partial x_1^2}\right]_s + c\left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]_s, & b\left[\frac{\partial^2 f}{\partial x_1^2}\right]_s + d\left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]_s \\ a\left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]_s + c\left[\frac{\partial^2 f}{\partial x_2^2}\right]_s, & b\left[\frac{\partial^2 f}{\partial x_1 \partial x_2}\right]_s + d\left[\frac{\partial^2 f}{\partial x_2^2}\right]_s \end{vmatrix}.$$

* Cf. Salmon, *loc. cit.*, p. 117.

It follows that

$$H(\phi) = (ad - bc)^2 \begin{vmatrix} \left[\dfrac{\partial^2 f}{\partial x_1^2}\right]_s, & \left[\dfrac{\partial^2 f}{\partial x_1 \partial x_2}\right]_s \\ \left[\dfrac{\partial^2 f}{\partial x_1 \partial x_2}\right]_s, & \left[\dfrac{\partial^2 f}{\partial x_2^2}\right]_s \end{vmatrix};$$

and consequently

$$H(\phi) = (ad - bc)^2 \Big[H(f)\Big]_{\substack{x_1 = ay_1 + by_2 \\ x_2 = cy_1 + dy_2}}.$$

ART. 180. We may consider more closely the meaning of the *covariant*. Suppose we have a binary form $f(x_1, x_2)$ of the nth degree. With its coefficients $a_0, a_1, \ldots, a_n$ and with x_1, x_2 we form an expression

$$C\{a_0, a_1, \ldots, a_n;\ x_1, x_2\},$$

C denoting a functional sign which with respect to x_1, x_2 is of the νth degree, and in regard to the a's it is of the ρth order.

Suppose further that by the substitution

$$\begin{array}{c|c} x_1 & ay_1 + by_2, \\ x_2 & cy_1 + dy_2, \end{array}$$

the function $f(x_1, x_2)$ becomes $\phi(y_1, y_2)$.

With the coefficients $a_0', a_1', \ldots, a_n'$ of $\phi(y_1, y_2)$ and with y_1, y_2 we form the same function

$$C\{a_0', a_1', \ldots, a_n';\ y_1, y_2\}.$$

If then

$$C\{a_0', a_1', \ldots, a_n';\ y_1, y_2\} = (ad - bc)^\mu [C\{a_0, a_1, \ldots, a_n; x_1, x_2\}]_{\substack{x_1 = ay_1 + by_2 \\ x_2 = cy_1 + dy_2}}$$

where $\mu = \frac{1}{2}(n\rho - \nu)$,

we say that C is a *covariant* * of the binary form $f(x_1, x_2)$.

ART. 181. In the theory of covariants it is shown *that for every binary form $f(x_1, x_2)$ there is a finite number of independent covariants, through which all the other covariants may be expressed.*†

If $f(x_1, x_2)$ is a binary form of the fourth degree, say

$$f(x_1, x_2) = a_0 x_1^4 + 4 a_1 x_1^3 x_2 + 6 a_2 x_1^2 x_2^2 + 4 a_3 x_1 x_2^3 + a_4 x_2^4,$$

there are two fundamental covariants (Salmon, *loc. cit.*, p. 192): The one is the Hessian, where

$$\nu = n - 2 + n - 2 = 2n - 4, \quad \rho = 2;$$

and consequently

$$\mu = \tfrac{1}{2}[2n - 2n + 4] = 2.$$

* Salmon, *loc. cit.*, p. 135; Burnside and Panton, *loc. cit.*, p. 376.

† Salmon, *loc. cit.*, pp. 132, 175, 176; and see also Clebsch, *Theorie der binären algebraischen Formen*, pp. 255 *et seq.*

This covariant is

$$H(f)=(a_0a_2-a_1^2)x_1^4+2(a_0a_3-a_1a_2)x_1^3x_2+(a_0a_4+2a_1a_3-3a_2^2)x_1^2x_2^2 + 2(a_1a_4-a_2a_3)x_1x_2^3+(a_2a_4-a_3^2)x_2^4.$$

The other fundamental covariant is the Jacobian of the quartic and its Hessian:

$$T=\frac{1}{8}\begin{vmatrix} \dfrac{\partial f}{\partial x_1}, & \dfrac{\partial f}{\partial x_2} \\ \dfrac{\partial H(f)}{\partial x_1}, & \dfrac{\partial H(f)}{\partial x_2} \end{vmatrix}.$$

For this covariant it is seen that

$$\nu = n-1+2n-5=3n-6,\quad \rho=1+2=3,$$

and therefore

$$\mu=\tfrac{1}{2}(n\rho-\nu)=3,$$

so that

$$T'=[T]_s(ad-bc)^3.$$

ART. 182. Between the two covariants T and $H(f)$ there exists the relation *

$$-T^2=I_3f^3-I_2f^2H(f)+4H(f)^3.$$

This formula is given by Cayley in *Crelle's Journal* (April 9, 1856, Bd. 50, p. 287). The formula, however, as stated by Cayley, is due to a communication from Hermite.†

We have at once

$$-\frac{2T^2}{f^3}=2I_3-2I_2\frac{H(f)}{f}+8\frac{H(f)^3}{f^3};$$

or writing $\dfrac{2H(f)}{f}=\zeta$, it is seen at once that

$$-\frac{2T^2}{f^3}=2I_3-I_2\zeta+\zeta^3.$$

ART. 183. Consider next the determinant

$$\Delta=\begin{vmatrix} H(f), & f \\ dH(f), & df \end{vmatrix}=\begin{vmatrix} H(f), & f \\ \dfrac{\partial H(f)}{\partial x_1}dx_1+\dfrac{\partial H(f)}{\partial x_2}dx_2, & \dfrac{\partial f}{\partial x_1}dx_1+\dfrac{\partial f}{\partial x_2}dx_2 \end{vmatrix}.$$

The functions f and $H(f)$ being homogeneous of the fourth degree in x_1, x_2, it follows that

$$\frac{\partial f}{\partial x_1}x_1+\frac{\partial f}{\partial x_2}x_2=4f,$$

$$\frac{\partial H(f)}{\partial x_1}x_1+\frac{\partial H(f)}{\partial x_2}x_2=4H(f).$$

* Cf. Salmon, *loc. cit.*, p. 195; Halphen, *Fonctions Elliptiques*, t. II, p. 362; Clebsch, *loc. cit.*, §62.

† Similar relations have been derived by Hermite for the quintic and for every form of odd degree (cf. Salmon, p. 249).

We therefore have

$$\Delta = \frac{1}{4}\begin{vmatrix} \frac{\partial H(f)}{\partial x_1}x_1 + \frac{\partial H(f)}{\partial x_2}x_2, & \frac{\partial f}{\partial x_1}x_1 + \frac{\partial f}{\partial x_2}x_2 \\ \frac{\partial H(f)}{\partial x_1}dx_1 + \frac{\partial H(f)}{\partial x_2}dx_2, & \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 \end{vmatrix}.$$

$$= \frac{1}{4}\begin{vmatrix} \frac{\partial H(f)}{\partial x_1}, & \frac{\partial H(f)}{\partial x_2} \\ \frac{\partial f}{\partial x_1}, & \frac{\partial f}{\partial x_2} \end{vmatrix}\begin{vmatrix} x_1, & x_2 \\ dx_1, & dx_2 \end{vmatrix} = 2\,T(x_2dx_1 - x_1dx_2).$$

On the other hand

$$\Delta = \begin{vmatrix} H(f), & f \\ dH(f), & df \end{vmatrix} = H(f)df - fdH(f)$$

$$= -f^2 d\frac{H(f)}{f} = -\frac{1}{2}f^2 d\zeta.$$

It follows that

$$\tfrac{1}{2}f^2 d\zeta = -2\,T(x_2dx_1 - x_1dx_2)$$

or

$$f^{\frac{1}{2}}d\zeta = -4\,(x_2dx_1 - x_1dx_2)\left(\frac{T^2}{f^3}\right)^{\frac{1}{2}}$$

$$= -\frac{4}{(-2)^{\frac{1}{2}}}(x_2dx_1 - x_1dx_2)\left(\frac{-2\,T^2}{f^3}\right)^{\frac{1}{2}}$$

$$= -\frac{4}{(-2)^{\frac{1}{2}}}(x_2dx_1 - x_1dx_2)\,(2I_3 - I_2\zeta + \zeta^3)^{\frac{1}{2}}.$$

From this it is evident that

$$\frac{x_2dx_1 - x_1dx_2}{\sqrt{f(x_1, x_2)}} = \frac{(-2)^{\frac{1}{2}}}{-4}\,\frac{d\zeta}{\sqrt{2I_3 - I_2\zeta + \zeta^3}}.$$

Since $z = \frac{x_1}{x_2}$, it follows that

$$\frac{x_2dx_1 - x_1dx_2}{\sqrt{f(x_1, x_2)}} = \frac{dz}{\sqrt{R(z)}},$$

where

$$R(z) = a_0z^4 + 4\,a_1z^3 + 6\,a_2z^2 + 4\,a_3z + a_4.$$

We finally have

$$\int\frac{dz}{\sqrt{R(z)}} = \frac{(-2)^{\frac{1}{2}}}{-4}\int\frac{d\zeta}{\sqrt{\zeta^3 - I_2\zeta + 2I_3}}.$$

This is practically the transformation given by Cayley * in *Crelle Journal*, Bd. 55, p. 23.

* See also Cayley, *Elliptic Functions*, p. 317; and Burnside and Panton, *loc. cit.*, p. 474; Brioschi, *Sur une formule de M. Cayley*, *Crelle*, Bd. 53, p. 377, and *Crelle*, Bd. 63, p. 32. The Berlin lectures of the late Prof. Fuchs have been of great assistance in the derivation of this transformation.

The mode of procedure, however, as noted above, was suggested by Hermite (cf. Hermite in "*Lettre* 123" of the *Correspondence d'Hermite et de Stieltjes;* read also letters 124 and 125 of the above correspondence and Hermite, *Crelle,* Bd. 52; *Cambridge and Dublin Math. Journ.*, vol. IX, p. 172; and t. I of the *Comptes Rendus* for 1866).

If we write $2t$ for ζ in the above formula, it becomes

$$\int\frac{dz}{\sqrt{R(z)}}=-i\int\frac{dt}{2\sqrt{4t^3-I_2t+I_3}}.$$

ART. 184. Weierstrass employed a somewhat different notation. He put

$$I_2=g_2,\qquad I_3=-g_3,$$

and consequently introduced as his normal form of the elliptic integral of the first kind,

$$\int\frac{dt}{\sqrt{4t^3-g_2t-g_3}}.$$

He further wrote

$$4t^3-g_2t-g_3=4(t-e_1)(t-e_2)(t-e_3)=S(t),$$

so that (cf. Art. 172) between the e's and the g's we have the following relations:

$$e_1+e_2+e_3=0,$$
$$e_1e_2+e_2e_3+e_3e_1=-\tfrac{1}{4}g_2,$$
$$e_1e_2e_3=\tfrac{1}{4}g_3.$$

ART. 185. We may show as follows how the Hermite-Weierstrass normal form may be brought to the Legendre-Jacobi normal form.

In the expression

$$\frac{dt}{\sqrt{S(t)}}$$

write $t=A+\frac{B}{\mathbf{z}^2}$, where A and B are constants. It is seen at once that

$$\frac{dt}{\sqrt{S(t)}}=\frac{-Bd\mathbf{z}}{\sqrt{\{(A-e_1)\mathbf{z}^2+B\}\{(A-e_2)\mathbf{z}^2+B\}\{(A-e_3)\mathbf{z}^2+B\}}}.$$

Under the root sign there is an expression of the sixth degree which contains only even powers of t. But by writing

$$A=e_3,$$

this reduces to

$$\frac{dt}{\sqrt{S(t)}}=\frac{-Bd\mathbf{z}}{\sqrt{B\{(e_3-e_1)\mathbf{z}^2+B\}\{(e_3-e_2)\mathbf{z}^2+B\}}}.$$

If further we give to B the value

$$B = e_1 - e_3,$$

and put

$$\frac{e_2 - e_3}{e_1 - e_3} = k^2,$$

we have

$$\frac{dt}{\sqrt{S(t)}} = -\frac{1}{\sqrt{e_1 - e_3}} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

It has thus been shown that *through the substitution*

$$t = e_3 + \frac{e_1 - e_3}{z^2},$$

the Weierstrassian normal form is changed into that of Legendre.

Other methods of effecting this transformation will be found in Volume II.

ART. 186. If we write *

$$e_1 - e_3 = \frac{1}{\varepsilon},$$

then is

$$e_1 = \frac{1}{\varepsilon} + e_3. \tag{1}$$

Further, since

$$e_2 - e_3 = \frac{k^2}{\varepsilon},$$

we have

$$e_2 = \frac{k^2}{\varepsilon} + e_3; \tag{2}$$

and using the relation

$$e_1 + e_2 + e_3 = 0,$$

we also have

$$e_3 = -\frac{1}{3}\,\frac{1 + k^2}{\varepsilon}. \tag{3}$$

This value of e_3 written in (1) and (2) gives

$$e_1 = \frac{1}{3\,\varepsilon}(2 - k^2), \tag{4}$$

$$e_2 = \frac{1}{3\,\varepsilon}(2\,k^2 - 1). \tag{5}$$

From the equations $e_1e_2 + e_2e_3 + e_3e_1 = -\frac{1}{4}g_2$ and $e_1e_2e_3 = \frac{1}{4}g_3$ it follows with the use of (3), (4) and (5) that

$$-\frac{9}{4}\varepsilon^2 g_2 = (2 - k^2)\,(2\,k^2 - 1) - (2 - k^2)\,(1 + k^2) - (2\,k^2 - 1)\,(1 + k^2),$$

and

$$\frac{27\,\varepsilon^3}{4} g_3 = -(2 - k^2)\,(2\,k^2 - 1)\,(1 + k^2),$$

and from these two relations †

$$\frac{g_2{}^3}{g_3{}^2} = \frac{108\{1 - k^2 + k^4\}^3}{[(2 - k^2)\,(2\,k^2 - 1)\,(1 + k^2)]^2}.$$

* Cf. Halphen, *Fonctions Elliptiques*, t. I, p. 25.

† Cf. Felix Müller, *Schlömilch Zeit.*, Bd. 18, pp. 282–287.

We shall next show that the above expression is an *absolute invariant*,* that is, it remains entirely unchanged by a linear substitution.

We have

$$\frac{g_2^3}{g_3^2} = \frac{I_2^3}{I_3^2},$$

and we saw in Art. 176 that

$$I_2'^3 = I_2^3(ad - bc)^{12}$$

and

$$I_3'^2 = I_3^2\,(ad - bc)^{12}.$$

It follows that

$$\frac{g_2^3}{g_3^2} = \frac{I_2^3}{I_3^2} = \frac{I_2'^3}{I_3'^2}.$$

From this it is seen *that k is the root of an algebraic equation of the* 12*th degree whose coefficients depend rationally upon the absolute invariant* $\frac{g_2^3}{g_3^2}$.

ART. 187. *Riemann's Normal Form.*† If in Legendre's normal form

$$\int\frac{dz}{\sqrt{(1-z^2)\,(1-k^2z^2)}}$$

we put $z^2 = t, k^2 = \lambda$, it becomes

$$\frac{1}{2}\int\frac{dt}{\sqrt{t\,(1 - t)(1 - \lambda t)}}.$$

If in the latter integral we write

$$\rho = \lambda + \frac{1}{\lambda},$$

we have, neglecting a constant multiplier,

$$\int\frac{d\tau}{\sqrt{\tau(1 - \rho\tau + \tau^2)}}.$$

(Kronecker, *Berlin Sitz.*, July, 1886.)

In Volume II the transformation of the general integral into its normal forms will be resumed and the discussion for the most part will be restricted to *real* variables.

ART. 188. In connection with the realms of rationality we may consider more closely the integrals that have been introduced in this Chapter.

Let R denote any rational function of its arguments, and write the integral

$$\int R(z, \sigma)dz,$$

where $\sigma = \sqrt{az + b}$. If we put $\sigma = \psi(t)$, where ψ is a rational function, then $z = \phi(t)$ is a rational function. The above integral becomes

$$\int R[\phi(t), \psi(t)]\phi'(t)dt,$$

* Salmon, *loc. cit.*, p. 111.

† Cf. Klein, *Math. Ann.*, Bd. 14, p. 116, and *Theorie der Elliptischen Modulfunctionen*, Bd. I, p. 25.

where the integrand is a *rational* function of t. For example, put $\sigma = \sqrt{az+b} = t$, then $z = \frac{t^2 - b}{a}$. In this case the realm (z, σ) is evidently the same as the realm (t), since (z, t) is the same as (t), the presence of z within the realm adding nothing to it, as z is a rational function of t.

Consider next the integral

$$\int R(z, \sigma) dz,$$

where $$\sigma = \sqrt{(z-a_1)(z-a_2)} = (z-a_2)\sqrt{\frac{z-a_1}{z-a_2}}.$$

By writing $t^2 = \frac{z-a_1}{z-a_2}$, it is seen that $\sigma = (z-a_2)t$ and $z = \frac{a_1 - a_2 t^2}{1-t^2}$.

We note that both σ and z are rational functions of t and that t is a rational function of σ and z. Hence every rational function of σ and z is a rational function of t and any rational function of t may be expressed rationally through z and σ. In this case we may say that the two realms (z, σ) and (t) are equivalent and write

$$(z, \sigma) \sim (t).$$

In the case of the integral

$$\int R(x, \sqrt{ax^2 + 2bx + c})dx,$$

if we put $y^2 = ax^2 + 2bx + c$, we have the equation of a conic section.

This conic section is cut by the line

$$y - \eta = t(x - \xi),$$

where t is the tangent of the angle that the line makes with the x-axis, at the point ξ, η, say, and at another point

$$x = \frac{a\xi + 2b - 2\eta t + \xi t^2}{t^2 - a},$$

$$y = \frac{\eta t^2 - 2a\xi t - 2bt + a\eta}{t^2 - a}.$$

Hence as above

$$(x, y) \sim (t).$$

In the case of the integral

$$\int R(z, s) dz,$$

where s is the square root of an expression of the third or fourth degree in z, it was shown by both Abel and Liouville that the integrand cannot be expressed as a rational function of t. This we know *à priori* from our previous investigations; for we saw that an elliptic integral of the first

kind nowhere becomes infinite, while the integral of a rational function must become infinite for either finite or infinite values of the variable.

In Art. 166 it is seen that z and s may be rationally expressed through $\mathbf{z}$ and $\mathbf{s} = \sqrt{(1-\mathbf{z}^2)(1-k^2\mathbf{z}^2)}$ and at the same time $\mathbf{z}$ and $\mathbf{s}$ may be rationally expressed through z and s so that

$$(z, s) \sim (\mathbf{z}, \mathbf{s}),$$

and consequently any element of one realm is an element of the other.

It is also seen that if $\tau = \sqrt{4\,t^3 - g_2 t - g_3}$, then

$$(z, s) \sim (\mathbf{z}, \mathbf{s}) \sim (t, \tau).$$

We note that by these transformations the order of the Riemann surface remains unchanged.

The above three realms of rationality being equivalent, the name *elliptic realm of rationality* may be applied indifferently to them all.

EXAMPLES

1. In the homographic transformation,

$$\alpha + \beta t + \gamma z + \delta t z = 0$$

for $\qquad z = a_1, \qquad z = a_2, \qquad z = a_3, \qquad z = a_4,$

let $\qquad t = 0, \qquad t = 1, \qquad t = \frac{1}{\lambda}, \qquad t = \infty.$

We thus have

$$\alpha + \gamma a_1 = 0, \qquad \alpha + \beta + \gamma a_2 + \delta a_2 = 0, \qquad \alpha\lambda + \beta + \gamma\lambda a_3 + \delta a_3 = 0, \qquad \beta + \delta a_4 = 0.$$

The vanishing of the determinant of these equations gives

$$\lambda = \frac{a_1 - a_2}{a_1 - a_3} \cdot \frac{a_3 - a_4}{a_2 - a_4}.$$

Show that $\dfrac{dz}{\sqrt{R(z)}}$ is thereby transformed into Riemann's normal form.

2. In a similar manner transform $\dfrac{dz}{\sqrt{R(z)}}$ into Legendre's normal form and from the resulting determinant derive the 12 values of k given in Art. 171. [*Thomae.*]

3. Show that the substitutions

$$t = \frac{z - a_1}{z - a_2} \cdot \frac{a_2 - a_4}{a_2 - a_1}, \quad k^2 = \frac{a_3 - a_4}{a_3 - a_1} \cdot \frac{a_2 - a_1}{a_2 - a_4}$$

transform

$$\int_0^t \frac{dt}{\sqrt{t(1-t)(1-k^2t)}} \text{ into } \pm\sqrt{(a_4 - a_2)(a_1 - a_3)} \int_{a_1}^{z} \frac{dz}{\sqrt{(z-a_1)(z-a_2)(z-a_3)(z-a_4)}}.$$

[Riemann-Stahl, *Ell. Funct.*, p. 16.]

4. Show that the substitution

$$\frac{z-a_1}{z-a_2}:\frac{a_3-a_1}{a_3-a_2}=\frac{t-a_2}{t-a_1}:\frac{a_4-a_2}{a_4-a_1}$$

transforms

$$\int\frac{dz}{\sqrt{A(z-a_1)(z-a_2)(z-a_3)(z-a_4)}}\text{ into }\int\frac{dt}{\sqrt{A(t-a_1)(t-a_2)(t-a_3)(t-a_4)}}.$$

[Burkhardt, *Ell. Funct.*]

Derive two other such substitutions.

5. Show that the substitution

$$t=e_1+\frac{(e_2-e_1)(e_3-e_1)}{\tau-e_1}$$

transforms Weierstrass's integral into itself.

6. If α is a root of $az^3+3\,bz^2+3\,cz+d=0$, by writing $z-\alpha=z^2$ transform $\dfrac{dz}{\sqrt{az^3+3\,bz^2+3\,cz+d}}$ into Legendre's normal form.

7. If $f(x)=x^4+6\,mx^2+1$, show that

$$4\int\frac{dx}{\sqrt{x^4+6\,mx^2+1}}=\int\frac{d\xi}{\sqrt{(\xi+m)\left(\xi-\frac{m-1}{2}\right)\left(\xi-\frac{m+1}{2}\right)}},$$

where $$\xi=-\frac{m(x^4+1)+(1-3\,m^2)x^2}{x^4+6\,mx^2+1}=-\frac{H(f)}{f}.$$

[Appell et Lacour, *Fonc. Ellip.*, p. 268.]

CHAPTER IX

THE MODULI OF PERIODICITY FOR THE NORMAL FORMS OF LEGENDRE AND OF WEIERSTRASS

ARTICLE 189. The Riemann surface for the elliptic integral of the first kind in Legendre's normal form,

$$\int \frac{d\mathbf{z}}{\sqrt{\mathbf{Z}}}, \text{ where } \mathbf{Z} = (1 - \mathbf{z}^2)(1 - k^2\mathbf{z}^2) = \mathbf{s}^2$$

has the branch-points $+1, -1, +\frac{1}{k}, -\frac{1}{k}$.

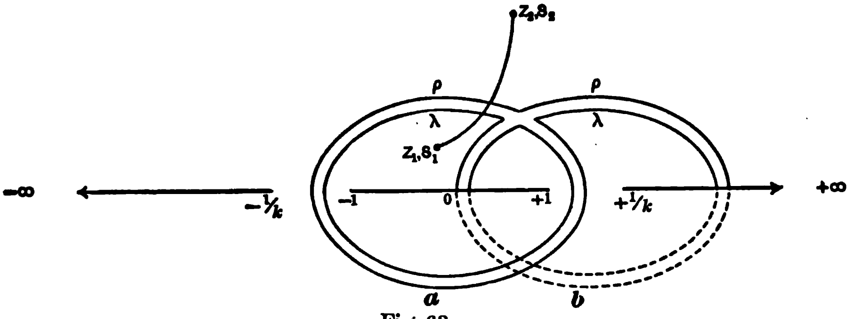


Fig. 63.

In the figure* we join the points $+1$ and -1 with a canal and also the points $+\frac{1}{k}$ and $-\frac{1}{k}$ with a canal which passes through infinity. Here we have taken the modulus k, which may be any arbitrary complex quantity, as a real quantity, positive and less than unity. In the following discussion we make no use, however, of this special assumption.

In Art. 142 we saw that

$$A = \int_b \frac{dz}{s}, \quad B = \int_a \frac{dz}{s}.$$

The corresponding quantities here are, say,

$$A(k) = 2\int_1^{+\frac{1}{k}} \frac{d\mathbf{z}}{\sqrt{\mathbf{Z}}}$$

and

$$B(k) = 2\int_{-1}^{+1} \frac{d\mathbf{z}}{\sqrt{\mathbf{Z}}}.$$

* Cf. Koenigsberger, *Ellipt. Funct.*, pp. 299 *et seq.*

For any integral in the T'-surface we shall take as lower limit the point $\mathbf{z}_0 = 0$, $\mathbf{s}_0 = +1$; that is, the origin in the upper leaf.

We then have

$$\bar{u}(\mathbf{z}, \mathbf{s}) = \int_{0,1}^{\mathbf{z},\mathbf{s}} \frac{d\mathbf{z}}{\sqrt{\mathbf{Z}}} \text{ in } T'.$$

If we let the upper limit coincide also with the point 0, 1, then, however the curve be drawn in the T'-surface, we have always

$$\text{(I)} \qquad \bar{u}(0, 1) = 0.$$

ART. 190. In Art. 139 we saw that

$$\textit{on the canal } \boldsymbol{a},\ \bar{u}(\lambda) - \bar{u}(\rho) = A(k),$$

and

$$\textit{on the canal } \boldsymbol{b},\ \bar{u}(\rho) - \bar{u}(\lambda) = B(k).$$

We form the integral between arbitrary limits, $\mathbf{z}_2$, $\mathbf{s}_2$ and $\mathbf{z}_1$, $\mathbf{s}_1$, where the path of integration is *free*, that is, taken without regard to the canals $\boldsymbol{a}$ and $\boldsymbol{b}$.

If the path of integration crosses the canal $\boldsymbol{a}$ (see Fig. 63) we have

$$\int_{\mathbf{z}_2,\mathbf{s}_2}^{\mathbf{z}_1,\mathbf{s}_1} = \int_{\mathbf{z}_2,\mathbf{s}_2}^{\rho} + \int_{\rho}^{\lambda} + \int_{\lambda}^{\mathbf{z}_1,\mathbf{s}_1},$$

the integrand for all these integrals being $\dfrac{d\mathbf{z}}{\sqrt{(1-\mathbf{z}^2)(1-k^2\mathbf{z}^2)}}$.

Noting that the second integral on the right-hand side is indefinitely small in T, it is seen that

$$\begin{aligned}\int_{\mathbf{z}_2,\mathbf{s}_2}^{\mathbf{z}_1,\mathbf{s}_1} \frac{d\mathbf{z}}{\sqrt{\mathbf{Z}}} &= \bar{u}(\rho) - \bar{u}(\mathbf{z}_2, \mathbf{s}_2) + \bar{u}(\mathbf{z}_1, \mathbf{s}_1) - \bar{u}(\lambda) \text{ in } T' \\ &= \bar{u}(\mathbf{z}_1, \mathbf{s}_1) - \bar{u}(\mathbf{z}_2, \mathbf{s}_2) - A(k).\end{aligned}$$

If, however, the integration is taken in the opposite direction, we have

$$\int_{\mathbf{z}_1,\mathbf{s}_1}^{\mathbf{z}_2,\mathbf{s}_2} \frac{d\mathbf{z}}{\sqrt{\mathbf{Z}}} = \bar{u}(\mathbf{z}_2, \mathbf{s}_2) - \bar{u}(\mathbf{z}_1, \mathbf{s}_1) + A(k).$$

We may form the following rule: *If the path of integration for the integral*

$$\int \frac{d\mathbf{z}}{\sqrt{(1-\mathbf{z}^2)(1-k^2\mathbf{z}^2)}}$$

crosses the canal once in the direction from ρ to λ, this integral with free path is equal to the integral taken in T' decreased by $A(k)$; but if we cross the canal $\boldsymbol{a}$ in the direction from λ to ρ, then this integral with free path is equal to the integral in T' increased by $A(k)$. Upon crossing the canal $\boldsymbol{b}$ we have the opposite result: If $\boldsymbol{b}$ is crossed in the direction from ρ to λ, then $B(k)$ is to be added to the integral in T'.

We may apply this rule in order to derive a number of formulas, which give the value of $\bar{u}(z, s)$ at certain points. In Fig. 63 it is seen that in the upper leaf of T'

$$\int_{-\frac{1}{k}}^{-1} \frac{dz}{\sqrt{Z}} = \bar{u}(-1) - \bar{u}\left(-\frac{1}{k}\right) - A(k).$$

But in the lower leaf where the path of integration is taken congruent to the one in the upper leaf, there being no canal between the points -1 and $-\frac{1}{k}$,

$$\int_{-\frac{1}{k}}^{-1} \frac{dz}{\sqrt{Z}} = \bar{u}(-1) - \bar{u}\left(-\frac{1}{k}\right).$$

If we add these two integrals and note that the elements of integration are equal in pairs and of opposite sign, it is seen that the two integrals on the left-hand side cancel, so that

$$0 = 2\left\{\bar{u}(-1) - \bar{u}\left(-\frac{1}{k}\right)\right\} - A(k),$$

or

$$\text{(II)} \qquad \frac{A(k)}{2} = \bar{u}(-1) - \bar{u}\left(-\frac{1}{k}\right).$$

Consider further the integral from -1 to $+1$ in the upper leaf and on the upper bank of the canal from -1 to $+1$ (the upper bank being the one nearest the top of the page)

$$\int_{-1}^{+1} \frac{dz}{\sqrt{Z}} = \bar{u}(+1) - \bar{u}(-1) + B(k).$$

The same integral in the lower leaf and on the upper bank of the canal is

$$\int_{-1}^{+1} \frac{dz}{\sqrt{Z}} = \bar{u}(+1) - \bar{u}(-1).$$

It follows, as above, that

$$\text{(III)} \qquad -\frac{B(k)}{2} = u(+1) - \bar{u}(-1).$$

Next forming the integral from $+1$ to $+\frac{1}{k}$ in the upper leaf and upper bank, we have

$$\int_{+1}^{+\frac{1}{k}} \frac{dz}{\sqrt{Z}} = \bar{u}\left(+\frac{1}{k}\right) - \bar{u}(+1) + A(k);$$

and in the lower leaf, upper bank,

$$\int_{+1}^{+\frac{1}{k}} \frac{dz}{\sqrt{Z}} = \bar{u}\left(+\frac{1}{k}\right) - \bar{u}(+1).$$

We therefore have

(IV) $$-\frac{A(k)}{2}=\bar{u}\left(+\frac{1}{k}\right)-\bar{u}(+1).$$

We then form in the upper leaf, upper bank,

$$\int_{\frac{1}{k}}^{+\infty,+\infty}\frac{dz}{\sqrt{Z}}=\bar{u}(\infty,+\infty)-\bar{u}\left(\frac{1}{k}\right)-B(k);$$

and on the lower leaf, upper bank,

$$\int_{\frac{1}{k}}^{+\infty,-\infty}\frac{dz}{\sqrt{Z}}=\bar{u}(\infty,-\infty)-\bar{u}\left(\frac{1}{k}\right).$$

Adding these two integrals we have

(V) $$B(k)=\bar{u}(\infty,+\infty)+\bar{u}(\infty,-\infty)-2\,\bar{u}\left(\frac{1}{k}\right).$$

ART. 191. If we form the integral

$$\int_{-\infty}^{-\frac{1}{k}}\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$$

in the upper leaf of T' and take the integration along the upper bank of the canal, it is seen that the path of integration is congruent to the one from $+\frac{1}{k}$ to $+\infty$. At two corresponding points of the paths the absolute values of z are the same, but the signs are opposite. This difference of sign, however, does not appear in the expression $(1-z^2)(1-k^2z^2)$. The differential dz is the same along both the paths and positive, and consequently the elements of integration are equal in pairs and we have

(M) $$\int_{-\infty}^{-\frac{1}{k}}\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}=\int_{\frac{1}{k}}^{\infty}\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

In a similar manner we have

(N) $$\int_{-\frac{1}{k}}^{-1}\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}=\int_{+1}^{+\frac{1}{k}}\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

We form the integration over the path indicated in Fig. 64 which lies wholly in the upper leaf and passes twice through infinity.

The integral $\int\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}$ taken over this path must be zero, since the path of integration does not include a branch-point.

We therefore have

$$\underset{\substack{\text{upper}\\ \text{bank}}}{\int_{-1}^{+1}} + \int_{+1}^{+\frac{1}{k}} + \underset{\substack{\text{upper}\\ \text{bank}}}{\int_{+\frac{1}{k}}^{+\infty}} + \underset{\substack{\text{upper}\\ \text{bank}}}{\int_{-\infty}^{-\frac{1}{k}}} + \underset{\substack{\text{lower}\\ \text{bank}}}{\int_{-\frac{1}{k}}^{-\infty}} + \underset{\substack{\text{lower}\\ \text{bank}}}{\int_{+\infty}^{+\frac{1}{k}}} + \int_{+\frac{1}{k}}^{+1} + \underset{\substack{\text{lower}\\ \text{bank}}}{\int_{+1}^{-1}} = 0.$$

We note that the two integrals

$$\underset{\text{upper bank}}{\int_{+\frac{1}{k}}^{+\infty} \frac{dz}{\sqrt{Z}}} \quad \text{and} \quad \underset{\text{lower bank}}{\int_{+\infty}^{+\frac{1}{k}} \frac{dz}{\sqrt{Z}}}$$

are equal, for the sign of dz is different in both integrals, and as both integrals are in the upper leaf but upon different banks, there is a difference

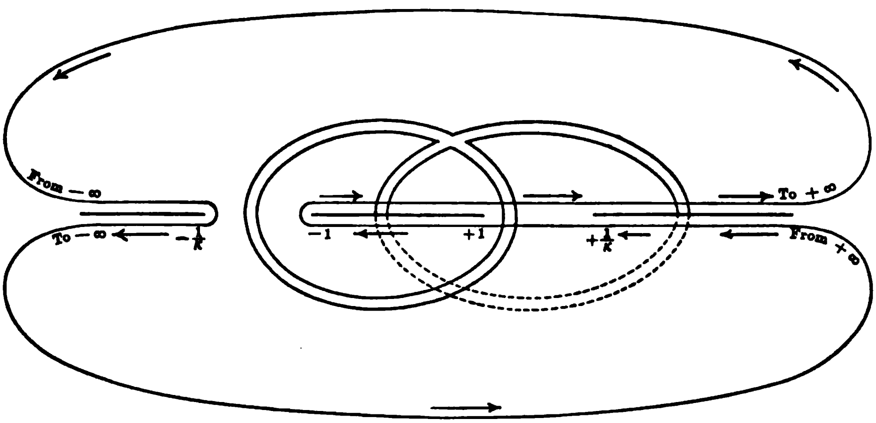


Fig. 64.

in sign and also a difference in sign due to the limits of integration. On the other hand the two integrals

$$\int_{+1}^{+\frac{1}{k}} \frac{dz}{\sqrt{Z}} \quad \text{and} \quad \int_{+\frac{1}{k}}^{+1} \frac{dz}{\sqrt{Z}}$$

are equal with opposite sign, since there is no canal between the two paths over which the integration is taken.

It follows that the sum of the above integrals reduces to

$$2\int_{-1}^{+1} \frac{dz}{\sqrt{Z}} + 2\int_{+\frac{1}{k}}^{+\infty} \frac{dz}{\sqrt{Z}} + 2\int_{-\infty}^{-\frac{1}{k}} \frac{dz}{\sqrt{Z}} = 0,$$

where the integration is on the upper bank for all the integrals.

Owing to the relation (M) above, this sum of integrals further reduces, after division by 2, to

$$\int_{-1}^{+1}\frac{dz}{\sqrt{Z}}+2\int_{+\frac{1}{k}}^{+\infty}\frac{dz}{\sqrt{Z}}=0.$$

It follows at once that

$$\bar{u}(+1)-\bar{u}(-1)+B(k)+2\left[\bar{u}(\infty,+\infty)-\bar{u}\left(+\frac{1}{k}\right)-B(k)\right]=0,$$

or, owing to (III),

$$\text{(VI)}\qquad \frac{3}{4}B(k)=\bar{u}(\infty,+\infty)-\bar{u}\left(+\frac{1}{k}\right).$$

If we take the congruent path of integration in the lower leaf, we again have, since no canals are crossed,

$$\bar{u}(+1)-\bar{u}(-1)+2\left[\bar{u}(\infty,-\infty)-\bar{u}\left(\frac{1}{k}\right)\right]=0,$$

or

$$-\frac{B(k)}{2}+2\left[\bar{u}(\infty,-\infty)-\bar{u}\left(\frac{1}{k}\right)\right]=0.$$

We have thus the formula

$$\text{(VII)}\qquad \frac{B(k)}{4}=\bar{u}(\infty,-\infty)-\bar{u}\left(\frac{1}{k}\right).$$

ART. 192. We compute the integral from 0 to 1 in the upper leaf of T' on the upper bank of the canal and then the integral taken over the congruent path in the lower leaf.

It is clear that

$$\underbrace{\int_{0,1}^{+1}\frac{dz}{\sqrt{Z}}}_{\text{upper leaf}}+\underbrace{\int_{0,-1}^{+1}\frac{dz}{\sqrt{Z}}}_{\text{lower leaf}}=0.$$

It follows that

$$\bar{u}(+1)-\bar{u}(0,1)+B(k)+\bar{u}(+1)-\bar{u}(0,-1)=0,$$

or, since $\bar{u}(0,1)=0$ from (I), we have

$$\text{(VIII)}\qquad 2\,\bar{u}(+1)-\bar{u}(0,-1)=-B(k).$$

Further, it is seen that

$$\underbrace{\int_{-1}^{0,+1}\frac{dz}{\sqrt{Z}}}_{\text{upper bank}}=\underbrace{\int_{0,+1}^{+1}\frac{dz}{\sqrt{Z}}}_{\text{upper bank}}$$

and consequently, multiplying by 2, we have

$$\underbrace{\int_{-1}^{+1}\frac{dz}{\sqrt{Z}}}_{\text{upper bank}}=2\underbrace{\int_{0,1}^{+1}\frac{dz}{\sqrt{Z}}}_{\text{upper bank}}.$$

From this it follows that

$$\bar{u}(+1)-\bar{u}(-1)+B(k)=2\{\bar{u}(+1)-\bar{u}(0,1)+B(k)\},$$

or, owing to (I) and (III),

$$\frac{B(k)}{2}=2\left[\bar{u}(+1)+B(k)\right].$$

We thus obtain

(IX) $$-\tfrac{3}{4}B(k)=\bar{u}(+1).$$

We have thus derived the following nine formulas:

(I) $\bar{u}(0,1)=0,$

(II) $\bar{u}(-1)-\bar{u}\left(-\dfrac{1}{k}\right)=\dfrac{A(k)}{2},$

(III) $\bar{u}(+1)-\bar{u}(-1)=-\dfrac{B(k)}{2},$

(IV) $\bar{u}\left(+\dfrac{1}{k}\right)-\bar{u}(+1)=-\dfrac{A(k)}{2},$

(V) $\bar{u}(\infty,+\infty)+\bar{u}(\infty,-\infty)-2\bar{u}\left(\dfrac{1}{k}\right)=B(k),$

(VI) $\bar{u}(\infty,+\infty)-\bar{u}\left(\dfrac{1}{k}\right)=\dfrac{3}{4}B(k),$

(VII) $\bar{u}(\infty,-\infty)-\bar{u}\left(\dfrac{1}{k}\right)=\dfrac{1}{4}B(k),$

(VIII) $2\bar{u}(+1)-\bar{u}(0,-1)=-B(k),$

(IX) $\bar{u}(+1)=-\tfrac{3}{4}B(k).$

From these formulas we have at once:

$$\bar{u}(+1)=-\tfrac{3}{4}B(k),\qquad \bar{u}(-1)=-\tfrac{1}{4}B(k),$$

$$\bar{u}\left(\frac{1}{k}\right)=-\frac{A(k)}{2}-\frac{3}{4}B(k),\qquad \bar{u}\left(-\frac{1}{k}\right)=-\frac{A(k)}{2}-\frac{1}{4}B(k),$$

$$\bar{u}(\infty,+\infty)=-\frac{A(k)}{2},\qquad \bar{u}(\infty,-\infty)=-\frac{A(k)}{2}-\frac{1}{2}B(k),$$

$$\bar{u}(0,1)=0,\qquad \bar{u}(0,-1)=-\frac{B(k)}{2}.$$

ART. 193. Legendre * and Jacobi † did not use the quantities $A(k)$ and $B(k)$ but instead two other quantities K and K'. These quantities are connected with $A(k)$ and $B(k)$ as follows:

$$4K=B(k)=2\int_{-1}^{+1}\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}};$$

or, since

$$\int_{-1}^{+1}\frac{dz}{\sqrt{Z}}=2\int_{0,+1}^{+1}\frac{dz}{\sqrt{Z}},\quad \text{(Art. 192).}$$

$$K=\frac{1}{4}B(k)=\int_{0,+1}^{+1}\frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

* Legendre, *Fonctions Elliptiques* (1825), t. I, p. 90.

† Jacobi, Werke, Bd. I, p. 82 (1829).

If further we write

$$z = \frac{-1}{\sqrt{1-k'^2v^2}}, \qquad k'^2 = 1 - k^2,$$

$$dz = \frac{-k'^2 v dv}{(1-k'^2v^2)^{\frac{3}{2}}}, \quad 1 - z^2 = \frac{-k'^2v^2}{1-k'^2v^2}, \quad 1 - k^2z^2 = \frac{k'^2(1-v^2)}{1-k'^2v^2},$$

it is seen that

$$A(k) = 2\int_1^{\frac{1}{k}} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = 2i\int_0^1 \frac{dv}{\sqrt{(1-v^2)(1-k'^2v^2)}}. \qquad \text{[Jacobi.]}$$

If then we write

$$K' = \int_0^1 \frac{dv}{\sqrt{(1-v^2)(1-k'^2v^2)}},$$

we have $\qquad A(k) = 2iK'.$

The quantity k' is called the *complementary* modulus.

Since $B(k) = 4K$ and $A(k) = 2iK'$, the formulas of the preceding article become

$$\bar{u}(+1) = -3K, \qquad \bar{u}(-1) = -K,$$

$$\bar{u}\left(+\frac{1}{k}\right) = -3K - iK', \quad \bar{u}\left(-\frac{1}{k}\right) = -K - iK',$$

$$\bar{u}(\infty, +\infty) = -iK', \qquad \bar{u}(\infty, -\infty) = -2K - iK',$$

$$\bar{u}(0, 1) = 0, \qquad \bar{u}(0, -1) = -2K.$$

Anticipating what follows, if we write

$$u = \int_0^z \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}},$$

and if z considered as a function of u is written

$$z = sn\, u,$$

we have from the above formulas

$$sn(-3K) = 1, \quad sn(-3K - iK') = \frac{1}{k}, \quad sn(-iK') = \infty, \text{ etc.}$$

ART. 194. We shall consider next the moduli of periodicity for Weierstrass's normal form of integral of the first kind.

We note that the point at infinity is a branch-point (Art. 115) for the integral

$$\int \frac{dt}{2\sqrt{(t-e_1)(t-e_2)(t-e_3)}} = \int \frac{dt}{\sqrt{S(t)}},$$

where $\qquad S(t) = S = 4(t-e_1)(t-e_2)(t-e_3).$

In the Riemann surface T without the canals $\mathfrak{a}$ and $\mathfrak{b}$ let

$$\bar{u}(t, \sqrt{S}) = \int_{\infty}^{t, \sqrt{S}} \frac{dt}{\sqrt{S}},$$

and let $\bar{u}(t, \sqrt{S})$ denote the corresponding integral in T'.

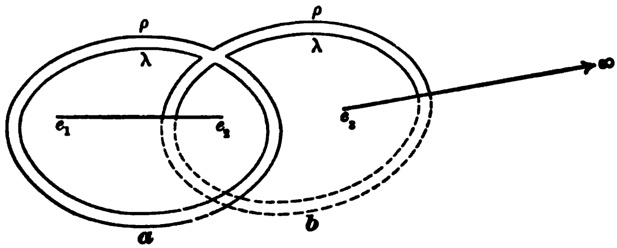


Fig. 65.

We may here write (cf. Art. 139)

$$\bar{u}(\lambda) - \bar{u}(\rho) = A' \textit{ on the canal } \mathfrak{a}, \text{ and}$$
$$\bar{u}(\rho) - \bar{u}(\lambda) = B' \text{ on } \textit{the canal } \mathfrak{b}.$$

The quantities $\bar{u}(e_1)$, $\bar{u}(e_2)$, $\bar{u}(e_3)$ may be computed as follows. In the figure we note that, when the integration is taken in the upper leaf,

$$\int_{\infty}^{e_1} \frac{dt}{\sqrt{S}} = \int_{\infty}^{\rho} \frac{dt}{\sqrt{S}} + \int_{\lambda}^{e_1} \frac{dt}{\sqrt{S}} = \bar{u}(\rho) + \bar{u}(e_1) - \bar{u}(\lambda) = \bar{u}(e_1) - A'.$$

In the lower leaf along the congruent path of integration,

$$\int_{\infty}^{e_1} \frac{dt}{\sqrt{S}} = \bar{u}(e_1).$$

Through subtraction it follows that*

$$\text{(I)} \qquad 2\underset{\text{upper leaf}}{\int_{\infty}^{e_1}} \frac{dt}{\sqrt{S}} = -A'.$$

Similarly along the upper bank of the upper leaf of T',

$$\int_{e_1}^{e_2} \frac{dt}{\sqrt{S}} = \int_{e_1}^{\rho} \frac{dt}{\sqrt{S}} + \int_{\lambda}^{e_2} \frac{dt}{\sqrt{S}} = \bar{u}(\rho) - \bar{u}(e_1) + \bar{u}(e_2) - \bar{u}(\lambda) = B' + \bar{u}(e_2) - \bar{u}(e_1);$$

while for the congruent path in the lower bank,

$$\int_{e_1}^{e_2} \frac{dt}{\sqrt{S}} = \bar{u}(e_2) - \bar{u}(e_1).$$

* Cf. Riemann-Stahl, *Ellipt. Funct.*, p. 134. In comparing the results given by different authors it must be noted that in most cases the sign of *equality* may b· replaced by that of *congruence*.

Hence through subtraction it is seen that in the upper leaf

$$2\int_{e_1}^{e_3}\frac{dt}{\sqrt{S}} = B'.$$

We may therefore write

$$\text{(II)} \qquad 2\int_{\infty}^{e_3}\frac{dt}{\sqrt{S}} = 2\int_{e_1}^{e_3}\frac{dt}{\sqrt{S}} + 2\int_{\infty}^{e_1}\frac{dt}{\sqrt{S}} = -A' + B'.$$

We further have in the upper leaf of T',

$$\int_{e_2}^{e_3}\frac{dt}{\sqrt{S}} = \int_{e_2}^{\lambda}\frac{dt}{\sqrt{S}} + \int_{\rho}^{e_3}\frac{dt}{\sqrt{S}} = \bar{u}(\lambda) - \bar{u}(e_2) + \bar{u}(e_3) - \bar{u}(\rho)$$
$$= A' + \bar{u}(e_3) - \bar{u}(e_2);$$

while in the lower leaf

$$\int_{e_2}^{e_3}\frac{dt}{\sqrt{S}} = \bar{u}(e_3) - \bar{u}(e_2).$$

Through subtraction we have in the upper leaf

$$2\int_{e_2}^{e_3}\frac{dt}{\sqrt{S}} = A'.$$

It is also evident that

$$2\int_{\infty}^{e_3}\frac{dt}{\sqrt{S}} = 2\int_{\infty}^{e_2}\frac{dt}{\sqrt{S}} + 2\int_{e_2}^{e_3}\frac{dt}{\sqrt{S}} = -A' + B' + A',$$

or

$$\text{(III)} \qquad \int_{\infty}^{e_3}\frac{dt}{\sqrt{S}} = \frac{B'}{2}.$$

ART. 195. It follows at once from (I), (II) and (III) that in T'

$$\bar{u}(e_1) = \int_{\infty}^{e_1}\frac{dt}{\sqrt{S}} = -\frac{A'}{2} = -\omega, \text{ say,}$$
$$\bar{u}(e_2) = \int_{\infty}^{e_2}\frac{dt}{\sqrt{S}} = \frac{-A' + B'}{2} = -\omega'',$$
$$\bar{u}(e_3) = \int_{\infty}^{e_3}\frac{dt}{\sqrt{S}} = \frac{B'}{2} = -\omega'.$$

From these definitions of ω, ω', ω'', it is seen that

$$\omega'' = \omega + \omega'.$$

Again (cf. Art. 185), if we write*

$$-u = \int_{\infty}^{t}\frac{dt}{\sqrt{S}},$$

and write the upper limit, considered as a function of the integral u,

$$t = \wp(u),$$

* The sign of the integral is changed in order to retain the notation of Weierstrass. It is seen in Chapter XV that $\wp u$ is an *even* function. It is called the *Pe-function*.

we have

$$\begin{cases} e_1 = \wp(\omega), \\ e_2 = \wp(\omega''), \\ e_3 = \wp(\omega'). \end{cases}$$

Art. 196. In Art. 185 we derived the relation

$$\frac{dt}{\sqrt{S}} = -\sqrt{\epsilon}\,\frac{d\mathrm{z}}{\sqrt{\mathrm{Z}}},$$

where

$$t = e_3 + \frac{1}{\epsilon \mathrm{z}^2}.$$

If we write

$$-du = \frac{dt}{\sqrt{S}}, \quad \text{or} \quad -u = \int_\infty^t \frac{dt}{\sqrt{S}},$$

we have

$$\frac{du}{\sqrt{\epsilon}} = \frac{d\mathrm{z}}{\sqrt{\mathrm{Z}}}, \quad \text{or} \quad \frac{u}{\sqrt{\epsilon}} = \int_{0,1}^{\mathrm{z}} \frac{d\mathrm{z}}{\sqrt{\mathrm{Z}}}.$$

It follows that

$$t = \wp(u), \quad \mathrm{z} = sn\left(\frac{u}{\sqrt{\epsilon}}\right)$$

and

$$\wp(u) = e_3 + \frac{1}{\epsilon\, sn^2\left(\frac{u}{\sqrt{\epsilon}}\right)}.$$

It is also evident that

$$K = \int_{0,1}^{1} \frac{d\mathrm{z}}{\sqrt{\mathrm{Z}}} = -\frac{1}{\sqrt{\epsilon}} \int_\infty^{e_1} \frac{dt}{\sqrt{S}} = \frac{\omega}{\sqrt{\epsilon}},$$

or $\omega = \sqrt{\epsilon}K$, and similarly $\omega' = \sqrt{\epsilon}iK'$.

Art. 197. *The conformal representation of the T'-surface.*

In Chapter VII we saw that if

$$u = \int_{z_0, s_0}^{z, s} \frac{dz}{\sqrt{R(z)}},$$

a one-valued function of u. We also saw that if the path of n is unrestricted, more than one value of u correspond to every The collectivity of these values was expressed by

$$u = \bar{u}(z, s) + mA + lB,$$

, s) represented the above integral in the simply connected ' and m and l were integers.

rite $z = \phi(u)$, then $\phi(u)$ is a one-valued function of u. We that $s = \dfrac{dz}{du} = \sqrt{R(z)}$ is a one-valued function of u. Further, quantity $\bar{u}$ is uniquely determined as soon as the upper limit own. Therefore for every value z, s in the surface T' we may

compute the corresponding value of $\bar{u}$ and lay it off in the plane of the complex variable $\bar{u}$. Since the integral $\bar{u}$ never becomes infinite (Art. 136), it follows that all the values of $\bar{u}$ which correspond to the collectivity of values z, s in the surface T' may be laid off within a finite portion of the $\bar{u}$-plane.

It cannot happen that to two different values of z, s on the surface T' there corresponds the same value $\bar{u}$. For if this were possible, then reciprocally to this value of $\bar{u}$ *either* there would correspond two different values of z in the T'-surface and z would not be a one-valued function of u, *or* there would correspond two different values of s, and then s would not be a one-valued function of $\bar{u}$. The points in the $\bar{u}$-plane follow one another in a continuous manner and the region which they fill is *simply* covered. It follows that the portion of surface in the $\bar{u}$-plane and the simply connected Riemann surface T' are *conformal representations* of each other, since to every point of the one structure there corresponds one and only one point of the other structure and *vice versa*.

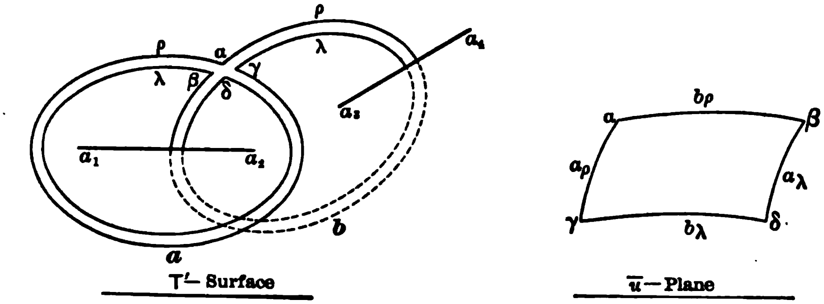


Fig. 66.

We may next investigate the form of the portion of surface in the $\bar{u}$-plane which is the *image* of the surface included within the canals $\boldsymbol{a}$ and $\boldsymbol{b}$. We compute the value of $\bar{u}$ for the point β which is the intersection of the left bank of the canal $\boldsymbol{a}$ with the right bank of the canal $\boldsymbol{b}$. The value of $\bar{u}$ at this point we also call β and lay it off in the $\bar{u}$-plane. We compute for every point of the left bank of $\boldsymbol{a}$ the corresponding value of $\bar{u}$ and lay it off in the $\bar{u}$-plane. We thus have a curve a_λ in the $\bar{u}$-plane which does not cross itself. Let the other end-point be denoted by δ in the $\bar{u}$-plane, which point corresponds to the point δ on the surface T'. If next starting from δ we traverse the bank λ of $\boldsymbol{b}$ and lay off the corresponding values in the plane $\bar{u}$, we have a curve b_λ which ends at γ, say. Then starting from γ in T' we go along the bank ρ of $\boldsymbol{a}$ and form in the $\bar{u}$-plane the corresponding curve a_ρ. Finally we return along the bank ρ of $\boldsymbol{b}$ back to β, and the corresponding curve b_ρ in the $\bar{u}$-plane must lead back to the

initial point β. The canals $\boldsymbol{a}$ and $\boldsymbol{b}$ are thus conformally represented on the $\overline{u}$-plane.

Since the canals $\boldsymbol{a}$ and $\boldsymbol{b}$ are the boundaries of T', the curve $a_\lambda b_\lambda a_\rho b_\rho$ must bound the surface which is the conformal representation of T' in the $\overline{u}$-plane. The *interior* of the figure is this conformal representation, for $\overline{u}$ cannot be infinite for any value of z, s, which may be the case if the surface without the curve $a_\lambda b_\lambda a_\rho b_\rho$ represented conformally T'.

Remark. — The curves a_λ and a_ρ are parallel curves, that is, to every point on a_λ there corresponds a point on a_ρ, so that lines joining such pairs of points are equal and parallel. For if we take on the canal $\boldsymbol{a}$ in T' two points opposite each other on the left and the right banks respectively, then we have

$$\overline{u}(\lambda) - \overline{u}(\rho) = A.$$

Consequently the complex quantity A represents the length and the direction in the $\overline{u}$-plane of the distance between two points lying on opposite banks of the canal $\boldsymbol{a}$, which conformally in the $\overline{u}$-plane lie on the curves a_λ and a_ρ. Since A is a constant the two curves a_λ and a_ρ must be parallel.

Similarly b_λ and b_ρ are parallel curves and the distance between them is B.

If the variable crosses a canal $\boldsymbol{a}$ or $\boldsymbol{b}$ in T', we have values of $\overline{u}$ which lie in a period-parallelogram that is congruent to the first parallelogram, and by crossing the canals $\boldsymbol{a}$ and $\boldsymbol{b}$ arbitrarily often in either direction we have more and more parallelograms which completely fill out the $\overline{u}$-plane.

ART. 198. The form of the two canals $\boldsymbol{a}$ and $\boldsymbol{b}$ was arbitrary. We shall show that they may be taken so that the corresponding parallelogram in the $\overline{u}$-plane is straight-lined. As a somewhat special case take Legendre's normal form and let the modulus k be real, positive and less than unity.

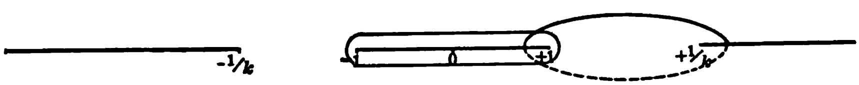


Fig. 67.

We draw the canals $\boldsymbol{a}$ and $\boldsymbol{b}$ so that they lie indefinitely near the real axis and indefinitely close to the points $-1, +1, +\dfrac{1}{k}$, as shown in the figure.

We had in T'

$$\overline{u} = \int_{0,1}^{z,s} \frac{dz}{\sqrt{Z}}.$$

The differential dz is here real, being taken along the right and left banks of the canals which are supposed to lie indefinitely near the real axis.

For the bank λ of $\mathbf{a}$ we have $1 - \mathbf{z}^2 > 0$ for all points except $\mathbf{z} = 1$, or $\mathbf{z} = -1$, and consequently also

$$1 - k^2\mathbf{z}^2 > 0.$$

It follows that a_λ is real in the $\overline{u}$-plane, since $\overline{u}$ is real for all points on the bank λ of $\mathbf{a}$. Hence in the $\overline{u}$-plane a_λ coincides with the real axis.

On the bank λ of $\mathbf{b}$ we have

$$1 - \mathbf{z}^2 < 0 \quad \text{and} \quad 1 - k^2\mathbf{z}^2 > 0.$$

The elements of integration are therefore all pure imaginaries along this bank and consequently $\overline{u}$ is purely imaginary along this bank. It follows that b_λ in the $\overline{u}$-plane is a straight line that stands perpendicular to the axis of the real.

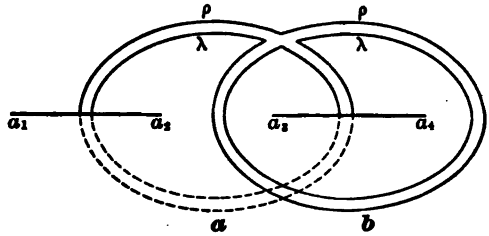


Fig. 68.

Since a_ρ is parallel to a_λ and b_ρ to b_λ, the conformal representation of T' on the $\overline{u}$-plane is a rectangle with the sides

$$\beta\delta = B(k) = 4K,$$
$$\delta\gamma = A(k) = 2iK'.$$

We may represent the integral in Weierstrass's normal form conformally in a like manner. This is left as an exercise for the student.

As another exercise derive the results of this Chapter by taking the Riemann surface as indicated in Fig. 68.

EXAMPLES

1. Show that $$\int_{\frac{1}{k}}^{\infty,-\infty} \frac{d\mathbf{z}}{\sqrt{\mathbf{Z}}} \equiv K.$$

2. Show that $$\int_0^{\frac{1}{k}} \frac{d\mathbf{z}}{\sqrt{\mathbf{Z}}} \equiv K + iK'.$$

3. Prove that $$\omega' \equiv \int_{e_3}^{e_2} \frac{dt}{\sqrt{S}} \equiv \int_{e_1}^{\infty} \frac{dt}{\sqrt{S}}.$$

4. The substitution $$\frac{e_1 - e_3}{s - e_3} = \frac{t - e_3}{e_2 - e_3}$$

transforms

$$\omega = \int_{e_1}^{\infty} \frac{dt}{\sqrt{4(t - e_1)(t - e_2)(t - e_3)}} \quad \text{into} \quad \int_{e_3}^{e_2} \frac{ds}{\sqrt{4(s - e_1)(s - e_2)(s - e_3)}}.$$

How does this result compare with the one derived by the methods of this Chapter?

5. Derive by means of the Riemann surface the formula

$$\omega'' = \omega + \omega' \equiv \int_{e_3}^{\infty} \frac{dt}{\sqrt{S}} + \int_{e_2}^{e_1} \frac{dt}{\sqrt{S}} \equiv \int_{e_3}^{e_1} \frac{dt}{\sqrt{S}}.$$

CHAPTER X

THE JACOBI THETA-FUNCTIONS

ARTICLE 199. We saw in Chapter V that the Φ-functions of the second degree satisfied the two functional equations

$$\Phi(u + a) = \Phi(u),$$

$$\Phi(u + b) = e^{-\frac{2\pi i}{a}(2u+b)}\,\Phi(u).$$

If $Q = e^{\pi i\frac{b}{a}}$, we saw in Art. 87 that

$$\Theta_1(u) = \sum_{m=-\infty}^{m=+\infty} Q^{2m^2} e^{\frac{4\pi i}{a}mu},$$

$$H_1(u) = \sum_{m=-\infty}^{m=+\infty} Q^{\left(\frac{2m+1}{2}\right)^2} e^{(2m+1)\frac{2\pi i}{a}u}.$$

We have now to write: $\quad a = B(k) = 4K,$

$$b = A(k) = 2iK'.$$

It follows that $Q = e^{-\frac{\pi}{2}\frac{K'}{K}}$.
If further we write

$$q = Q^2 = e^{-\pi\frac{K'}{K}},$$

we have *

$$\Theta_1(u) = \sum_{m=-\infty}^{m=+\infty} q^{m^2} e^{\frac{\pi i}{K}mu},$$

$$H_1(u) = \sum_{m=-\infty}^{m=+\infty} q^{\left(\frac{2m+1}{2}\right)^2} e^{(2m+1)\frac{\pi i u}{2K}}.$$

When K and K' are introduced into the functional equations, they are

$$\Phi(u + 4K) = \Phi(u),$$

$$\Phi(u + 2iK') = e^{-\frac{\pi i}{K}(u+iK')}\Phi(u).$$

In $\Theta_1(u)$ the term which corresponds to $m = 0$ is unity. If we take this term without the summation and then combine under the summa-

* Cf. Jacobi, Werke, Bd. I, pp. 224 *et seq.;* and in particular Hermite, *Cours rédigé en* 1882 *par* M. Andoyer, p. 235 (Quatrième édition).

tion the term which corresponds to $+m$ with the term that corresponds to $-m$, we have

$$\Theta_1(u) = 1 + \sum_{m=1}^{m=\infty} q^{m^2}\left\{e^{\frac{\pi i m u}{K}} + e^{-\frac{\pi i m u}{K}}\right\},$$

or

$$\Theta_1(u) = 1 + 2\sum_{m=1}^{m=\infty} q^{m^2}\cos\left(\frac{m\pi u}{K}\right).$$

The terms in $H_1(u)$ may be combined as follows:

$$H_1(u) = \sum_{m=0}^{m=\infty} q^{\left(\frac{2m+1}{2}\right)^2}\left\{e^{(2m+1)\frac{\pi i u}{2K}} + e^{-(2m+1)\frac{\pi i u}{2K}}\right\},$$

or

$$H_1(u) = 2\sum_{m=0}^{m=\infty} q^{\left(\frac{2m+1}{2}\right)^2}\cos\left(\frac{(2m+1)\pi u}{2K}\right).$$

It follows immediately, as we have already seen in Art. 148, that

$$\Theta_1(-u) = \Theta_1(u),$$
$$H_1(-u) = H_1(u).$$

ART. 200. We introduce two new functions,* the first of which is defined by the relation

$$H(u) = H_1(u - K).$$

We have at once

$$H(u) = 2\sum_{m=0}^{m=\infty} q^{\left(\frac{2m+1}{2}\right)^2}\cos\left\{\frac{(2m+1)\pi u}{2K} - (2m+1)\frac{\pi}{2}\right\}.$$

But, since $\cos\left[A - (2m+1)\frac{\pi}{2}\right] = (-1)^m \sin A$, it is seen that

$$H(u) = 2\sum_{m=0}^{m=\infty} (-1)^m q^{\left(\frac{2m+1}{2}\right)^2}\sin\left\{\frac{(2m+1)\pi u}{2K}\right\}.$$

The second function is defined by the relation

$$\Theta(u) = \Theta_1(u - K);$$

and consequently we have

$$\Theta(u) = 1 + 2\sum_{m=1}^{m=\infty} (-1)^m q^{m^2}\cos\left(\frac{m\pi u}{K}\right).$$

It is seen at once that

$$\Theta(-u) = \Theta(u).$$
$$H(-u) = -H(u).$$

The functions $\Theta_1(u)$, $H_1(u)$, $\Theta(u)$, $H(u)$ are known in mathematics as the Θ-functions. Excepting $H(u)$ they are all *even* functions, and it is seen that they are more rapidly convergent than a geometrical progression.

* Cf. Jacobi, *loc. cit.*, p. 235, and Werke, II, p. 293; see also Hermite, *loc. cit.*, p. 235.

ART. 201. From the equation

$$\mathrm{H}_1(u-K)=\mathrm{H}(u)$$

it follows that $\mathrm{H}_1(u+K)=\mathrm{H}(u+2K)$,

and therefore $\mathrm{H}_1(u+K)=-\mathrm{H}(u)$.

In a similar manner

$$\Theta_1(u+K)=\Theta(u+2K)=\Theta(u).$$

We also have

$$\mathrm{H}(u+K)=\mathrm{H}_1(u)$$

and $\Theta(u+K)=\Theta_1(u)$.

We thus have the four formulæ

$$\text{(I)}\quad\begin{cases}\mathrm{H}_1(u+K)=-\mathrm{H}(u),\\ \Theta_1(u+K)=\quad\Theta(u),\\ \mathrm{H}(u+K)=\quad\mathrm{H}_1(u),\\ \Theta(u+K)=\quad\Theta_1(u).\end{cases}$$

From these formulæ we derive at once

$$\text{(II)}\quad\begin{cases}\mathrm{H}_1(u+2K)=-\mathrm{H}_1(u),\\ \Theta_1(u+2K)=\quad\Theta_1(u),\\ \mathrm{H}(u+2K)=-\mathrm{H}(u),\\ \Theta(u+2K)=\quad\Theta(u).\end{cases}$$

From (I) and (II) we again have

$$\text{(III)}\quad\begin{cases}\mathrm{H}_1(u+3K)=\quad\mathrm{H}(u),\\ \Theta_1(u+3K)=\quad\Theta(u),\\ \mathrm{H}(u+3K)=-\mathrm{H}_1(u),\\ \Theta(u+3K)=\quad\Theta_1(u);\end{cases}$$

and finally

$$\text{(IV)}\quad\begin{cases}\mathrm{H}_1(u+4K)=\mathrm{H}_1(u),\\ \Theta_1(u+4K)=\Theta_1(u),\\ \mathrm{H}(u+4K)=\mathrm{H}(u),\\ \Theta(u+4K)=\Theta(u).\end{cases}$$

ART. 202. We shall next increase the argument of the Θ-functions by iK'.

We have

$$\begin{aligned}\Theta_1(u+iK')&=\sum_{m=-\infty}^{m=+\infty}q^{m^2}e^{m\frac{\pi i}{K}(u+iK')}\\ &=\sum_{m=-\infty}^{m=+\infty}q^{m^2}e^{m\frac{\pi iu}{K}}e^{-m\pi\frac{K'}{K}}=\sum_{m=-\infty}^{m=+\infty}q^{m^2+m}e^{\frac{m\pi iu}{K}}\\ &=\sum_{m=-\infty}^{m=+\infty}q^{\left(\frac{2m+1}{2}\right)^2}q^{-\frac{1}{4}}e^{\frac{(2m+1)\pi iu}{2K}}e^{-\frac{\pi iu}{2K}}=q^{-\frac{1}{4}}e^{-\frac{\pi iu}{2K}}\mathrm{H}_1(u).\end{aligned}$$

If further we write *

$$e^{\frac{\pi}{4}\frac{K'}{K}-\frac{\pi i u}{2K}}=\lambda=\lambda(u),$$

we have

$$\Theta_1(u+iK')=\lambda(u)\mathrm{H}_1(u).$$

We may also note that

$$\mathrm{H}_1(u+iK')=\sum_{m=-\infty}^{m=+\infty}q^{\left(\frac{2m+1}{2}\right)^2}e^{\frac{(2m+1)\pi i(u+iK')}{2K}}$$

$$=\sum_{m=-\infty}^{m=+\infty}q^{\left(\frac{2m+1}{2}\right)^2}e^{\frac{(2m+1)\pi i u}{2K}}e^{-\frac{(2m+1)\pi K'}{2K}}$$

$$=\sum_{m=-\infty}^{m=+\infty}q^{\left(\frac{2m+1}{2}\right)^2+\frac{2m+1}{2}}e^{\frac{(2m+1)\pi i u}{2K}}$$

$$=\sum_{m=-\infty}^{m=+\infty}q^{(m+1)^2}q^{-\frac{1}{4}}e^{\frac{(m+1)\pi i u}{K}}e^{-\frac{\pi i u}{2K}}$$

$$=\sum_{m'=-\infty}^{m'=+\infty}q^{m'^2}e^{m'\frac{\pi i u}{K}}e^{\frac{\pi}{4}\frac{K'}{K}-\frac{\pi i u}{2K}},$$

where $m+1=m'$.

It is seen at once that

$$\mathrm{H}_1(u+iK')=\lambda(u)\,\Theta_1(u).$$

Since

$$\Theta(u)=\Theta_1(u-K),$$

we have

$$\Theta(u+iK')=\Theta_1(u-K+iK')$$

$$=e^{\frac{\pi K'}{4K}-\frac{\pi i}{2K}(u-K)}\mathrm{H}_1(u-K)$$

$$=\lambda(u)e^{\frac{\pi i}{2}}\mathrm{H}(u)=i\lambda(u)\,\mathrm{H}(u).$$

In a similar manner it is seen that

$$\mathrm{H}(u+iK')=i\lambda(u)\,\Theta(u).$$

We may therefore write

$$(\mathrm{V})\quad\begin{cases}\mathrm{H}_1(u+iK')=\lambda(u)\,\Theta_1(u),\\ \Theta_1(u+iK')=\lambda(u)\,\mathrm{H}_1(u),\\ \mathrm{H}(u+iK')=i\lambda(u)\,\Theta(u),\\ \Theta(u+iK')=i\lambda(u)\,\mathrm{H}(u).\end{cases}$$

It follows from (I) and (V) that

$$(\mathrm{VI})\quad\begin{cases}\mathrm{H}_1(u+K+iK')=-i\lambda(u)\,\Theta(u),\\ \Theta_1(u+K+iK')=i\lambda(u)\,\mathrm{H}(u),\\ \mathrm{H}(u+K+iK')=\lambda(u)\,\Theta_1(u),\\ \Theta(u+K+iK')=\lambda(u)\,\mathrm{H}_1(u).\end{cases}$$

* Hermite, *loc. cit.*, p. 236.

It is clear that

$$\mathrm{H}_1(u + 2\,iK') = \mathrm{H}_1[(u + iK') + iK']$$
$$= e^{\frac{\pi}{4}\frac{K'}{K} - \frac{\pi i}{2K}(u+iK')}\,\Theta_1(u + iK')$$
$$= e^{\frac{\pi}{4}\frac{K'}{K} - \frac{\pi i}{2K}(u+iK')}\,\lambda(u)\,\mathrm{H}_1(u).$$

If we put

$$\mu = \mu(u) = e^{-\frac{\pi i}{K}(u+iK')},$$

it follows that

$$\mathrm{H}_1(u + 2\,iK') = \mu(u)\,\mathrm{H}_1(u).$$

We have the following formulæ:

$$\text{(VII)}\quad \begin{cases} \mathrm{H}_1(u + 2\,iK') = \mu(u)\,\mathrm{H}_1(u), \\ \Theta_1(u + 2\,iK') = \mu(u)\,\Theta_1(u), \\ \mathrm{H}(u + 2\,iK') = -\,\mu(u)\,\mathrm{H}(u), \\ \Theta(u + 2\,iK') = -\,\mu(u)\,\Theta(u). \end{cases}$$

It is seen that H and Θ satisfy the functional equations

$$\Phi(u + 4\,K) = \Phi(u), \quad \Phi(u + 2\,iK') = -\,\mu(u)\,\Phi(u);$$

while H_1 and Θ_1 satisfy

$$\Phi(u + 4\,K) = \Phi(u), \quad \Phi(u + 2\,iK') = +\,\mu(u)\,\Phi(u).$$

We note in particular that the four theta-functions belong to two categories of functions of essentially different nature.

ART. 203. *The Zeros.* — The Θ-functions being Φ-functions of the second degree vanish at two incongruent points (*congruent* points being those which differ from one another by multiples of $4\,K$ and $2\,iK'$).

We saw in Art. 200 that $\mathrm{H}(u)$ was an odd function and therefore vanishes for $u = 0$. We also had

$$\mathrm{H}(u + 2\,K) = -\,\mathrm{H}(u),$$

and consequently

$$\mathrm{H}(2\,K) = -\,\mathrm{H}(0) = 0.$$

The points 0, $2\,K$ are therefore the two incongruent zeros of this function; *i.e.*, the function $\mathrm{H}(u)$ vanishes on all points of the form

$$0 + m_1\,4\,K + l_1\,2\,iK',$$
$$2\,K + m_2\,4\,K + l_2\,2\,iK',$$

where m_1, m_2, l_1, l_2 are integers.

Hence all the points at which $\mathrm{H}(u)$ vanishes are had for the values of the argument

$$u = m\,2\,K + n\,2\,iK',$$

where m and n are integers.

Further, since

$$\Theta(u + iK') = i\lambda(u)\, H(u),$$

when $u = 0$, we see that

$$\Theta(iK') = 0;$$

and since

$$\Theta(u + 2K) = \Theta(u),$$

we also have

$$\Theta(iK' + 2K) = 0.$$

The zeros of $\Theta(u)$ are consequently

$$m\,2K + (2n + 1)iK'.$$

By definition we have

$$H(u) = H_1(u - K),$$

so that the zeros of $H_1(u)$ are

$$(2m + 1)K + n\,2iK'.$$

Finally, since

$$\Theta(u) = \Theta_1(u - K),$$

the zeros of $\Theta_1(u)$ are

$$(2m + 1)K + (2n + 1)iK'.$$

ART. 204. Write

$$\Theta_1(u) = \sum_{m=-\infty}^{m=+\infty} q^{2} e^{\frac{\pi i}{K}mu} = \Theta_1(u; K, iK'),$$

where

$$q = e^{-\pi\frac{K'}{K}};$$

and

$$\sum_{m=-\infty}^{m=+\infty} q_0{}^{2} e^{\frac{\pi imu}{K'}} = \Theta_1(u; K', iK),$$

where

$$q_0 = e^{-\pi\frac{K}{K'}}.$$

It is seen that the latter series fulfills the requirements of convergence given in Art. 86.

We also note, cf. formulas (II) and (VII), that

$$\Theta_1(u; K', iK) \text{ and } e^{-\frac{\pi u^2}{4KK'}}\Theta_1(iu; K, iK')$$

satisfy the same functional equations

$$\Phi(u + 2K') = \Phi(u),$$

$$\Phi(u + 2iK) = e^{-\frac{\pi i}{K'}(u+iK)}\Phi(u).$$

The two functions have also the same zeros

$$u = (2m + 1)K' + (2n + 1)iK.$$

It follows that the ratio of the two functions is a constant.

We therefore have (cf. Jacobi's Werke, Bd. I, p. 214)

$$e^{-\frac{\pi u^2}{4KK'}}\Theta_1(iu; K, iK') = C\Theta_1(u; K', iK),$$

$$e^{-\frac{\pi u^2}{4KK'}}\mathrm{H}_1(iu; K, iK') = C\Theta(u; K', iK),$$

$$e^{-\frac{\pi u^2}{4KK'}}\mathrm{H}(iu; K, iK') = iC\mathrm{H}(u; K', iK),$$

$$e^{-\frac{\pi u^2}{4KK'}}\Theta(iu; K, iK') = C\mathrm{H}_1(u; K', iK).$$

Expression of the Theta-functions in the Form of Infinite Products.

Art. 205. With Hermite* consider the two functions

$$\Phi(u) = \phi(u + iK')\,\phi(u + 3\,iK')\,\phi(u + 5\,iK') \cdots \\ \phi(-u + iK')\,\phi(-u + 3\,iK')\,\phi(-u + 5\,iK') \cdots;$$

and

$$\Phi_1(u) = \phi(u)\,\phi(u + 2\,iK')\,\phi(u + 4\,iK') \cdots \\ \phi(-u + 2\,iK')\,\phi(-u + 4\,iK') \cdots.$$

It is seen at once that if $\phi(u)$ has the period $2\,K$, then

$$\Phi(u + 2\,K) = \Phi(u),$$

$$\Phi_1(u + 2\,K) = \Phi_1(u).$$

It is also evident that

$$\Phi_1(u + iK') = \Phi(u),$$

$$\Phi(u + iK') = \Phi_1(u)\,\frac{\phi(-u)}{\phi(u)};$$

and consequently

$$\Phi(u + 2\,iK') = \Phi(u)\,\frac{\phi(-u - iK')}{\phi(u + iK')},$$

$$\Phi_1(u + 2\,iK') = \Phi_1(u)\,\frac{\phi(-u)}{\phi(u)}.$$

If next we put

$$\phi(u) = 1 + e^{\frac{\pi iu}{K}},$$

we have

$$\frac{\phi(-u)}{\phi(u)} = e^{-\frac{\pi iu}{K}},$$

and also

$$\phi(u + niK')\,\phi(-u + niK') = 1 + 2\,q^n \cos\frac{\pi u}{K} + q^{2n}.$$

* See *Note sur la théorie des fonctions elliptiques* placed at the end of Serret's *Calc\[u\]l Différentiel et Intégral*, pp. 753 *et seq.*; Œuvres, t. 2, pp. 123 *et seq.*

It is thus seen that

$$\Phi(u) = \prod (1 + 2q^n \cos \frac{\pi u}{K} + q^{2n})$$
$$(n = 1, 3, 5, \ldots)$$

and

$$\Phi_1(u) = \left(1 + e^{\frac{\pi i u}{K}}\right) \prod (1 + 2q^n \cos \frac{\pi u}{K} + q^{2n}).$$
$$(n = 2, 4, 6, \ldots.)$$

These products are convergent (cf. Art. 17) if $|q| < 1$ (see Art. 81).

ART. 206. The two functions $\Phi(u)$, $\Phi_1(u)$ both have the period $2K$ and they satisfy the functional equations

$$\Phi_1(u + iK') = \Phi(u),$$
$$\Phi(u + iK') = e^{-\frac{\pi i u}{K}} \Phi_1(u).$$

Let us introduce a function $\Psi(u)$ defined by the equation

$$\Phi_1(u) = e^{\frac{\pi i}{2K}(u - \frac{1}{2} iK')} \Psi(u).$$

We have at once

$$\Psi(u + iK') = e^{-\frac{\pi i}{2K}\left(u + \frac{iK'}{2}\right)} \Phi(u),$$
$$\Phi(u + iK') = e^{-\frac{\pi i}{2K}\left(u + \frac{iK'}{2}\right)} \Psi(u);$$
$$\Phi(u + 2K) = \Phi(u), \quad \Psi(u + 2K) = -\Psi(u).$$

It is evident from formulas (II) and (V) that we may write (cf. Art. 83)

$$\Theta_1(u) = A\Phi(u),$$
$$H_1(u) = A\Psi(u),$$

where A is a constant.

Noting also that

$$\Theta(u) = \Theta_1(K - u),$$
$$H(u) = H_1(K - u),$$

it is seen that

$$\Theta_1\left(\frac{2Ku}{\pi}\right) = A(1 + 2q\cos 2u + q^2)(1 + 2q^3 \cos 2u + q^6)(1 + 2q^5 \cos 2u + q^{10}) \cdots,$$

$$H_1\left(\frac{2Ku}{\pi}\right) = 2A\sqrt[4]{q}\cos u\,(1 + 2q^2 \cos 2u + q^4)(1 + 2q^4 \cos 2u + q^8) \cdots,$$

$$\Theta\left(\frac{2Ku}{\pi}\right) = A(1 - 2q\cos 2u + q^2)(1 - 2q^3 \cos 2u + q^6) \cdots,$$

$$H\left(\frac{2Ku}{\pi}\right) = 2A\sqrt[4]{q}\sin u\,(1 - 2q^2 \cos 2u + q^4)(1 - 2q^4 \cos 2u + q^8) \cdots,$$

where A is a constant.

ART. 207. To determine the constant A of the preceding article, we follow a method due to Biehler.*

Consider the product composed of a finite number of factors

$$(1)\qquad f(t) = (1 + qt)\,(1 + q^3t)\;.\;.\;.\;(1 + q^{2n-1}t)$$
$$\left(1 + \frac{q}{t}\right)\left(1 + \frac{q^3}{t}\right)\;.\;.\;.\;\left(1 + \frac{q^{2n-1}}{t}\right).$$

This expression developed according to positive and negative powers of t is of the form

$$(2)\qquad f(t) = A_0 + A_1\left(t + \frac{1}{t}\right) + A_2\left(t^2 + \frac{1}{t^2}\right) + \cdot\cdot\cdot + A_n\left(t^n + \frac{1}{t^n}\right).$$

The following identity, which may be at once verified,

$$f(q^2t)\,(q^{2n} + qt) = f(t)\,(1 + q^{2n+1}t),$$

gives between two consecutive coefficients A_i and A_{i-1} the relation

$$A_i(1 - q^{2n+2i}) = A_{i-1}\,(q^{2i-1} - q^{2n+1}).$$

We thus have

$$A_1 = A_0\frac{q(1 - q^{2n})}{1 - q^{2n+2}},$$

$$A_2 = A_1\frac{q^3(1 - q^{2n-2})}{1 - q^{2n+4}},$$

$$.\quad.\quad.\quad.\quad.\quad.\quad.\quad.$$

$$A_i = A_{i-1}\frac{q^{2i-1}(1 - q^{2n-2i+2})}{1 - q^{2n+2i}},$$

$$.\quad.\quad.\quad.\quad.\quad.\quad.\quad.$$

$$A_n = A_{n-1}\,\frac{q^{2n-1}(1 - q^2)}{1 - q^{4n}}.$$

If these equations are multiplied together, we find that

$$A_n = A_0q^{n^2}\frac{(1 - q^{2n})\,(1 - q^{2n-2})\;.\;.\;.\;(1 - q^2)}{(1 - q^{2n+2})\,(1 - q^{2n+4})\;.\;.\;.\;(1 - q^{4n})}.$$

But it follows directly from (1) and (2) that

$$A_n = q^{n^2}.$$

We therefore have

$$(1 - q^{4n})^n \quad A_0 \underset{\leq}{=} \frac{(1 - q^{2n+2})\,(1 - q^{2n+4})\;.\;.\;.\;(1 - q^{4n})}{(1 - q^2)\,(1 - q^4)\;.\;.\;.\;(1 - q^{2n})}. \leq 1$$

When n becomes indefinitely large, it is seen that

$$A_0 = \frac{1}{(1 - q^2)\,(1 - q^4)\,(1 - q^6)\;.\;.\;.}.$$

* Biehler, *Crelle*, Bd. 88, pp. 185–204; see also Hermite, *loc. cit.*, pp. 770–772; Appell et Lacour, *Fonctions Elliptiques*, pp. 398–399. Jacobi gives two methods of determining this constant (Werke, I, p. 230, § 63 and § 64) and a third proof (Werke, II, pp. 153, 160).

$\lim_{x\to\infty}(1 - e^{-ax})^x = 1$

Further, since

$$A_i = A_0 q^{i^2},$$

it follows from the equation (2) that

$$(1 + qt)(1 + q^3t)(1 + q^5t) \ldots \left(1 + \frac{q}{t}\right)\left(1 + \frac{q^3}{t}\right)\left(1 + \frac{q^5}{t}\right) \ldots$$
$$= \frac{1 + q\left(t + \frac{1}{t}\right) + q^4\left(t^2 + \frac{1}{t^2}\right) + \cdots + q^{i^2}\left(t^i + \frac{1}{t^i}\right) + \cdots}{(1 - q^2)(1 - q^4)(1 - q^6) \ldots}.$$

Writing $t = e^{2iu}$, this formula becomes

$$(1 + 2q\cos 2u + q^2)(1 + 2q^3\cos 2u + q^6) \cdots$$
$$= \frac{1 + 2q\cos 2u + 2q^4\cos 4u + \cdots}{(1 - q^2)(1 - q^4)(1 - q^6) \ldots}.$$

From this we conclude that the constant A of the previous Article is

$$A = (1 - q^2)(1 - q^4)(1 - q^6) \cdots;$$

and at the same time it is shown that Θ_1 as defined in the last Article as an infinite product is

$$\Theta_1\left(\frac{2Ku}{\pi}\right) = 1 + 2q\cos 2u + 2q^4\cos 4u + \cdots,$$

or

$$\Theta_1(u) = \sum_{m=-\infty}^{m=+\infty} e^{\frac{\pi i}{K}(mu + m^2 iK')},$$

which is the original definition of this Θ-function.

Example. — By means of the infinite products prove the formulas (I), (II), (III), (IV) and (V) of this Chapter, and therefrom derive the expressions in infinite series of $\mathrm{H}_1\left(\frac{2Ku}{\pi}\right)$, $\mathrm{H}\left(\frac{2Ku}{\pi}\right)$ and $\Theta\left(\frac{2Ku}{\pi}\right)$.

The Small Theta-Functions.

Art. 208. Jacobi (Werke, Bd. I, pp. 499 *et seq.*) introduced a notation similar to the following (see Art. 210):

$$\Theta(2Ku) = \vartheta_0(u) = \sum_{m=-\infty}^{m=+\infty} (-1)^m q^{m^2} e^{2m\pi i u},$$

$$\mathrm{H}(2Ku) = \vartheta_1(u) = \frac{1}{i}\sum_{m=-\infty}^{m=+\infty} (-1)^m q^{\left(\frac{2m+1}{2}\right)^2} e^{(2m+1)\pi i u},$$

$$\mathrm{H}_1(2Ku) = \vartheta_2(u) = \sum_{m=-\infty}^{m=+\infty} q^{\left(\frac{2m+1}{2}\right)^2} e^{(2m+1)\pi i u},$$

$$\Theta_1(2Ku) = \vartheta_3(u) = \sum_{m=-\infty}^{m=+\infty} q^{m^2} e^{2m\pi i u}.$$

It follows at once [cf. formulas (I) and (V)] that

$$(\mathrm{I}')\quad\begin{cases}\vartheta_0(u+\tfrac{1}{2})=\vartheta_3(u),\\ \vartheta_1(u+\tfrac{1}{2})=\vartheta_2(u),\\ \vartheta_2(u+\tfrac{1}{2})=-\vartheta_1(u),\\ \vartheta_3(u+\tfrac{1}{2})=\vartheta_0(u);\end{cases}$$

and if $\tau=\dfrac{iK'}{K}$,

$$(\mathrm{V}')\quad\begin{cases}\vartheta_0(u+\tfrac{1}{2}\tau)=iq^{-\frac{1}{4}}e^{-\pi iu}\vartheta_1(u),\\ \vartheta_1(u+\tfrac{1}{2}\tau)=iq^{-\frac{1}{4}}e^{-\pi iu}\vartheta_0(u),\\ \vartheta_2(u+\tfrac{1}{2}\tau)=q^{-\frac{1}{4}}e^{-\pi iu}\vartheta_3(u),\\ \vartheta_3(u+\tfrac{1}{2}\tau)=q^{-\frac{1}{4}}e^{-\pi iu}\vartheta_2(u).\end{cases}$$

The other formulas given in the Table of Formulas, No. XXXIII, are left as examples to be worked.

Art. 209. For brevity we may write

$$Q_0=\prod_{m=1}^{m=\infty}(1-q^{2m}),\qquad Q_1=\prod_{m=1}^{m=\infty}(1+q^{2m}),$$

$$Q_2=\prod_{m=1}^{m=\infty}(1+q^{2m-1}),\quad Q_3=\prod_{m=1}^{m=\infty}(1-q^{2m-1}).$$

It follows at once that

$$\vartheta_0(u)=Q_0\prod_{m=1}^{m=\infty}(1-2q^{2m-1}\cos 2\pi u+q^{4m-2}),$$

$$\vartheta_1(u)=2Q_0q^{\frac{1}{4}}\sin\pi u\prod_{m=1}^{m=\infty}(1-2q^{2m}\cos 2\pi u+q^{4m}),$$

$$\vartheta_2(u)=2Q_0q^{\frac{1}{4}}\cos\pi u\prod_{m=1}^{m=\infty}(1+2q^{2m}\cos 2\pi u+q^{4m}),$$

$$\vartheta_3(u)=Q_0\prod_{m=1}^{m=\infty}(1+2q^{2m-1}\cos 2\pi u+q^{4m-2}).$$

If we write $z=e^{iu\pi}$, we have

$$\frac{1-2q^{2m+1}\cos 2\pi u+q^{4m+2}}{(1-q^{2m+1})^2}=\frac{1-q^{2m+1}z^{-2}}{1-q^{2m+1}}\cdot\frac{1-q^{2m+1}z^2}{1-q^{2m+1}}$$

$$=\frac{\sin\pi\left(\dfrac{2m+1}{2}\tau-u\right)}{\sin\dfrac{2m+1}{2}\pi\tau}e^{-iu\pi}\,\frac{\sin\pi\left(\dfrac{2m+1}{2}\tau+u\right)}{\sin\dfrac{2m+1}{2}\pi\tau}e^{iu\pi}$$

$$=1-\frac{\sin^2\pi u}{\sin^2\left(\dfrac{2m+1}{2}\pi\tau\right)}.$$

We therefore have

$$\vartheta_0(u) = Q_0 Q_3{}^2 \prod_{m=1}^{m=\infty}\left(1 - \frac{\sin^2 \pi u}{\sin^2\left(\dfrac{2m-1}{2}\pi\tau\right)}\right),$$

$$\vartheta_1(u) = 2q^{\frac{1}{4}} \sin \pi u Q_0 Q_0{}^2 \prod_{m=1}^{m=\infty}\left(1 - \frac{\sin^2 \pi u}{\sin^2(m\pi\tau)}\right),$$

$$\vartheta_2(u) = 2q^{\frac{1}{4}} \cos \pi u Q_0 Q_1{}^2 \prod_{m=1}^{m=\infty}\left(1 - \frac{\sin^2 \pi u}{\cos^2(m\pi\tau)}\right),$$

$$\vartheta_3(u) = Q_0 Q_2{}^2 \prod_{m=1}^{m=\infty}\left(1 - \frac{\sin^2 \pi u}{\cos^2\left(\dfrac{2m-1}{2}\pi\tau\right)}\right).$$

ART. 210. *Jacobi's fundamental theorem.* If we write $\pi u = x$ on the right-hand side of the equations above, the theta-functions as given by Jacobi* are

$$\begin{aligned}
\vartheta_0(x, q) &= \vartheta(x, q) = \sum_{m=-\infty}^{m=+\infty} (-1)^m q^{m^2} e^{2mxi} \\
&= 1 - 2q\cos 2x + 2q^4 \cos 4x - 2q^9 \cos 6x + \cdots, \\
\vartheta_1(x, q) &= \frac{1}{i}\sum_{m=-\infty}^{m=+\infty} (-1)^m q^{\frac{1}{4}(2m+1)^2} e^{(2m+1)xi} \\
&= 2q^{\frac{1}{4}} \sin x - 2q^{\frac{9}{4}} \sin 3x + 2q^{\frac{25}{4}} \sin 5x - \cdots, \\
\vartheta_2(x, q) &= \sum_{m=-\infty}^{m=+\infty} q^{\frac{1}{4}(2m+1)^2} e^{(2m+1)xi} \\
&= 2q^{\frac{1}{4}} \cos x + 2q^{\frac{9}{4}} \cos 3x + 2q^{\frac{25}{4}} \cos 5x + 2q^{\frac{49}{4}} \cos 7x + \cdots, \\
\vartheta_3(x, q) &= \sum_{m=-\infty}^{m=+\infty} q^{m^2} e^{2mxi} \\
&= 1 + 2q\cos 2x + 2q^4 \cos 4x + 2q^9 \cos 6x + \cdots.
\end{aligned}$$

We have at once

$$\begin{array}{l|l|l}
\vartheta_0(x + \frac{1}{2}\pi) = \vartheta_3(x) & \vartheta_0(x + \pi) = \vartheta_0(x) & \vartheta_0(-x) = \vartheta_0(x) \\
\vartheta_1(x + \frac{1}{2}\pi) = \vartheta_2(x) & \vartheta_1(x + \pi) = -\vartheta_1(x) & \vartheta_1(-x) = -\vartheta_1(x) \\
\vartheta_2(x + \frac{1}{2}\pi) = -\vartheta_1(x) & \vartheta_2(x + \pi) = -\vartheta_2(x) & \vartheta_2(-x) = \vartheta_2(x) \\
\vartheta_3(x + \frac{1}{2}\pi) = \vartheta_0(x) & \vartheta_3(x + \pi) = \vartheta_3(x) & \vartheta_3(-x) = \vartheta_3(x)
\end{array}$$

$$\begin{array}{l|l}
\vartheta_0(x + \frac{1}{2}\log q \cdot i) = -iq^{-\frac{1}{4}} e^{xi}\vartheta_1(x) & \vartheta_0(x + \log q \cdot i) = -q^{-1}e^{2xi}\vartheta_0(x) \\
\vartheta_1(x + \frac{1}{2}\log q \cdot i) = -iq^{-\frac{1}{4}} e^{xi}\vartheta_0(x) & \vartheta_1(x + \log q \cdot i) = -q^{-1}e^{2xi}\vartheta_1(x) \\
\vartheta_2(x + \frac{1}{2}\log q \cdot i) = q^{-\frac{1}{4}} e^{xi}\vartheta_3(x) & \vartheta_2(x + \log q \cdot i) = q^{-1}e^{2xi}\vartheta_2(x) \\
\vartheta_3(x + \frac{1}{2}\log q \cdot i) = q^{-\frac{1}{4}} e^{xi}\vartheta_2(x) & \vartheta_3(x + \log q \cdot i) = q^{-1}e^{2xi}\vartheta_3(x)
\end{array}$$

$$\begin{aligned}
\vartheta_0(x + \tfrac{1}{2}\pi + \tfrac{1}{2}\log q \cdot i) &= q^{-\frac{1}{4}} e^{xi}\vartheta_2(x) \\
\vartheta_1(x + \tfrac{1}{2}\pi + \tfrac{1}{2}\log q \cdot i) &= q^{-\frac{1}{4}} e^{xi}\vartheta_3(x) \\
\vartheta_2(x + \tfrac{1}{2}\pi + \tfrac{1}{2}\log q \cdot i) &= iq^{-\frac{1}{4}} e^{xi}\vartheta_0(x) \\
\vartheta_3(x + \tfrac{1}{2}\pi + \tfrac{1}{2}\log q \cdot i) &= -iq^{-\frac{1}{4}} e^{xi}\vartheta_1(x)
\end{aligned}$$

* Jacobi, Werke, I, pp. 497–538.

We next observe that if the quantities a, b, c, d; a', b', c', d' are connected by the equations

$$(1)\quad \begin{cases} a' = \tfrac{1}{2}(a+b+c+d), \\ b' = \tfrac{1}{2}(a+b-c-d), \\ c' = \tfrac{1}{2}(a-b+c-d), \\ d' = \tfrac{1}{2}(a-b-c+d), \end{cases}$$

it follows that

$$(2)\quad \begin{cases} a = \tfrac{1}{2}(a'+b'+c'+d'), \\ b = \tfrac{1}{2}(a'+b'-c'-d'), \\ c = \tfrac{1}{2}(a'-b'+c'-d'), \\ d = \tfrac{1}{2}(a'-b'-c'+d'), \end{cases}$$

and also that

$$(3)\quad a^2+b^2+c^2+d^2 = a'^2+b'^2+c'^2+d'^2.$$

We shall next show that if a', b', c', d' are either all even integers or all odd integers, then also a, b, c, d are all either even or odd integers.

This may be seen at once from the following table.*

We note that all integers, positive or negative, belong to one or the other of the four forms

$$4p, \qquad 4p+1, \qquad 4p+2, \qquad 4p+3,$$

where p is an integer or zero.

For four even integers we may write

$a=$	$b=$	$c=$	$d=$
4α	4β	4γ	4δ
4α	4β	4γ	$4\delta+2$
4α	4β	$4\gamma+2$	$4\delta+2$
4α	$4\beta+2$	$4\gamma+2$	$4\delta+2$
$4\alpha+2$	$4\beta+2$	$4\gamma+2$	$4\delta+2$

where the numbers in any column may be permuted among one another.

If for brevity we put

$$\alpha+\beta+\gamma+\delta=\alpha' \qquad \alpha+\beta-\gamma-\delta=\beta'$$
$$\alpha-\beta+\gamma-\delta=\gamma' \qquad \alpha-\beta-\gamma+\delta=\delta'$$

it follows from equations (1) that

$a'=$	$b'=$	$c'=$	$d'=$
$2\alpha'$	$2\beta'$	$2\gamma'$	$2\delta'$
$2\alpha'+1$	$2\beta'-1$	$2\gamma'-1$	$2\delta'+1$
$2\alpha'+2$	$2\beta'-2$	$2\gamma'$	$2\delta'$
$2\alpha'+3$	$2\beta'-1$	$2\gamma'-1$	$2\delta'-1$
$2\alpha'+4$	$2\beta'$	$2\gamma'$	$2\delta'$

* See Enneper, *Elliptische Funktionen*, p. 136.

For four odd integers we may write

$a=$	$b=$	$c=$	$d=$
$4\alpha+1$	$4\beta+1$	$4\gamma+1$	$4\delta+1$
$4\alpha+1$	$4\beta+1$	$4\gamma+1$	$4\delta+3$
$4\alpha+1$	$4\beta+1$	$4\gamma+3$	$4\delta+3$
$4\alpha+1$	$4\beta+3$	$4\gamma+3$	$4\delta+3$
$4\alpha+3$	$4\beta+3$	$4\gamma+3$	$4\delta+3$

where the corresponding values of a', b', c', d' are, owing to equations (1),

$a'=$	$b'=$	$c'=$	$d'=$
$2\alpha'+2$	2β	$2\gamma'$	$2\delta'$
$2\alpha'+3$	$2\beta'-1$	$2\gamma'-1$	$2\delta'+1$
$2\alpha'+4$	$2\beta'-2$	$2\gamma'$	$2\delta'$
$2\alpha'+5$	$2\beta'-1$	$2\gamma'-1$	$2\delta'-1$
$2\alpha'+6$	$2\beta'$	$2\gamma'$	$2\delta'$

If for example we write

$$a=1,\quad b=3,\quad c=5,\quad d=7,$$

we have $a'=8,\quad b'=-4,\quad c'=-2,\quad d'=0;$

and reciprocally if $a=8,\quad b=-4,\quad c=-2,\quad d=0,$

we have $a'=1,\quad b'=3,\quad c'=5,\quad d'=7.$

It follows that if for a, b, c, d we write all possible combinations including all systems of four even integers and all systems of four odd integers, the corresponding integers a', b', c', d' will take the same systems of values in a different order and in such a way that none of the systems will be omitted or doubled.

Since

$$\vartheta_3(x)=\sum q^{m^2}e^{2mxi}=\sum e^{m^2\log q+2mxi}=e^{\frac{x^2}{\log q}}\sum e^{\frac{1}{\log q}[2m\frac{1}{2}\log q+xi]^2}$$

and

$$\vartheta_2(x)=e^{\frac{1}{\log q}x^2}\sum e^{\frac{1}{\log q}[(2m+1)\frac{1}{2}\log q+xi]^2},$$

it follows that

$$\vartheta_3(w)\vartheta_3(x)\vartheta_3(y)\vartheta_3(z)=e^{\frac{1}{\log q}(w^2+x^2+y^2+z^2)}\sum e^{\frac{L}{\log q}},$$

and

$$\vartheta_2(w)\vartheta_2(x)\vartheta_2(y)\vartheta_2(z)=e^{\frac{1}{\log q}(w^2+x^2+y^2+z^2)}\sum e^{\frac{M}{\log q}},$$

where

$$L=(2\nu\tfrac{1}{2}\log q+wi)^2+(2\nu'\tfrac{1}{2}\log q+xi)^2+(2\nu''\tfrac{1}{2}\log q+yi)^2+(2\nu'''\tfrac{1}{2}\log q+zi)^2,$$

and $$M=\left(\frac{2\nu+1}{2}\log q+wi\right)^2+\left(\frac{2\nu'+1}{2}\log q+xi\right)^2+\left(\frac{2\nu''+1}{2}\log q+yi\right)^2+\left(\frac{2\nu'''+1}{2}\log q+zi\right)^2,$$

the summation in the first equation to be taken over all positive and negative even integers 2ν, $2\nu'$, $2\nu''$, $2\nu'''$ and in the second equation over all positive and negative odd integers $2\nu+1$, $2\nu'+1$, $2\nu''+1$, $2\nu'''+1$.

Adding the two expressions we have

(4) $\vartheta_3(w)\vartheta_3(x)\vartheta_3(y)\vartheta_3(z)+\vartheta_2(w)\vartheta_2(x)\vartheta_2(y)\vartheta_2(z)=e^{\frac{1}{\log q}(w^2+x^2+y^2+z^2)}\sum e^{\frac{N}{\log q}}$,

where

$$N=[a\tfrac{1}{2}\log q+wi]^2+[b\tfrac{1}{2}\log q+xi]^2+[c\tfrac{1}{2}\log q+yi]^2+[d\tfrac{1}{2}\log q+zi]^2,$$

the summation to be taken over all systems of four even integers a, b, c, d *plus* the summation over all systems of four odd integers a, b, c, d.

We note that N may be written in the form

$$(5)\quad \begin{aligned} N=&\left[\frac{a+b+c+d}{2}\frac{\log q}{2}+\frac{w+x+y+z}{2}i\right]^2\\ &+\left[\frac{a+b-c-d}{2}\frac{\log q}{2}+\frac{w+x-y-z}{2}i\right]^2\\ &+\left[\frac{a-b+c-d}{2}\frac{\log q}{2}+\frac{w-x+y-z}{2}i\right]^2\\ &+\left[\frac{a-b-c+d}{2}\frac{\log q}{2}+\frac{w-x-y+z}{2}i\right]^2. \end{aligned}$$

We define w', x', y', z' through the equations

$$(6)\quad \begin{cases} w'=\tfrac{1}{2}(w+x+y+z), & y'=\tfrac{1}{2}(w-x+y-z),\\ x'=\tfrac{1}{2}(w+x-y-z), & z'=\tfrac{1}{2}(w-x-y+z). \end{cases}$$

It follows at once that

$$w'^2+x'^2+y'^2+z'^2=w^2+x^2+y^2+z^2.$$

If further we put accents on all the letters in equation (4) and note that the summation taken over all systems of four even integers a', b', c', d' plus the summation over all systems of odd integers a', b', c', d' is in virtue of (1) and (5) the same as those above over a, b, c, d, it follows that

$$\begin{aligned}\vartheta_3(w)\vartheta_3(x)\vartheta_3(y)\vartheta_3(z)&+\vartheta_2(w)\vartheta_2(x)\vartheta_2(y)\vartheta_2(z)\\ &=\vartheta_3(w')\vartheta_3(x')\vartheta_3(y')\vartheta_3(z')+\vartheta_2(w')\vartheta_2(x')\vartheta_2(y')\vartheta_2(z').\end{aligned}$$

Jacobi (*loc. cit.*) made this formula the foundation of the theory of elliptic functions.

ART. 211. If for w we write $w+\pi$, we have

$$\vartheta_3(w+\pi)=\vartheta_3(w),\quad \vartheta_2(w+\pi)=-\vartheta_2(w),$$

while at the same time w', x', y', z' are increased by $\frac{1}{2}\pi$ so that $\vartheta_3(w'+\frac{1}{2})$ becomes $\vartheta_0(w_1)$ and $\vartheta_2(w'+\frac{1}{2})$ becomes $-\vartheta_1(w')$.

The formula above becomes

$$\begin{aligned}\vartheta_3(w)\vartheta_3(x)\vartheta_3(y)\vartheta_3(z)&-\vartheta_2(w)\vartheta_2(x)\vartheta_2(y)\vartheta_2(z)\\ &=\vartheta_0(w')\vartheta_0(x')\vartheta_0(y')\vartheta_0(z')+\vartheta_1(w')\vartheta_1(x')\vartheta_1(y')\vartheta_1(z').\end{aligned}$$

The number of formulas which we may derive in this manner is thirty-five, which fall into two categories, namely, changes in w, x, y, z which

produce corresponding changes of $\frac{1}{2}\pi$ and $\frac{1}{2}\log q \cdot i$ in $w,' x', y', z'$ and secondly changes in w, x, y, z which cause changes of $\frac{1}{4}\pi$ and $\frac{1}{4}\log q \cdot i$ in w', x', y', z'.

The following eleven formulas belong to the first category, where for brevity we write

$$(\lambda\mu\nu\rho) \text{ for } \vartheta_\lambda(w)\vartheta_\mu(x)\vartheta_\nu(y)\vartheta_\rho(z)$$

and

$$(\lambda\mu\nu\rho)' \text{ for } \vartheta_\lambda(w')\vartheta_\mu(x')\vartheta_\nu(y')\vartheta_\rho(z').$$

(A).

$$
\begin{array}{rl}
(1) & (3333)+(2222)=(3333)'+(2222)' \\
(2) & (3333)-(2222)=(0000)'+(1111)' \\
(3) & (0000)+(1111)=(3333)'-(2222)' \\
(4) & (0000)-(1111)=(0000)'-(1111)'
\end{array}
$$

$$
\begin{array}{rl}
(5) & (0033)+(1122)=(0033)'+(1122)' \\
(6) & (0033)-(1122)=(3300)'+(2211)' \\
(7) & (0022)+(1133)=(0022)'+(1133)' \\
(8) & (0022)-(1133)=(2200)'+(3311)' \\
(9) & (3322)+(0011)=(3322)'+(0011)' \\
(10) & (3322)-(0011)=(2233)'+(1100)'
\end{array}
$$

$$
\begin{array}{rl}
(11) & (3201)+(2310)=(1023)'-(0132)' \\
(12) & (3201)-(2310)=(3201)'-(2310)'
\end{array}
$$

Equations (11) and (12) are counted as one equation, since (11) becomes (12) when x, w, z, y are written for w, x, y, z.

We also note that the equations

(5)	(7)	(9)	(11) are transformed into
(6)	(8)	(10)	(12) and *vice versa*,

when $-x, -y$ are written for x, y, and consequently also w' becomes z' and x' becomes y'.

If we put $$w = x + y + z,$$

it follows that $\quad w' = x + y + z, \quad x' = x, \quad y' = y, \quad z' = z;$

while if we write $$w = -(x + y + z),$$

we have $\quad w' = 0, \quad x' = -(y + z), \quad y' = -(x + z), \quad z' = -(x + y).$

Equations (A) may then be combined into double equations. If for brevity we denote $\vartheta_0(0)\vartheta_\lambda(y + z)\vartheta_\mu(x + z)\vartheta_\nu(x + y)$ by $|\,0\lambda\mu\nu\,|$ and $\vartheta_\lambda(x + y + z)\vartheta_\mu(x)\vartheta_\nu(y)\vartheta_\rho(z)$ by $\{\lambda\mu\nu\rho\}$, the five most interesting of these double formulas are given in the following table.

(B).

$$
\begin{aligned}
|0000| &= \{3333\} - \{2222\} = \{0000\} + \{1111\}\\
|0033| &= \{0033\} - \{1122\} = \{3300\} + \{2211\}\\
|0022| &= \{0022\} - \{1133\} = \{2200\} + \{3311\}\\
|0011| &= \{3322\} - \{2233\} = \{0011\} + \{1100\}\\
|0123| &= \{3210\} + \{2301\} = \{1032\} - \{0123\}
\end{aligned}
$$

We may derive a more special system of formulas if in the formulas in table (A) we put

$$w = x, \qquad y = z,$$
$$w' = x + y, \qquad x' = x - y, \qquad y' = 0, \qquad z' = 0;$$

or if we put

$$w = - x, \qquad y = - z,$$
$$w' = 0, \qquad x' = 0, \qquad y' = -(x - y), \qquad z' = - (x + y).$$

Similar formulas, making in all thirty-six, are had by writing

$$
\begin{array}{llllll}
w = y, & x = z; & w' = x+y, & x' = 0, & y' = -(x-y), & z' = 0,\\
w = -y, & x = -z; & w' = 0, & x' = x-y, & y' = 0, & z' = -(x+y),\\
w = z, & x = y; & w' = y+z, & x' = 0, & y' = 0, & z' = -(y-z),\\
w = -z, & x = -y; & w' = 0, & x' = -(y+z), & y' = y-z, & z' = 0.
\end{array}
$$

Using the notations *

$$[\lambda\mu\nu\rho] = \vartheta_\lambda\vartheta_\mu\vartheta_\nu(x + y)\vartheta_\rho(x - y),$$
$$(\lambda\mu\nu\rho) = \vartheta_\lambda(x)\vartheta_\mu(x)\vartheta_\nu(y)\vartheta_\rho(y),$$

these thirty-six formulas are included in the following table.

(C).

(1)	$[3333] = (3333) + (1111) = (0000) + (2222)$
(2)	$[3300] = (0033) + (2211) = (3300) + (1122)$
(3)	$[3322] = (2233) - (0011) = (3322) - (1100)$
(4)	$[3311] = (1133) - (3311) = (0022) - (2200)$
(5)	$[0033] = (0033) - (1122) = (3300) - (2211)$
(6)	$[0000] = (3333) - (2222) = (0000) - (1111)$
(7)	$[0022] = (0022) - (1133) = (2200) - (3311)$
(8)	$[0011] = (3322) - (2233) = (1100) - (0011)$

* Koenigsberger, *Elliptische Functionen*, p. 379.

(9) $[2233] = (3322) + (0011) = (2233) + (1100)$

(10) $[2200] = (0022) + (3311) = (1133) + (2200)$

(11) $[2222] = (2222) - (1111) = (3333) - (0000)$

(12) $[2211] = (1122) - (2211) = (0033) - (3300)$

---

(13) $[0202] = (0202) + (1313);\quad [0220] = (0202) - (1313)$

(14) $[3232] = (3232) + (0101);\quad [3223] = (3232) - (0101)$

(15) $[0303] = (0303) + (1212);\quad [0330] = (0303) - (1212)$

---

(16) $[0213] = (1302) + (0213);\quad [0231] = (1302) - (0213)$

(17) $[3210] = (0132) + (3201);\quad [3201] = (0132) - (3201)$

(18) $[0312] = (1203) + (0312);\quad [0321] = (1203) - (0312)$

If in the above formulas we put $x = y$, we have from (1), (2) and (11) the following:

$$\vartheta_3{}^3\vartheta_3(2x) = \vartheta_3{}^4(x) + \vartheta_1{}^4(x) = \vartheta_0{}^4(x) + \vartheta_2{}^4(x)$$

$$\vartheta_3{}^2\vartheta_0\vartheta_0(2x) = \vartheta_0{}^2(x)\vartheta_3{}^2(x) + \vartheta_1{}^2(x)\vartheta_2{}^2(x)$$

$$\vartheta_2{}^3\vartheta_2(2x) = \vartheta_2{}^4(x) - \vartheta_1{}^4(x) = \vartheta_3{}^4(x) - \vartheta_0{}^4(x).$$

If we write $y = 0$ in the formulas (C), (1), (2) and (11), we have the formulas of the following table.

(D).

(1) $\vartheta_3{}^2\vartheta_3{}^2(x) = \vartheta_0{}^2\vartheta_0{}^2(x) + \vartheta_2{}^2\vartheta_2{}^2(x)$

(1′) $\vartheta_3{}^2\vartheta_0{}^2(x) = \vartheta_0{}^2\vartheta_3{}^2(x) + \vartheta_2{}^2\vartheta_1{}^2(x)$

(2) $\vartheta_3{}^2\vartheta_2{}^2(x) = \vartheta_2{}^2\vartheta_3{}^2(x) - \vartheta_0{}^2\vartheta_1{}^2(x)$

(3) $\vartheta_3{}^2\vartheta_1{}^2(x) = \vartheta_2{}^2\vartheta_0{}^2(x) - \vartheta_0{}^2\vartheta_2{}^2(x)$

If in equation (1) we put $x = 0$, we have

$$\vartheta_3{}^4 = \vartheta_0{}^4 + \vartheta_2{}^4,$$

or

$$[1 + 2q + 2q^4 + 2q^9 + \cdots]^4 = [1 - 2q + 2q^4 - 2q^9 + \cdots]^4 + 16q[1 + q^{1.2} + q^{2.3} + q^{3.4} + \cdots]^4.$$

ART. 212. We have defined and developed the theta-functions by means of infinite power series. These functions being integral transcendents are susceptible of the treatment indicated in Chapter I and performed there for $\sin u$.

It will be shown later (Chapter XIV) that these theta-functions are to a constant factor the same as the Weierstrassian *sigma-functions*.

In order to observe the general theory from another point of view and at the same time study Weierstrass's presentation of the subject, we shall develop the sigma-functions by means of infinite binomial products as has been suggested in Chapter I for $\sin u$. It is therefore superfluous here to express the theta-functions through these infinite binomial products.

EXAMPLES

1. Show that

$$e^{\frac{\pi u^2}{4KK'}}\Theta(u) = (-1)^m e^{\frac{\pi(u+2miK')^2}{4KK'}}\Theta(u+2miK'),$$

$$e^{\frac{\pi u^2}{4KK'}}\mathrm{H}(u) = (-i)^{2m+1} e^{\frac{\pi[u+(2m+1)iK']^2}{4KK'}}\Theta[u+(2m+1)iK'],$$

$$e^{\frac{\pi u^2}{4KK'}}\mathrm{H}(u) = (-1)^m e^{\frac{\pi(u+2miK')^2}{4KK'}}\mathrm{H}(u+2miK'),$$

$$e^{\frac{\pi u^2}{4KK'}}\Theta(u) = (-i)^{2m+1} e^{\frac{\pi[u+(2m+1)iK']^2}{4KK'}}\mathrm{H}[u+(2m+1)iK'].$$

[Jacobi, Werke I, p. 226.]

2. Derive the corresponding formulas for Θ_1 and H_1.

3. If

$$q = e^{-\pi\frac{K'}{K}},\ q_0 = e^{-\pi\frac{K}{K'}},$$

so that q, q_0 are interchanged when K, K' change places, and if

$$\Theta(u, q) = 1 - 2q\cos 2u + 2q^4\cos 4u - 2q^9\cos 6u + \cdots,$$

$$\mathrm{H}(u, q) = 2\sqrt[4]{q}\sin u - 2\sqrt[4]{q^9}\sin 3u + 2\sqrt[4]{q^{25}}\sin 5u - \cdots,$$

prove that

$$\Theta(iu, q) = \sqrt{\frac{K}{K'}}\, e^{\frac{Ku^2}{\pi K'}}\mathrm{H}\left(\frac{Ku}{K'} + \frac{\pi}{2}, q_0\right),$$

$$\mathrm{H}(iu, q) = i\sqrt{\frac{K}{K'}}\, e^{\frac{Ku^2}{\pi K'}}\mathrm{H}\left(\frac{Ku}{K'}, q_0\right).$$

[Jacobi, Werke, I, p. 264.]

4. Using the Jacobi notation show that

$$\vartheta_0(u + mi\log q) = (-1)^m q^{-m^2}e^{2mui}\vartheta_0(u),$$

$$\vartheta_1(u + mi\log q) = (-1)^m q^{-m^2}e^{2mui}\vartheta_1(u),$$

$$\vartheta_2(u + mi\log q) = q^{-m^2}e^{2mui}\vartheta_2(u),$$

$$\vartheta_3(u + mi\log q) = q^{-m^2}e^{2mui}\vartheta_3(u).$$

5. Show that, if n and m are integers,

$$\vartheta_0\left(n\pi + \frac{2m+1}{2}i\log q\right) = 0,\quad \vartheta_1(n\pi + mi\log q) = 0,$$

$$\vartheta_2\left(\frac{2n+1}{2}\pi + mi\log q\right) = 0,\quad \vartheta_3\left(\frac{2n+1}{2}\pi + \frac{2m+1}{2}i\log q\right) = 0.$$

CHAPTER XI

THE FUNCTIONS *sn u, cn u, dn u*

ARTICLE 213. It was shown in Art. 152 that z may be expressed as the quotient of two Φ-functions in the form

$$z = \frac{\Phi(u)}{\Phi_1(u)},$$

where
$$u = \int_{z_0, s_0}^{z, s} \frac{dz}{\sqrt{R(z)}}.$$

If we put
$$u = \int_0^{\mathbf{z}} \frac{d\mathbf{z}}{\sqrt{(1-\mathbf{z}^2)(1-k^2\mathbf{z}^2)}},$$

and study a quotient of Φ-functions, it is seen that $\dfrac{\Phi(u)}{\Phi_1(u)}$ must $= 0$, for $\mathbf{z} = 0$ in both the upper and the lower leaves of the Riemann surface; and further for $\mathbf{z} = \infty$, we must have $\dfrac{\Phi(u)}{\Phi_1(u)} = \infty$ in both leaves. It follows that

$$\frac{\Phi(u)}{\Phi_1(u)} = 0 \text{ for } \mathbf{z} = 0,\ \mathbf{s} = +1 \text{ and for } \mathbf{z} = 0,\ \mathbf{s} = -1;$$

and
$$\frac{\Phi(u)}{\Phi_1(u)} = \infty \text{ for } \mathbf{z} = \infty,\ \mathbf{s} = +\infty \text{ and for } \mathbf{z} = \infty,\ \mathbf{s} = -\infty.$$

In Art. 193 we saw that

$$\bar{u}(0, +1) = 0, \quad \bar{u}(0, -1) = -2K;$$

and consequently

$$\mathrm{H}[\bar{u}(0, +1)] = 0, \quad \mathrm{H}[\bar{u}(0, -1)] = \mathrm{H}(-2K) = 0.$$

Hence it is shown that $\mathrm{H}(u)$ becomes zero for $\mathbf{z} = 0$, $\mathbf{s} = +1$ and for $\mathbf{z} = 0$, $\mathbf{s} = -1$. We may therefore take $\mathrm{H}(u)$ as the numerator in the quotient of Φ-functions.

On the other hand we have

$$\bar{u}(\infty, +\infty) = -iK', \quad \bar{u}(\infty, -\infty) = -2K - iK';$$

and since
$$\Theta(-iK') = 0, \quad \Theta(-2K - iK') = 0,$$

we may use $\Theta(u)$ as the denominator of the above quotient. If then for u we write Legendre's normal integral of the first kind, it is evident that

the quotient $\dfrac{\mathrm{H}(u)}{\Theta(u)}$ has the desired zeros and infinities, and has besides no other such points.

It follows that

$$\mathbf{z} = C\frac{\mathrm{H}(u)}{\Theta(u)},$$

where C is a constant.

To determine the constant C, write $\mathbf{z}=1$ and we have

$$1 = C\frac{\mathrm{H}[\bar{u}(1)]}{\Theta[\bar{u}(1)]}.$$

But since (Art. 193) $\bar{u}(1) = -3K$, we have

$$1 = C\frac{\mathrm{H}(-3K)}{\Theta(-3K)}.$$

In Art. 201 we saw that

$$\mathrm{H}_1(u+3K) = \mathrm{H}(u),$$

or

$$\mathrm{H}_1(0) = \mathrm{H}(-3K).$$

In a similar manner it may be shown that

$$\Theta_1(0) = \Theta(-3K).$$

We thus have

$$1 = C\frac{\mathrm{H}_1(0)}{\Theta_1(0)}, \quad \text{or} \quad C = \frac{\Theta_1(0)}{\mathrm{H}_1(0)}. \tag{i}$$

It therefore follows that

$$\text{(M)} \qquad C = \frac{\sum\limits_{m=-\infty}^{m=+\infty} q^{m^2}}{\sum\limits_{m=-\infty}^{m=+\infty} q^{\left(\frac{2m+1}{2}\right)^2}}.$$

This transcendental expression, however, may be expressed algebraically in terms of k.

If we write $\mathbf{z} = \dfrac{1}{k}$ in the formula $\mathbf{z} = C\dfrac{\mathrm{H}(u)}{\Theta(u)}$, we have

$$\frac{1}{k} = C\frac{\mathrm{H}\left[u\left(\frac{1}{k}\right)\right]}{\Theta\left[u\left(\frac{1}{k}\right)\right]} = C\frac{\mathrm{H}[-3K-iK']}{\Theta[-3K-iK']}$$

$$= -C\frac{\mathrm{H}[3K+iK']}{\Theta[3K+iK']} = C\frac{\mathrm{H}[K+iK']}{\Theta[K+iK']} = C\frac{[\lambda(u)]_{u=0}\Theta_1(0)}{[\lambda(u)]_{u=0}\mathrm{H}_1(0)}.$$

It follows that

$$C = \frac{1}{k}\frac{\mathrm{H}_1(0)}{\Theta_1(0)}. \tag{ii}$$

But from (*i*)

$$C = \frac{\Theta_1(0)}{H_1(0)};$$

so that

$$C^2 = \frac{1}{k} \quad \text{or} \quad C = \frac{1}{\sqrt{k}},$$

where the sign is to be taken positive since it is definitely determined from the expression (M) above.

We thus have

$$\sqrt{k} = \frac{2\sqrt[4]{q} + 2\sqrt[4]{q^9} + 2\sqrt[4]{q^{25}} + \cdots}{1 + 2q + 2q^4 + 2q^9 + \cdots}. \quad \text{[Jacobi, Bd. I, p. 236.]}$$

If in the integral of the first kind

$$u = \int_{0,1}^{\mathbf{z},\mathbf{s}} \frac{d\mathbf{z}}{\sqrt{(1-\mathbf{z}^2)(1-k^2\mathbf{z}^2)}}$$

we write

$$\mathbf{z} = \sin\phi,$$

it becomes

$$u = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}}.$$

Jacobi * wrote

$$\phi = \text{am}\, u \text{ (amplitude of } u),$$

so that

$$\mathbf{z} = \sin\phi = \sin \text{am}\, u.$$

If the modulus k is zero, it is seen that am u becomes u and consequently $\mathbf{z}$ becomes sin u.

Somewhat later $\mathbf{z} = \sin \text{am}\, u$ was called the *modular sine* and written by Gudermann †

$$\mathbf{z} = sn\, u.$$

ART. 214. Consider next the quotient

$$\frac{H_1(u)}{\Theta(u)}.$$

We have (cf. Art. 140)

$$\frac{H_1(u)}{\Theta(u)} = \frac{H_1(\overline{u}(\mathbf{z},\mathbf{s}) + m4K + n2iK')}{\Theta[\overline{u}(\mathbf{z},\mathbf{s}) + m4K + n2iK']}.$$

Since $H_1(u)$ and $\Theta(u)$ have the period $4K$, it follows that

$$\frac{H_1(u)}{\Theta(u)} = \frac{H_1[\overline{u}(\mathbf{z},\mathbf{s}) + n2iK']}{\Theta[\overline{u}(\mathbf{z},\mathbf{s}) + n2iK']}.$$

If we take $n = 1$, we have

$$\frac{H_1(u)}{\Theta(u)} = \frac{-\mu(\overline{u})H_1(\overline{u})}{\mu(\overline{u})\Theta(\overline{u})} = -\frac{H_1(\overline{u})}{\Theta(\overline{u})}.$$

* Jacobi, Werke, Bd. I, p. 81. Here Jacobi retained the word *amplitude* of Legendre [*Fonct. Ellip.*, t. I, p. 14].

† Gudermann, *Theorie der Modularfunctionen*, *Crelle*, Bd. 18.

Since we have the negative sign on the right, it is well to take the square of the quotient, so that

$$\left[\frac{H_1(u)}{\Theta(u)}\right]^2 = \left[\frac{H_1(\bar{u})}{\Theta(\bar{u})}\right]^2,$$

a formula which is true for any value of n.

ART. 215. All the Theta-functions have the property of becoming zero of the first order upon only two incongruent points. It follows that the quotient

$$\left[\frac{H_1(\bar{u})}{\Theta(\bar{u})}\right]^2$$

becomes zero of the second order upon two incongruent points, and upon two incongruent points it becomes infinite of the second order.

Since $\qquad H_1(u) = 0$ for $u = (2m+1)K + n\,2\,iK'$,

it is seen that

$$H_1(u) = 0 \text{ for } u = -K \text{ and } u = -3K;$$

and from above

$$\Theta(u) = 0 \text{ for } u = -iK' \text{ and } u = -iK' - 2K.$$

In Art. 193 it was found that

$$\begin{array}{ll} \text{when } \bar{u} = -K, & \text{then } \mathbf{z} = -1, \\ \text{when } \bar{u} = -3K, & \text{then } \mathbf{z} = +1, \\ \text{when } \bar{u} = -iK', & \text{then } \mathbf{z} = \infty, \mathbf{s} = \infty, \\ \text{when } \bar{u} = -iK' - 2K, & \text{then } \mathbf{z} = \infty, \mathbf{s} = -\infty. \end{array}$$

It follows from Art. 150 that $\left[\frac{H_1(u)}{\Theta(u)}\right]^2$ is a rational function of $\mathbf{z}$. It becomes zero of the second order on the positions $\mathbf{z} = -1$ and $\mathbf{z} = +1$, and infinite of the second order on the positions $\mathbf{z} = \infty, \mathbf{s} = \infty$ and $\mathbf{z} = \infty, \mathbf{s} = -\infty$.

We note that the function $\mathbf{z}^2 - 1$ has the same properties. We may therefore write

$$\sqrt{1 - \mathbf{z}^2} = C_1 \frac{H_1(u)}{\Theta(u)}.$$

The function $\sqrt{1-\mathbf{z}^2}$ is consequently like $\mathbf{z}$ a *one-valued* doubly periodic function of u. It has the period $4K$ but not the period $2iK'$; for when u is changed into $u + 2iK'$, the above quotient changes sign. Hence the other period is $4iK'$.

$$\sqrt{1-\mathbf{z}^2} = \sqrt{1 - sn^2 u} = \cos \operatorname{am} u = cn\, u,$$

$$cn\, u = C_1 \frac{H_1(u)}{\Theta(u)}.$$

choose the sign that $cn\, u$ has the value $+1$ when $\mathbf{z} = 0$.

This function $cn\, u$ is called the *modular cosine*. The analogue in trigonometry is naturally the cosine, where

$$\cos u = \sqrt{1 - \sin^2 u}.$$

In order to determine the constant C_1, we may write $\mathbf{z} = 0$, $\mathbf{s} = 0$, so that

$$1 = C_1 \frac{\mathrm{H}_1(0)}{\Theta(0)} \quad \text{or} \quad C_1 = \frac{\Theta(0)}{\mathrm{H}_1(0)} = \frac{1 - 2q + 2q^4 - 2q^9 + \cdots}{2\sqrt[4]{q} + 2\sqrt[4]{q^9} + \cdots}.$$

Again, if we write $\mathbf{z} = \frac{1}{k}$, then, since $\bar{u}\left(\frac{1}{k}\right) = -3K - iK'$, it follows that

$$\sqrt{1 - \frac{1}{k^2}} = C_1 \frac{\mathrm{H}_1(-3K - iK')}{\Theta(-3K - iK')} = C_1 \frac{\mathrm{H}_1(3K + iK')}{\Theta(3K + iK')}$$
$$= -C_1 \frac{\mathrm{H}_1(K + iK')}{\Theta(K + iK')} = C_1 \frac{[\lambda(u)]_{u=0} i\Theta(0)}{[\lambda(u)]_{u=0}\mathrm{H}_1(0)} = iC_1 \frac{\Theta(0)}{\mathrm{H}_1(0)}.$$

But, since $C_1 = \dfrac{\Theta(0)}{\mathrm{H}_1(0)}$, we see that

$$iC_1^2 = \sqrt{\frac{-(1 - k^2)}{k^2}}, \quad \text{or} \quad C_1 = \frac{\sqrt{k'}}{\sqrt{k}} \quad \text{(see Art. 193)},$$

the sign being definitely determined through $C_1 = \dfrac{\Theta(0)}{\mathrm{H}_1(0)}$.

In the preceding Article we saw that $\sqrt{k}$ was definitely determined and consequently here $\sqrt{k'}$ is also definite.

We may therefore write

$$cn\, u = \frac{\sqrt{k'}}{\sqrt{k}} \frac{\mathrm{H}_1(u)}{\Theta(u)}.$$

ART. 216. We saw in Art. 152 that $\dfrac{dz}{du}$ is a one-valued function of u and from above it is seen that $\sqrt{1 - \mathbf{z}^2}$ is also one-valued. It therefore follows from the expression

$$\frac{dz}{du} = \sqrt{(1 - \mathbf{z}^2)(1 - k^2\mathbf{z}^2)}$$

that $\sqrt{1 - k^2\mathbf{z}^2}$ must be a one-valued function of u. This function is called the *delta amplitude* u and written $\Delta \operatorname{am} u$, $dn\, u$ or $\Delta\phi$.

Since $\dfrac{du}{dz} = \dfrac{1}{\sqrt{(1 - \mathbf{z}^2)(1 - k^2\mathbf{z}^2)}}$, it follows, since $\mathbf{z} = \sin\phi$, that $du = \dfrac{d\phi}{\Delta\phi}$.

To investigate this function $dn\, u$, let us study the quotient

$$\left[\frac{\Theta_1(u)}{\Theta(u)}\right]^2 = \left[\frac{\Theta_1(\bar{u})}{\Theta(\bar{u})}\right]^2.$$

The zeros of the numerator are expressed through

$$\bar{u} = (2m + 1)K + (2n + 1)iK'.$$

We may therefore take as the two incongruent zeros the values

$$\bar{u} = -3K - iK' \quad \text{and} \quad \bar{u} = -K - iK'.$$

In Art. 193 we saw that

$$\bar{u}(\mathbf{z}, \mathbf{s}) = -3K - iK' \text{ for } \mathbf{z} = \frac{1}{k},$$

and

$$\bar{u}(\mathbf{z}, \mathbf{s}) = -K - iK' \text{ for } \mathbf{z} = -\frac{1}{k}.$$

Hence the above quotient becomes zero for $\mathbf{z} = \pm\frac{1}{k}$, and it becomes infinite for $\mathbf{z} = \infty$, $\mathbf{s} = +\infty$ and for $\mathbf{z} = \infty$, $\mathbf{s} = -\infty$.

The function $\sqrt{1 - k^2\mathbf{z}^2}$ has the same zeros and the same infinities. We may therefore write

$$\sqrt{1 - k^2\mathbf{z}^2} = C_2\frac{\Theta_1(u)}{\Theta(u)}.$$

We shall choose the sign so that when $\mathbf{z} = 0$ the root has the value $+1$. Hence for $\mathbf{z} = 0$ we have

$$1 = C_2\frac{\Theta_1(0)}{\Theta(0)} \quad \text{or} \quad C_2 = \frac{\Theta(0)}{\Theta_1(0)}.$$

If further we write $\mathbf{z} = 1$, we have

$$k' = C_2\frac{\Theta_1(-3K)}{\Theta(-3K)} = C_2\frac{\Theta_1(3K)}{\Theta(3K)} = C_2\frac{\Theta(0)}{\Theta_1(0)}.$$

It follows that $k' = C_2^2$ or $C_2 = \sqrt{k'}$, and consequently

$$\sqrt{k'} = \frac{\Theta(0)}{\Theta_1(0)} = \frac{1 - 2q + 2q^4 - 2q^9 + \cdots}{1 + 2q + 2q^4 + 2q^9 + \cdots}.$$

(Jacobi, Bd. I, p. 236.)

Finally we have

$$dn\, u = \sqrt{k'}\frac{\Theta_1(u)}{\Theta(u)}.$$

ART. 217. We may write* the three elliptic functions of u

$$\text{(VIII)} \quad \begin{cases} sn\, u = \dfrac{1}{\sqrt{k}}\dfrac{\mathrm{H}(u)}{\Theta(u)}, \\ cn\, u = \dfrac{\sqrt{k'}}{\sqrt{k}}\dfrac{\mathrm{H}_1(u)}{\Theta(u)}, \\ dn\, u = \sqrt{k'}\dfrac{\Theta_1(u)}{\Theta(u)}. \end{cases}$$

* Cf. Jacobi, Werke, Bd. I, pp. 225, 256 and 512; Hermite, *loc. cit.*, p. 794.

The first of these functions is *odd*, the other two are *even*. It follows at once that

$$\text{(VIII}'\text{)}\quad \begin{cases} sn\,0 = 0, \\ cn\,0 = 1, \\ dn\,0 = 1. \end{cases}$$

The zeros of sn u are . . . . $2\,mK + 2\,niK'$,

the zeros of cn u are . . . . . . $(2\,m + 1)K + 2\,niK'$,

the zeros of dn u are . . . . . . $(2\,m + 1)K + (2\,n + 1)iK'$;

the infinities of all three functions are . $2\,mK + (2\,n + 1)iK'$,

where m and n are integers including zero.

We will derive nothing new by forming other quotients of Theta-functions.

ART. 218. It follows at once from the above formulas that

$$sn(u + K) = \frac{1}{\sqrt{k}}\,\frac{\mathrm{H}(u+K)}{\Theta(u+K)} = \frac{1}{\sqrt{k}}\,\frac{\mathrm{H}_1(u)}{\Theta_1(u)}$$

$$= \frac{1}{\sqrt{k}}\,\frac{\dfrac{\mathrm{H}_1(u)}{\Theta(u)}}{\dfrac{\Theta_1(u)}{\Theta(u)}} = \frac{1}{\sqrt{k}}\,\frac{\sqrt{k}}{\sqrt{k'}}\,cn\,u\,\frac{\sqrt{k'}}{dn\,u};$$

or

$$sn(u + K) = \frac{cn\,u}{dn\,u}.$$

We may consequently write

$$\text{(IX)}\quad \begin{cases} sn(u + K) = \dfrac{cn\,u}{dn\,u}, \\ cn(u + K) = -\,k'\,\dfrac{sn\,u}{dn\,u}, \\ dn(u + K) = \dfrac{k'}{dn\,u}. \end{cases} \qquad \text{(IX}'\text{)}\quad \begin{cases} sn\,K = 1, \\ cn\,K = 0, \\ dn\,K = k'. \end{cases}$$

When the argument u is increased by $2\,K$, it follows that

$$sn(u + 2\,K) = \frac{1}{\sqrt{k}}\,\frac{\mathrm{H}(u + 2\,K)}{\Theta(u + 2\,K)} = -\,\frac{1}{\sqrt{k}}\,\frac{\mathrm{H}(u)}{\Theta(u)} = -\,sn\,u.$$

We thus have

$$\text{(X)}\quad \begin{cases} sn(u + 2\,K) = -\,sn\,u, \\ cn(u + 2\,K) = -\,cn\,u, \\ dn(u + 2\,K) = dn\,u. \end{cases}$$

Noting that

$$cn(u+iK')=\frac{\sqrt{k'}}{\sqrt{k}}\frac{H_1(u+iK')}{\Theta(u+iK')}=\frac{\sqrt{k'}}{\sqrt{k}}\frac{\lambda(u)\Theta_1(u)}{i\lambda(u)H(u)}=-\frac{i}{k}\frac{dn\,u}{sn\,u}, \text{ etc.,}$$

we may write

$$\text{(XI)}\quad \begin{cases} sn(u+iK')=\dfrac{1}{k\,sn\,u}, \\[2mm] cn(u+iK')=-\dfrac{i}{k}\dfrac{dn\,u}{sn\,u}, \\[2mm] dn(u+iK')=-i\dfrac{cn\,u}{sn\,u}; \end{cases}$$

and in a similar manner

$$\text{(XII)}\quad \begin{cases} sn(u+2iK')=sn\,u, \\ cn(u+2iK')=-cn\,u, \\ dn(u+2iK')=-dn\,u. \end{cases}$$

It is also seen that

$$sn(u+K+iK')=\frac{1}{k\,sn(u+K)}=\frac{dn\,u}{k\,sn\,u}.$$

We thus have

$$\text{(XIII)}\quad \begin{cases} sn(u+K+iK')=\dfrac{dn\,u}{k\,cn\,u}, \\[2mm] cn(u+K+iK')=-i\dfrac{k'}{k}\dfrac{1}{cn\,u}, \\[2mm] dn(u+K+iK')=ik'\dfrac{sn\,u}{cn\,u}. \end{cases}$$

All three functions have the periods $4K$ and $4iK'$, so that

$$\text{(XIV)}\quad \begin{cases} sn(u+4K)=sn\,u, \\ cn(u+4K)=cn\,u, \\ dn(u+4K)=dn\,u; \end{cases}$$

and

$$\text{(XV)}\quad \begin{cases} sn(u+4iK')=sn\,u, \\ cn(u+4iK')=cn\,u, \\ dn(u+4iK')=dn\,u. \end{cases}$$

The periods of $sn\,u$ are . . . . . $4K$ and $2iK'$,
the periods of $cn\,u$ are . . . . . $4K$ and $2K+2iK'$,
the periods of $dn\,u$ are . . . . . $2K$ and $4iK'$.

ART. 219. The fundamental formulas connecting the elliptic functions follow at once from their definitions.

From the relations

$$du=\frac{d\phi}{\Delta\phi}, \qquad \phi=\text{am}\,u,$$

we have

$$d\frac{\text{am}\,u}{du}=\Delta\phi=dn\,u.$$

It follows that

$$d\frac{snu}{du} = sn'u = cn\,u\,dn\,u,$$
$$cn'u = -\,sn\,u\,dn\,u,$$
$$dn'u = -\,k^2 sn\,u\,cn\,u.$$

The following two relations are also evident:

$$sn^2u + cn^2u = 1,$$
$$dn^2u + k^2sn^2u = 1.$$

Further, from the relations

$$\frac{dz}{du} = \sqrt{(1 - z^2)(1 - k^2z^2)} \quad \text{and} \quad z = sn\,u,$$

we have

$$sn'^2u = (1 - sn^2u)\,(1 - k^2sn^2u),$$

and similarly

$$cn'^2u = (1 - cn^2u)\,(1 - k^2 + k^2cn^2u),$$
$$dn'^2u = (1 - dn^2u)\,(dn^2u - 1 + k^2).$$

Art. 220. *Jacobi's imaginary transformation.** — If we put

$$\sin\phi = i\tan\psi,$$

it follows at once that

$$\sin\phi = i\tan\psi, \qquad \sin\psi = -\,i\tan\phi,$$
$$\cos\phi = \frac{1}{\cos\psi}, \qquad \cos\psi = \frac{1}{\cos\phi},$$
$$d\phi = i\frac{d\psi}{\cos\psi}, \qquad d\psi = -\,i\frac{d\phi}{\cos\phi},$$

and also that

$$\frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}} = \frac{id\psi}{\sqrt{1 - k'^2\sin^2\psi}}.$$

If next we write

$$\int_0^\phi \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}} = i\int_0^\psi \frac{d\psi}{\sqrt{1 - k'^2\sin^2\psi}} = iu, \text{ say},$$

then

$$\psi = \text{am}(u, k') \text{ and } \phi = \text{am}(iu, k).$$

From the relations above we have

$$\text{(XVI)} \quad \begin{cases} sn(iu, k) = i\dfrac{sn(u, k')}{cn(u, k')}, & sn(u, k') = -\,i\dfrac{sn(iu, k)}{cn(iu, k)}, \\[2ex] cn(iu, k) = \dfrac{1}{cn(u, k')}, & cn(u, k') = \dfrac{1}{cn(iu, k)}, \\[2ex] dn(iu, k) = \dfrac{dn(u, k')}{cn(u, k')}. & dn(u, k') = \dfrac{dn(iu, k)}{cn(iu, k)}. \end{cases}$$

* Jacobi, Werke, Bd. I, p. 85.

ART. 221. As a definition Jacobi wrote

$$\text{coam}\, u = \text{am}(K - u).$$

We have at once

$$\text{(XVII)} \quad \begin{cases} \sin \text{coam}\, u = \dfrac{cn\, u}{dn\, u}, \\ \cos \text{coam}\, u = \dfrac{k'\, sn\, u}{dn\, u}, \\ \Delta \text{coam}\, u = \dfrac{k'}{dn\, u}. \end{cases}$$

It also follows that

$$\text{(XVIII)} \quad \begin{cases} \sin \text{coam}(iu, k) = \dfrac{1}{dn(u, k')}, \\ \cos \text{coam}(iu, k) = \dfrac{ik'}{k} \cos \text{coam}(u, k'), \\ \Delta \text{coam}(iu, k) = k' \sin \text{coam}(u, k'). \end{cases}$$

ART. 222. From the two preceding Articles it is seen that

$$\text{(XIX)} \quad \begin{cases} sn(u + iK') = \dfrac{1}{k\, sn\, u}, \\ cn(u + iK') = -\dfrac{i\, dn\, u}{k\, sn\, u} = \dfrac{-ik'}{k \cos \text{coam}\, u}, \\ dn(u + iK') = -i \cot \text{am}\, u; \end{cases}$$

and also that

$$\text{(XX)} \quad \begin{cases} \sin \text{coam}(u + iK') = \dfrac{1}{k \sin \text{coam}\, u}, \\ \cos \text{coam}(u + iK') = \dfrac{ik'}{k\, cn\, u}, \\ \Delta \text{coam}(u + iK') = ik' \tan \text{am}\, u. \end{cases}$$

ART. 223. *Linear transformations.* — If with Jacobi (*loc. cit.*, p. 125) we put $t = k\mathbf{z}$, we have

$$\int_0^t \frac{dt}{\sqrt{(1 - t^2)\left(1 - \dfrac{t^2}{k^2}\right)}} = k \int_0^{\mathbf{z}} \frac{d\mathbf{z}}{\sqrt{(1 - \mathbf{z}^2)(1 - k^2\mathbf{z}^2)}}.$$

If further we write

$$u = \int_0^{\mathbf{z}} \frac{d\mathbf{z}}{\sqrt{(1-\mathbf{z}^2)(1-k^2\mathbf{z}^2)}},$$

$$ku = \int_0^{t} \frac{dt}{\sqrt{(1-t^2)\left(1-\frac{t^2}{k^2}\right)}},$$

we have $\mathbf{z} = sn(u, k)$, $t = sn\left(ku, \frac{1}{k}\right)$, and consequently*

$$\text{(XXI)} \quad \begin{cases} sn\left(ku, \frac{1}{k}\right) = k\, sn(u, k), \\ cn\left(ku, \frac{1}{k}\right) = dn(u, k), \\ dn\left(ku, \frac{1}{k}\right) = cn(u, k). \end{cases}$$

We also have

$$\text{(XXII)} \quad \begin{cases} \sin \operatorname{coam}\left(ku, \frac{1}{k}\right) = \frac{1}{\sin \operatorname{coam}(u, k)}, \\ \cos \operatorname{coam}\left(ku, \frac{1}{k}\right) = ik' \tan \operatorname{am}(u, k), \\ \Delta \operatorname{coam}\left(ku, \frac{1}{k}\right) = \frac{ik'}{k \cos \operatorname{am}(u, k)}. \end{cases}$$

Next put iu in the place of u and observing that the complementary modulus of $\frac{1}{k}$ is $\frac{ik'}{k}$, it is seen that

$$\text{(XXIII)} \quad \begin{cases} sn\left(ku, \frac{ik'}{k}\right) = \cos \operatorname{coam}(u, k'), \\ cn\left(ku, \frac{ik'}{k}\right) = \sin \operatorname{coam}(u, k'), \\ dn\left(ku, \frac{ik'}{k}\right) = \frac{1}{\Delta \operatorname{am}(u, k')}; \end{cases}$$

and

$$\text{(XXIV)} \quad \begin{cases} \sin \operatorname{coam}\left(ku, \frac{ik'}{k}\right) = \cos \operatorname{am}(u, k'), \\ \cos \operatorname{coam}\left(ku, \frac{ik'}{k}\right) = \sin \operatorname{am}(u, k'), \\ \Delta \operatorname{coam}\left(ku, \frac{ik'}{k}\right) = \frac{\Delta \operatorname{am}(u, k')}{k}; \end{cases}$$

$$\tan \operatorname{coam}\left(ku, \frac{ik'}{k}\right) = \cot \operatorname{am}(u, k').$$

* See also Hermite, Œuvres, t. II, p. 267.

ART. 224. It follows from Art. 204 that

$$\frac{\mathrm{H}(iu;\ K, iK')}{\Theta(iu;\ K, iK')} = i\frac{\mathrm{H}(u;\ K', iK)}{\mathrm{H}_1(u;\ K', iK)}$$

and

$$\frac{\Theta_1(0;\ K, iK')}{\mathrm{H}_1(0;\ K, iK')} = \frac{\Theta_1(0;\ K', iK)}{\Theta(0;\ K', iK)}.$$

We have at once (cf. also Art. 220)

$$(\mathrm{XVI})\quad \begin{cases} sn(iu;\ K, iK') = i\dfrac{sn(u;K', iK)}{cn(u;K', iK)}, \\ cn(iu;\ K, iK') = \dfrac{1}{cn(u;K', iK)}, \\ dn(iu;\ K, iK') = \dfrac{dn(u;\ K', iK)}{cn(u;\ K', iK)}. \end{cases}$$

ART. 225. *Quadratic transformations.* — If we write

$$t = \frac{(1+k)\mathbf{z}}{1+k\mathbf{z}^2},$$

we have

$$\frac{d\mathbf{z}}{\sqrt{(1-\mathbf{z}^2)(1-k^2\mathbf{z}^2)}} = \frac{M\,dt}{\sqrt{(1-t^2)(1-l^2t^2)}},$$

where

$$l = \frac{2\sqrt{k}}{1+k} \text{ and } M = \frac{1}{1+k}.$$

Writing

$$u = \int_0^{\mathbf{z}} \frac{d\mathbf{z}}{\sqrt{(1-\mathbf{z}^2)(1-k^2\mathbf{z}^2)}},$$

it follows that

$$(1+k)u = \int_0^t \frac{dt}{\sqrt{(1-t^2)(1-l^2t^2)}};$$

and consequently

$$(\mathrm{XXV})\quad \begin{cases} sn\left[(1+k)u, \dfrac{2\sqrt{k}}{1+k}\right] = \dfrac{(1+k)sn(u,k)}{1+k\,sn^2(u,k)}, \\ cn\left[(1+k)u, \dfrac{2\sqrt{k}}{1+k}\right] = \dfrac{cn(u,k)dn(u,k)}{1+k\,sn^2(u,k)}, \\ dn\left[(1+k)u, \dfrac{2\sqrt{k}}{1+k}\right] = \dfrac{1-k\,sn^2(u,k)}{1+k\,sn^2(u,k)}. \end{cases}$$

In a similar manner write

$$t = \frac{(1+k')\mathbf{z}\sqrt{1-\mathbf{z}^2}}{\sqrt{1-k^2\mathbf{z}^2}},$$

and we have

$$\frac{d\mathbf{z}}{\sqrt{(1-\mathbf{z}^2)(1-k^2\mathbf{z}^2)}} = \frac{M\,dt}{\sqrt{(1-t^2)(1-l^2t^2)}},$$

where

$$l = \frac{1-k'}{1+k'} \text{ and } M = \frac{1}{1+k'}.$$

It follows at once that

$$\text{(XXVI)}\quad \begin{cases} sn\left[(1+k')u, \dfrac{1-k'}{1+k'}\right] = \dfrac{(1+k')sn(u,k)cn(u,k)}{dn(u,k)}, \\ cn\left[(1+k')u, \dfrac{1-k'}{1+k'}\right] = \dfrac{1-(1+k')\,sn^2(u,k)}{dn(u,k)}, \\ dn\left[(1+k')u, \dfrac{1-k'}{1+k'}\right] = \dfrac{1-(1-k')sn^2(u,k)}{dn(u,k)}. \end{cases}$$

In formulas (XXVI) change k to $1/k$ and u to uk and observe formulas (XXI). It is seen that

$$\text{(XXVII)}\quad \begin{cases} sn\left[(k+ik')u, \dfrac{k-ik'}{k+ik'}\right] = \dfrac{(k+ik')sn(u,k)dn(u,k)}{cn(u,k)}, \\ cn\left[(k+ik')u, \dfrac{k-ik'}{k+ik'}\right] = \dfrac{1-(k+ik')k\,sn^2(u,k)}{cn(u,k)}, \\ dn\left[(k+ik')u, \dfrac{k-ik'}{k+ik'}\right] = \dfrac{1-(k-ik')k\,sn^2(u,k)}{cn(u,k)}. \end{cases}$$

The formulas just written are the very celebrated formulas due to John Landen (*Phil. Trans.*, LXV, p. 283, 1775; or *Mathematical Memoirs*, I, p. 32, London, 1780) and may be derived as follows:

Write

$$\sin(2\phi - \phi_1) = k_1 \sin\phi_1, \tag{1}$$

where
$$k_1 = \frac{1-k'}{1+k'}, \qquad (k^2 + k'^2 = 1).$$

Since
$$k_1 < k,$$

it is evident that

$$\sin(2\phi - \phi_1) < \sin\phi_1,$$
$$(2\phi - \phi_1) < \phi_1,$$
$$\phi < \phi_1.$$

Solving (1) for ϕ, we have

$$\sin^2 2\phi = (1+k_1)^2 \sin^2\phi_1\left[1 - \frac{4\,k_1}{(1+k_1)^2}\sin^2\phi\right];$$

or, since
$$\frac{4}{(1+k_1)^2} = (1+k')^2 \text{ and } \frac{4\,k_1}{(1+k_1)^2} = k^2,$$

it is seen that

$$\sin\phi_1 = (1+k')\frac{\sin\phi\cos\phi}{\Delta\phi}.$$

We further have

$$\frac{d\phi_1}{\sqrt{1-k_1{}^2\sin^2\phi_1}} = \frac{(1+k')d\phi}{\sqrt{1-k^2\sin^2\phi}} = (1+k')du.$$

ART. 226. *Development in powers of u.* — If we develop by Maclaurin's Theorem the three functions $sn\, u$, $cn\, u$, $dn\, u$, we obtain the following series:

$$sn\, u = u - (1 + k^2)\frac{u^3}{3!} + (1 + 14k^2 + k^4)\frac{u^5}{5!} - \cdots,$$

$$cn\, u = 1 - \frac{u^2}{2!} + (1 + 4k^2)\frac{u^4}{4!} - \cdots,$$

$$dn\, u = 1 - \frac{k^2u^2}{2!} + (k^4 + 4k^2)\frac{u^4}{4!} - \cdots,$$

where the coefficient of any term, say $\frac{u^{2n+1}}{(2n+1)!}$ or $\frac{u^{2n}}{(2n)!}$, is an integral function of k^2 with integral coefficients.

Following Hermite* we wish to determine these coefficients. From the formulas derived above

$$sn\left(ku, \frac{1}{k}\right) = k\, sn(u, k),$$

$$cn\left(ku, \frac{1}{k}\right) = dn(u, k),$$

it is seen that the coefficients of $sn(u, k)$ are reciprocal polynomials in k and that those of $dn(u, k)$ may be derived immediately from those of $cn(u, k)$.

Gudermann† has shown that the coefficients of $cn\, u$ are

$$1 + 4k^2,$$
$$1 + 44k^2 + 16k^4,$$
$$1 + 408k^2 + 912k^4 + 64k^6,$$
$$1 + 3688k^2 + 30768k^4 + 15808k^6 + 256k^8,$$
$$\cdots\cdots\cdots$$

We note that if we put $k = \cos\theta$ and introduce the multiple arcs instead of the powers of the cosines, the above coefficients when multiplied by k may be written

$$k + 4k^3 = 4\cos\theta + \cos 3\theta,$$

$$k + 44k^3 + 16k^5 = 44\cos\theta + 16\cos 3\theta + \cos 5\theta,$$

$$k + 408k^3 + 912k^5 + 64k^7 = 912\cos\theta + 408\cos 3\theta + 64\cos 5\theta + \cos 7\theta,$$

$$\cdots\cdots\cdots$$

In these equalities it is seen that the powers of k and the cosines of the multiples of θ have precisely the same coefficients.

* Cf. Hermite, *Comptes rendus*, t. LVII, 1863 (II), p. 613; or Œuvres, t. II, p. 264.

† Gudermann, *Crelle*, Bd. XIX, p. 80.

In general, if we denote the coefficient of $\dfrac{u^{2n+2}}{(2n+2)!}$ by

$$A_0 + A_1 k^2 + A_2 k^4 + \cdots + A_n k^{2n} = \sum_{i=0}^{i=n} A_i k^{2i} = cn^{(2n+2)}(0, k),$$

we will have the relation

$$\Sigma A_i \cos^{2i+1}\theta = \Sigma A_i \cos(2n + 1 - 4i)\theta,$$

which may be demonstrated as follows:

From formulas (XXVI) we have

$$cn\left[(k + ik')u, \frac{k - ik'}{k + ik'}\right] = \frac{1 - (k + ik')k\, sn^2(u, k)}{cn(u, k)},$$

and changing i to $-i$ it follows that

$$cn\left[(k - ik')u, \frac{k + ik'}{k - ik'}\right] = \frac{1 - (k - ik')k\, sn^2(u, k)}{cn(u, k)}.$$

From these two formulas it follows at once that

$$(k + ik')cn\left[(k - ik')u, \frac{k + ik'}{k - ik'}\right] + (k - ik')cn\left[(k + ik')u, \frac{k - ik'}{k + ik'}\right] = 2k\, cn(u, k).$$

In this formula write $k = \cos\theta$, $k' = \sin\theta$, and we have

$$e^{i\theta}cn(e^{-i\theta}u, e^{2i\theta}) + e^{-i\theta}cn(e^{i\theta}u, e^{-2i\theta}) = 2\cos\theta\, cn(u, k).$$

Noting that

$$cn^{(2n+2)}(0) = 1 + A_1 k^2 + A_2 k^4 + \cdots + A_n k^{2n},$$

it is seen by equating the coefficients of $\dfrac{u^{2n+2}}{(2n+2)!}$ on either side of this equation, when expanded by Maclaurin's Theorem, that

$$\Sigma A_i \cos^{2i+1}\theta = \Sigma A_i \cos(2n + 1 - 4i)\theta.$$

From this formula the quantities $A_0 = 1, A_1, A_2, \ldots$, may be determined at once.

For example, let $n = 4$ and for brevity put $A_i = 4^i a_i$. If the multiple arcs are replaced by the powers of the cosine, we have

$$\begin{aligned}
&\cos\theta + 4a_1\cos^3\theta + 16a_2\cos^5\theta + 64a_3\cos^7\theta + 256a_4\cos^9\theta \\
&= \cos\theta + a_1(\cos 3\theta + 3\cos\theta) + a_2(\cos 5\theta + 5\cos 3\theta + 10\cos\theta) \\
&+ a_3(\cos 7\theta + 7\cos 5\theta + 21\cos 3\theta + 35\cos\theta) \\
&+ a_4(\cos 9\theta + 9\cos 7\theta + 36\cos 5\theta + 84\cos 3\theta + 126\cos\theta) \\
&= \cos 9\theta + 4a_1\cos 5\theta + 16a_2\cos\theta + 64a_3\cos 3\theta + 256a_4\cos 7\theta.
\end{aligned}$$

We thus have among the a's the five equations

$$\begin{aligned} 1 &= a_4, \\ 4a_1 &= a_2 + 7a_3 + 36a_4, \\ 16a_2 &= 1 + 3a_1 + 10a_2 + 35a_3 + 126a_4, \\ 64a_3 &= a_1 + 5a_2 + 21a_3 + 84a_4, \\ 256a_4 &= a_3 + 9a_4. \end{aligned}$$

Since the sum of these equations leads to an identity, we may omit any one of them, say the third; and from the other four we have

$$a_1 = 922, \qquad a_2 = 1923, \qquad a_3 = 247, \qquad a_4 = 1,$$

which agree with the above results of Gudermann.

Since

$$dn\left(u, \frac{1}{k}\right) = cn\left(\frac{u}{k}, k\right),$$

the coefficients of $dn(u, k)$ are at once deduced from those of $cn(u, k)$; while those of $sn(u, k)$ may be obtained from the formula

$$sn'(u, k) = cn(u, k)\, dn(u, k).$$

[See Table of Formulas, LVII.]

Development of the Elliptic Functions in Simple Series of Sines and Cosines.

First Method.

ART. 227. In Art. 206 we saw that

$$\Theta\left(\frac{2Ku}{\pi}\right) = A(1 - 2q\cos 2u + q^2)(1 - 2q^3\cos 2u + q^6)(1 - 2q^5\cos 2u + q^{10})\cdots.$$

Noting that

$$\log(1 - t) = -\sum_{\lambda=1}^{\lambda=\infty} \frac{t^\lambda}{\lambda},$$

$$\log(1 + t) = -\sum_{\lambda=1}^{\lambda=\infty} (-1)^\lambda \frac{t^\lambda}{\lambda},$$

and that

$$1 - 2q\cos 2u + q^2 = (1 - qe^{2iu})(1 - qe^{-2iu}),$$

it is seen that

$$-\frac{1}{2}\log(1 - 2q\cos 2u + q^2) = q\cos 2u + \frac{q^2\cos 4u}{2} + \frac{q^3\cos 6u}{3} + \frac{q^4\cos 8u}{4} + \cdots.$$

We therefore have

$$\begin{aligned}\frac{1}{2}\log\Theta\left(\frac{2\,Ku}{\pi}\right) = \text{const.} &- \cos 2\,u\,(q + q^3 + q^5 + \cdots)\\ &- \frac{\cos 4\,u}{2}(q^2 + q^6 + q^{10} + \cdots)\\ &- \frac{\cos 6\,u}{3}(q^3 + q^9 + q^{15} + \cdots)\\ &- \frac{\cos 8\,u}{4}(q^4 + q^{12} + q^{20} + \cdots)\\ &\quad\cdots\cdots\cdots\end{aligned}$$

or

$$\begin{aligned}\frac{1}{2}\log\Theta\left(\frac{2\,Ku}{\pi}\right) &= \text{const.} - \frac{q\cos 2\,u}{1-q^2} - \frac{q^2\cos 4\,u}{2(1-q^4)}\\ &\quad - \frac{q^3\cos 6\,u}{3(1-q^6)} - \frac{q^4\cos 8\,u}{4(1-q^8)} - \cdots.\end{aligned}$$

The logarithms of the other Theta-functions may be expressed in a similar manner.

ART. 228. Hermite (Œuvres, t. II, p. 216) gives the following method for the expressions of $sn\,u$, $cn\,u$, $dn\,u$ in terms of the sines and the cosines.

We have the formulas

$$k\,sn\,u = \frac{d\log(dn\,u - k\,cn\,u)}{du},$$

$$ik\,cn\,u = \frac{d\log(dn\,u + ik\,sn\,u)}{du},$$

$$i\,dn\,u = \frac{d\log(cn\,u + i\,sn\,u)}{du}.$$

We shall next derive the formulas

$$(1)\qquad dn\frac{2\,Ku}{\pi} - k\,cn\frac{2\,Ku}{\pi}$$

$$= \frac{1-2\sqrt{q}\cos u + q}{1+2\sqrt{q}\cos u + q}\cdot\frac{1-2\sqrt{q^3}\cos u + q^3}{1+2\sqrt{q^3}\cos u + q^3}\cdot\frac{1-2\sqrt{q^5}\cos u + q^5}{1+2\sqrt{q^5}\cos u + q^5}\cdots,$$

$$(2)\qquad dn\frac{2\,Ku}{\pi} + ik\,sn\frac{2\,Ku}{\pi}$$

$$= \frac{1-2\sqrt{-q}\sin u - q}{1+2\sqrt{-q}\sin u - q}\cdot\frac{1-2\sqrt{-q^3}\sin u - q^3}{1+2\sqrt{-q^3}\sin u - q^3}\cdot\frac{1-2\sqrt{-q^5}\sin u - q^5}{1+2\sqrt{-q^5}\sin u - q^5}\cdots,$$

$$(3)\qquad cn\frac{2\,Ku}{\pi} + i\,sn\frac{2\,Ku}{\pi}$$

$$= e^{iu}\frac{(1-qe^{-2iu})(1-q^3e^{2iu})(1-q^5e^{-2iu})\cdots}{(1-qe^{2iu})(1-q^3e^{-2iu})(1-q^5e^{2iu})\cdots}.$$

Jacobi [Werke, I, p. 143, formula (5)] has implicitly derived formulas (1) and (2) above, the first being had when in Jacobi's formula u is changed to $\frac{\pi}{2} - u$, and the second when $-q$ is written for q.

These two formulas may be derived directly in the manner which we now give for the formula (3) above.

Write as in Art. 205, $\phi(u) = 1 - e^{\frac{\pi i u}{K}}$; the expression which we wish to demonstrate equal to

$$cn\, u + i\, sn\, u$$

will take the form

$$\frac{e^{\frac{\pi i u}{2K}}\phi(-u+iK')\phi(u+3iK')\phi(-u+5iK')\ldots}{\phi(u+iK')\phi(-u+3iK')\phi(u+5iK')\ldots}.$$

Multiplying numerator and denominator of this expression by

$$A\phi(-u+iK')\phi(u+3iK')\phi(-u+5iK')\ldots,$$

where A is a constant, and putting

$$\Phi(u) = Ae^{\frac{\pi i u}{2K}}\phi^2(-u+iK')\phi^2(u+3iK')\phi^2(-u+5iK')\ldots,$$

we have to demonstrate the formula

$$cn\, u + i\, sn\, u = \frac{\Phi(u)}{\Theta(u)}.$$

We further note that

$$\Phi(u+2K) = -\Phi(u),$$

$$\Phi(u+4iK') = \Phi(u)\, q^2 \frac{\phi^2(-u-3iK')}{\phi^2(u+3iK')},$$

or

$$\Phi(u+4iK') = e^{-\frac{2i\pi}{K}(u+2iK')}\Phi(u).$$

The same functional equations are satisfied by $\mathrm{H}(u)$ and $\mathrm{H}_1(u)$. In Art. 90 it was shown that any three intermediary functions of the second order were connected by a linear relation, so that here we may write

$$\Phi(u) = C\mathrm{H}(u) + C_1\mathrm{H}_1(u).$$

Divide this expression by $\Theta(u)$, and we have

$$\frac{e^{\frac{\pi i u}{2K}}\phi(-u+iK')\phi(u+3iK')\phi(-u+5iK')\ldots}{\phi(u+iK')\phi(-u+3iK')\phi(u+5iK')\ldots} = \frac{C\mathrm{H}(u) + C_1\mathrm{H}_1(u)}{\Theta(u)}$$

$$= D\, cn\, u + iB\, sn\, u.$$

Writing $u = 0$ and $u = K$ respectively in this formula we have $D = 1$ and $B = 1$, which we wished to demonstrate.

From the formulas (1), (2) and (3) we have (see Jacobi, Werke, II, p. 296)

$$\frac{kK}{2\pi} sn \frac{2Ku}{\pi} = \frac{\sqrt{q}\sin u}{1-q} + \frac{\sqrt{q^3}\sin 3u}{1-q^3} + \frac{\sqrt{q^5}\sin 5u}{1-q^5} + \cdots,$$

$$\frac{kK}{2\pi} cn \frac{2Ku}{\pi} = \frac{\sqrt{q}\cos u}{1+q} + \frac{\sqrt{q^3}\cos 3u}{1+q^3} + \frac{\sqrt{q^5}\cos 5u}{1+q^5} + \cdots,$$

$$\frac{K}{2\pi} dn \frac{2Ku}{\pi} = \frac{1}{4} + \frac{q\cos 2u}{1+q^2} + \frac{q^2\cos 4u}{1+q^4} + \frac{q^3\cos 6u}{1+q^6} + \cdots.$$

Second Method.

ART. 229. Suppose with Briot and Bouquet (*Fonct. Ellipt.*, p. 286) that $f(u)$ is a doubly periodic function of the $2n$th order with periods $4K$ and $2iK'$ such that

$$f(u+2K) = -f(u)$$

and further suppose that $f(u)$ has n infinities α_h *within* (see Art. 91) the period-parallelogram $ABDC$, where A is an arbitrary point u_0, while B and C are the two points u_0+2K and u_0+2iK'. Form the parallelogram $EFGH$ whose vertices E and H are the points $u_0-2m'iK'$ and $u_0+2m'iK'$, while F and G are the points $u_0+2K-2m'iK'$ and $u_0+2K+2m'iK'$. The infinities of $f(u)$ situated within the parallelogram $EFGH$ may be represented by $\alpha = \alpha_h + 2miK'$, where m varies from $-m'$ to $m'-1$.

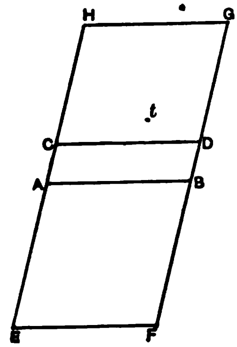


Fig. 69.

Let t be any point situated within this parallelogram. The function

$$\frac{f(u)}{\sin\frac{\pi}{2K}(u-t)}$$

has the period $2K$; its poles are the point t and the points $\alpha = \alpha_h + 2miK'$.

It follows from Cauchy's Theorem that the definite integral

$$\frac{1}{2\pi i}\int \frac{f(u)}{\sin\frac{\pi}{2K}(u-t)}\, du,$$

where the integration is taken over the sides of the parallelogram $EFGH$, is equal to the sum of the residues relative to the poles that are situated within this parallelogram. The two sides FG and HE give values that are equal and of opposite sign, while on the sides EF and GH the function $f(u)$ has a finite value and mod. $\dfrac{1}{\sin\frac{\pi}{2K}(u-t)}$ tends towards zero * when m' becomes very large.

Thus when m' becomes very large the definite integral tends towards zero and consequently the sum of the residues is zero.

The residue relative to t being $\dfrac{2K}{\pi}f(t)$, we have the equation

$$f(t) = \frac{\pi}{2K}\sum \underset{(\alpha)}{\text{Res}} \frac{f(u)}{\sin\frac{\pi}{2K}(t-u)}.$$

* In $\dfrac{1}{\sin u}$ write $u = x + iy$ and note that

$$\left|\frac{1}{\sin u}\right| = \left|\frac{2i}{e^{ix\mp y} - e^{-ix\pm y}}\right| = 0 \text{ for } y = \infty.$$

If $f(u)$ has only simple infinities, which case alone is necessary for our investigation, the above equation becomes

$$f(t)=\frac{\pi}{2K}\sum_{m=-\infty}^{m=+\infty}\sum_{h=1}^{h=n}\frac{A_h}{\sin\frac{\pi}{2K}(t-\alpha_h-2miK')},$$

where A_h is the residue of $f(u)$ relative to α_h. The series is convergent in both directions. This equality is thus demonstrated for all points t situated within two indefinitely long parallel lines EH and FG. Since both sides of this equation change signs when t is replaced by $t+2K$, the equality is true for all values of t; and consequently we have for the finite portion of the u-plane

$$f(u)=\frac{\pi}{2K}\sum_{m=-\infty}^{m=+\infty}\sum_{h=1}^{h=n}\frac{A_h}{\sin\frac{\pi}{2K}(u-\alpha_h-2miK')}.$$

ART. 230. Consider next a doubly periodic function $f(u)$ with periods $2K$ and $2iK'$ and having n infinities α_h within the parallelogram $ABDC$ of the preceding Article.

The function

$$\frac{f(u)}{\tan\frac{\pi}{2K}(u-t)}$$

admits the period $2K$, and the definite integral

$$\frac{1}{2\pi i}\int\frac{f(u)}{\tan\frac{\pi}{2K}(u-t)}du$$

relative to the contour of the parallelogram $EFGH$ is equal to the sum of the residues with respect to the poles situated within the parallelogram, that is, for the point t and the points $\alpha=\alpha_h+2miK'$, where m varies from $-m'$ to $m'-1$. The sides FG and HE give equal results with contrary sign. If we represent by u a point on the line AB, the congruent points on HG and EF are $u+2m'iK'$ and $u-2miK'$, and the parts of the integral relative to these two sides are

$$\frac{1}{2\pi i}\int_{AB}\left[\frac{1}{\tan\frac{\pi}{2K}(u-t-2m'iK')}-\frac{1}{\tan\frac{\pi}{2K}(u-t+2m'iK')}\right]f(u)du.$$

When m' becomes very large the first tangent tends towards $-i$ (see Art. 25) and the second tangent towards i, so that the integral just written tends towards a limit equal to the rectilinear integral

$$M=\frac{1}{\pi}\int_{u_0}^{u_0+2K}f(u)du$$

along the line AB. The residue of the function relative to the point t being $\frac{2K}{\pi}f(t)$, we have, as in the preceding Article,

$$f(t)=\frac{\pi M}{2K}+\frac{\pi}{2K}\sum_{(\alpha)}\text{Res}\frac{f(u)}{\tan\frac{\pi}{2K}(t-u)},$$

and consequently if the function has only simple infinities

$$f(t)=\frac{\pi M}{2K}+\frac{\pi}{2K}\sum_{m=-\infty}^{m=+\infty}\sum_{h=1}^{h=n}\frac{A_h}{\tan\frac{\pi}{2K}(t-\alpha_h-2miK')},$$

where t is any point in the finite portion of the u-plane, and A_h is the residue of $f(u)$ relative to α_h.

ART. 231. To make application of the results of the two preceding Articles consider the ratios of the four Theta-functions. Of these twelve ratios eight satisfy the relation $f(u+2K)=-f(u)$ and four the relation $f(u+2K)=f(u)$. Take the two functions $\frac{\Theta(u)}{\mathrm{H}(u)}$ and $\frac{\Theta_1(u)}{\mathrm{H}(u)}$. Form a parallelogram $EFGH$ with the origin as center and vertices $\pm K\pm(2m'+1)iK'$. The infinities of these two functions are the zeros of $\mathrm{H}(u)$. Those infinities within the parallelogram are represented by the formula $\alpha=2miK'$, m varying from $-m'$ to $+m'$; all these infinities are simple.

The residue of $\frac{\Theta(u)}{\mathrm{H}(u)}$ relative to the infinity $2miK'$ is $\frac{\Theta(0)}{\mathrm{H}'(0)}$; that of $\frac{\Theta_1(u)}{\mathrm{H}(u)}$ is $(-1)^m\frac{\Theta_1(0)}{\mathrm{H}'(0)}$.

We therefore have

$$(1)\qquad \frac{\Theta(u)}{\mathrm{H}(u)}=\frac{\pi}{2K}\frac{\Theta(0)}{\mathrm{H}'(0)}\sum_{m=-\infty}^{m=+\infty}\frac{1}{\sin\frac{\pi}{2K}(u-2miK')},$$

$$(2)\qquad \frac{\Theta_1(u)}{\mathrm{H}(u)}=\frac{\pi}{2K}\frac{\Theta_1(0)}{\mathrm{H}'(0)}\sum_{m=-\infty}^{m=+\infty}\frac{(-1)^m}{\sin\frac{\pi}{2K}(u-2miK')}.$$

Replacing in these two formulas u by the quantities $u+K$, $u+iK'$, $u+K+iK'$ we have six additional formulas including

$$(3)\qquad \frac{\mathrm{H}(u)}{\Theta(u)}=\frac{\pi}{2K}\frac{\Theta(0)}{\mathrm{H}'(0)}\sum_{m=-\infty}^{m=+\infty}\frac{1}{\sin\frac{\pi}{2K}[u-(2m-1)iK']},$$

$$(4)\qquad \frac{\mathrm{H}_1(u)}{\Theta(u)}=\frac{\pi i}{2K}\frac{\Theta_1(0)}{\mathrm{H}'(0)}\sum_{m=-\infty}^{m=+\infty}\frac{(-1)^m}{\sin\frac{\pi}{2K}[u-(2m-1)iK']}.$$

To develop the function $\dfrac{\Theta_1(u)}{\Theta(u)}$, say, which admits the period $2K$, we apply the method of the preceding Article. We note that for congruent points on the sides EF and GH of the parallelogram $EFGH$, the difference of the values u being equal to $(2m'+1)2iK'$, the function $f(u)$ takes equal values with contrary signs; and the values of the tangent on these two sides being $\mp i$, the definite integral relative to these two sides is zero.

We therefore have

$$(5)\qquad \frac{\Theta_1(u)}{\Theta(u)}=\frac{\pi i}{2K}\frac{\mathrm{H}_1(0)}{\mathrm{H}'(0)}\sum_{m=-\infty}^{m=+\infty}\frac{(-1)^m}{\tan\dfrac{\pi}{2K}[u-(2m-1)iK']}.$$

Further, since

$$sn\,u=\frac{1}{\sqrt{k}}\frac{\mathrm{H}(u)}{\Theta(u)},\quad cn\,u=\sqrt{\frac{k'}{k}}\frac{\mathrm{H}_1(u)}{\Theta(u)},\quad dn\,u=\sqrt{k'}\frac{\Theta_1(u)}{\Theta(u)},$$

we have by differentiating $sn\,u$ with regard to u, and then writing $u=0$

$$\frac{\Theta(0)}{\mathrm{H}'(0)}=\frac{1}{\sqrt{k}};\text{ and since }\frac{\mathrm{H}_1(0)}{\Theta(0)}=\sqrt{\frac{k}{k'}},$$

$$\frac{\mathrm{H}_1(0)}{\mathrm{H}'(0)}=\frac{1}{\sqrt{k'}}\text{ and similarly }\frac{\Theta_1(0)}{\mathrm{H}'(0)}=\frac{1}{\sqrt{kk'}}.$$

It follows immediately from (3), (4) and (5) that

$$(6)\qquad sn\,u=\frac{\pi}{2kK}\sum_{m=-\infty}^{m=+\infty}\frac{1}{\sin\dfrac{\pi}{2K}[u-(2m-1)iK']},$$

$$(7)\qquad cn\,u=\frac{\pi i}{2kK}\sum_{m=-\infty}^{m=+\infty}\frac{(-1)^m}{\sin\dfrac{\pi}{2K}[u-(2m-1)iK']},$$

$$(8)\qquad dn\,u=\frac{\pi i}{2K}\sum_{m=-\infty}^{m=+\infty}\frac{(-1)^m}{\tan\dfrac{\pi}{2K}[u-(2m-1)iK']}.$$

If we group the terms two and two the equations (6) and (7) become

$$(9)\qquad sn\,u=\frac{2\pi\sqrt{q}}{kK}\sin\frac{\pi u}{2K}\sum_{m=1}^{m=\infty}\frac{q^{m-1}(1+q^{2m-1})}{1-2q^{2m-1}\cos\dfrac{2\pi u}{2K}+q^{4m-2}},$$

$$(10)\qquad cn\,u=\frac{2\pi\sqrt{q}}{kK}\cos\frac{\pi u}{2K}\sum_{m=1}^{m=\infty}\frac{(-1)^{m-1}q^{m-1}(1-q^{2m-1})}{1-2q^{2m-1}\cos\dfrac{2\pi u}{2K}+q^{4m-2}}.$$

The series (8) is not convergent in both directions; but if from $dn\,0$ we subtract $dn\,u$, we have the convergent series

$$(11)\qquad 1 - dn\,u = \frac{4\pi}{K}\sin^2\frac{\pi u}{2K}\sum_{m=1}^{m=\infty}\frac{(-1)^{m-1}q^{2m-1}\dfrac{1+q^{2m-1}}{1-q^{2m-1}}}{1-2q^{2m-1}\cos\dfrac{2\pi u}{2K}+q^{4m-2}}.$$

Observing that

$$\frac{(1+q)\sin t}{1-2q\cos 2t+t^2} = \sin t + q\sin 3t + q^2\sin 5t + \cdots,$$

it is evident that (9) and (10) may be written

$$(12)\qquad sn\,u = \frac{2\pi\sqrt{q}}{kK}\sum_{m=1}^{m=\infty}\frac{q^{m-1}}{1-q^{2m-1}}\sin(2m-1)\frac{\pi u}{2K},$$

$$(13)\qquad cn\,u = \frac{2\pi\sqrt{q}}{kK}\sum_{m=1}^{m=\infty}\frac{q^{m-1}}{1+q^{2m-1}}\cos(2m-1)\frac{\pi u}{2K}.$$

These values are the same as those given at the end of Art. 229, where the corresponding value of dnu is found.

By considering the quotient $\dfrac{\Theta(u)}{\mathrm{H}(u)}$ as given in equation (1) and also the quotient $\dfrac{\Theta(u)}{\mathrm{H}_1(u)}$, we may derive in a similar manner

$$(14)\qquad \frac{2K}{\pi}\frac{1}{sn\,u} = \frac{1}{\sin\dfrac{\pi u}{2K}} + 4\sum_{m=1}^{m=\infty}\frac{q^{2m-1}}{1-q^{2m-1}}\sin(2m-1)\frac{\pi u}{2K},$$

$$(15)\qquad \frac{2Kk'}{\pi}\frac{1}{cn\,u} = \frac{1}{\cos\dfrac{\pi u}{2K}} + 4\sum_{m=1}^{m=\infty}(-1)^m\frac{q^{2m-1}}{1+q^{2m-1}}\cos(2m-1)\frac{\pi u}{2K}.$$

[See Jacobi, Werke, I, p. 157.]

EXAMPLES

1. Prove that $sn(iu+K) = \dfrac{cn(iu)}{dn(iu)}$.
2. Show that

$$\sin\mathrm{am}\left(ik'u, \frac{1}{k'}\right) = \frac{ik'\sin\mathrm{am}\,u}{\cos\mathrm{am}\,u},$$

$$\cos\mathrm{am}\left(ik'u, \frac{1}{k'}\right) = \frac{\Delta\,\mathrm{am}\,u}{\cos\mathrm{am}\,u},$$

$$\Delta\,\mathrm{am}\left(ik'u, \frac{1}{k'}\right) = \frac{1}{\cos\mathrm{am}\,u}.$$

3. Show that

$$\sin \operatorname{am}\left(iku, \frac{ik'}{k}\right) = \frac{ik \sin \operatorname{am} u}{\Delta \operatorname{am} u},$$

$$\cos \operatorname{am}\left(iku, \frac{ik'}{k}\right) = \frac{1}{\Delta \operatorname{am} u},$$

$$\Delta \operatorname{am}\left(iku, \frac{ik'}{k}\right) = \frac{\cos \operatorname{am} u}{\Delta \operatorname{am} u}.$$

4. Prove that

$$\frac{1}{sn^2(iu, k)} + \frac{1}{sn^2(u, k)} = 1.$$

5. Derive the formulas

$$sn\left[(1 + k')iu, \frac{2\sqrt{k'}}{1 + k'}\right] = \frac{i(1 + k')\, sn(u, k)\, cn(u, k)}{1 - (1 + k')\, sn^2(u, k)},$$

$$cn\left[(1 + k')iu, \frac{2\sqrt{k'}}{1 + k'}\right] = \frac{dn(u, k)}{1 - (1 + k')\, sn^2(u, k)},$$

$$dn\left[(1 + k')iu, \frac{2\sqrt{k'}}{1 + k'}\right] = \frac{1 - (1 - k')\, sn^2(u, k)}{1 - (1 + k')\, sn^2(u, k)}.$$

Suggestion: apply formulas (XVI) to formulas (XXV).

6. Show that

$$sn\left[(k' + ik)u, \frac{2\sqrt{ikk'}}{k' + ik}\right] = \frac{(k' + ik)\, sn(u, k)\, dn(u, k)}{1 - (k - ik')k\, sn^2(u, k)},$$

$$cn\left[(k' + ik)u, \frac{2\sqrt{ikk'}}{k' + ik}\right] = \frac{cn(u, k)}{1 - (k - ik')\, k\, sn^2(u, k)},$$

$$dn\left[(k' + ik)u, \frac{2\sqrt{ikk'}}{k' + ik}\right] = \frac{1 - (k + ik')\, k\, sn^2(u, k)}{1 - (k - ik')\, k\, sn^2(u, k)}.$$

7. Show that

$$sn\left[(k - ik')u, \frac{k + ik'}{k - ik'}\right] = \frac{(k - ik')\, sn(u, k) dn(u, k)}{cn(u, k)},$$

$$cn\left[(k - ik')u, \frac{k + ik'}{k - ik'}\right] = \frac{1 - (k - ik')k\, sn^2(u, k)}{cn(u, k)},$$

$$dn\left[(k - ik')u, \frac{k + ik'}{k - ik'}\right] = \frac{1 - (k + ik')k\, sn^2(u, k)}{cn(u, k)}.$$

8. Show that

$$\mathrm{H}'(u) = \sqrt{k}\, cn\, u\, dn\, u\, \Theta(u) + \sqrt{k}\, sn\, u\, \Theta'(u).$$

9. Show that

$$\Theta(K) = \frac{\Theta(0)}{\sqrt{k'}},$$

$$\mathrm{H}(K) = \sqrt{\frac{k}{k'}}\, \Theta(0),$$

$$\sqrt{k} = \frac{\mathrm{H}(K)}{\Theta(K)}.$$

10. Prove the following relations:

$$\frac{\Theta(iu,k)}{\Theta(0,k)} = \sqrt{\frac{k}{k'}}\,e^{\frac{\pi u^2}{4KK'}}\,\frac{\mathrm{H}(u+K',k')}{\Theta(0,k')},$$

$$\frac{\mathrm{H}(iu,k)}{\Theta(0,k)} = i\sqrt{\frac{k}{k'}}\,e^{\frac{\pi u^2}{4KK'}}\,\frac{\mathrm{H}(u,k')}{\Theta(0,k')},$$

$$\frac{\mathrm{H}(iu+K,k)}{\Theta(0,k)} = \sqrt{\frac{k}{k'}}\,e^{\frac{\pi u^2}{4KK'}}\,\frac{\Theta(u,k')}{\Theta(0,k')},$$

$$\frac{\Theta(iu+K,k)}{\Theta(0,k)} = \sqrt{\frac{k}{k'}}\,e^{\frac{\pi u^2}{4KK'}}\,\frac{\Theta(u+K',k')}{\Theta(0,k')}.$$

11. Show that

$$\frac{d}{du}\left(\frac{cn\,u\;dn\,u}{sn\,u}\right) = dn^2(u+iK') - dn^2 u$$

and that

$$\frac{d}{du}\left(\frac{sn\,u\;dn\,u}{cn\,u}\right) = dn^2 u + dn^2(iu,k') - 1.$$

12. Prove that $sn\,u\,dn''u - sn''u\,dn\,u = sn\,u\,dn\,u$;
and that

$$\begin{vmatrix} (sn\,u)^2, & sn\,u\,sn'u, & (sn'u)^2 \\ (cn\,u)^2, & cn\,u\,cn'u, & (cn'u)^2 \\ (dn\,u)^2, & dn\,u\,dn'u, & (dn'u)^2 \end{vmatrix} = k'\,sn\,u\,cn\,u\,dn\,u.$$

(G. B. Mathews.)

13. Show that

$$\frac{2\,kK}{\pi}\cos\operatorname{coam}\frac{2\,Ku}{\pi} = \frac{4\sqrt{q}\sin u}{1+q} - \frac{4\sqrt{q^3}\sin 3\,u}{1+q^3} + \frac{4\sqrt{q^5}\sin 5\,u}{1+q^5} - \cdots.$$

14. Show that

$$\frac{2\,k'K}{\pi}\,\frac{1}{dn\,\frac{2\,Ku}{\pi}} = 1 - \frac{4\,q}{1+q^2}\cos 2\,u + \frac{4\,q^2}{1+q^4}\cos 4\,u - \frac{4q^3}{1+q^6}\cos 6\,u + \cdots.$$

CHAPTER XII

DOUBLY PERIODIC FUNCTIONS OF THE SECOND SORT

ARTICLE 232. From the formulas (X) and (XII) of the preceding Chapter it follows that $dn\, u$ has the period $2K$ and $sn\, u$ the period $2iK'$, although $2K$ is *not* a period of $sn\, u$ and $2iK'$ is *not* a period of $dn\, u$. There is consequently an irregularity in this respect. In order fully to understand this, it is well to consider the doubly periodic functions of the second sort which were introduced by Hermite.*

The Germans use the word "Art" for the word "espèce" which I translate by " sort " (see Art. 84 where the doubly periodic functions of the third sort were treated under the name "*Hermite's intermediary functions*"). In this connection see Jordan, *Cours d'Analyse*, t. II, No. 401, and Halphen, *Traité des fonctions elliptiques*, t. I, pp. 325–338, 411–426, 438–442, 463.

ART. 233. A doubly periodic function of the second sort with the primitive periods $2K$ and $2iK'$ is defined through the functional equations

$$f(u + 2K) = \nu f(u);$$
$$f(u + 2iK') = \nu' f(u),$$

where ν and ν' are constants called *factors* or *multipliers* and are independent of u. When $\nu = 1 = \nu'$, we have the doubly periodic functions properly so called, which belong to the category of *doubly periodic functions of the first sort.*

In the case before us of the preceding Article $sn\, u$, $cn\, u$, $dn\, u$ belong to the class of functions of the second sort, as appears from the formulas (X) and (XII).

For the function $sn\, u$ we have $\nu = -1$, $\nu' = 1$; for $cn\, u$ we have $\nu = -1$, $\nu' = -1$, while $\nu = +1$, $\nu' = -1$ for $dn\, u$. We may now consider more closely these doubly periodic functions of the second sort.

* Hermite, *Comptes Rendus*, t. 53, pp. 214–228, and t. 55, pp. 11–18 and pp. 85–91; Hermite, *Note sur la théorie des fonctions elliptiques*, in Lacroix's *Calcul*, t. 2 (6th ed.), pp. 484–491; see also *Cours de M. Hermite rédigé en* 1882, *par* M. Andoyer, p. 206; Appell, *Acta Math.*, Bd. 13, 1890; Picard, *Comptes Rendus*, t. 90, pp. 128–131 and 293–295; Picard, *Crelle*, Bd. 90, pp. 281–302; and in particular Forsyth, *Theory of Functions*, pp. 273–281, where references are made among others to Frobenius, *Crelle*, Bd. 93, pp. 53–68; Brioschi, *Comptes Rendus*, t. 92, pp. 323–328.

ART. 234. *Formation of the doubly periodic functions of the second sort which have prescribed factors ν and ν'.* — In the following Article it is shown that it is always possible to form a fundamental doubly periodic function of the second sort $f(u)$ with factors ν and ν', which function is infinite of the first order at only one point within the parallelogram with sides $2K$ and $2iK'$. The infinity of this fundamental function is denoted by $u = c$.

This admitted for the moment, let $F(u)$ be an arbitrary doubly periodic function of the second sort which has the periods $2K$ and $2iK'$ and has the same factors ν and ν' as $f(u)$. Further we shall assume that $F(u)$ is determinate at every point of the period-parallelogram.

Suppose that the function $F(u)$ is infinite of the λ_i order at the points α_i $(i = 1, 2, \ldots, n)$, where the points $\alpha_1, \alpha_2, \ldots, \alpha_n$ all lie within the period-parallelogram.

We shall show that $F(u)$ may be expressed in terms of $f(u)$.

For simplicity suppose that the parallelogram is so situated (Art. 91) that $F(u)$ does not become infinite upon its sides.

Consider next the function

$$\psi(\xi) = F(\xi) f(u - \xi),$$

where u is any point within the period-parallelogram, while ξ is to be regarded as the independent variable.

Instead of ξ write $\xi + 2K$. It follows that

$$\psi(\xi + 2K) = F(\xi + 2K) f(u - \xi - 2K).$$

But we have

$$f(u + 2K) = \nu f(u),$$

$$f(u - \xi + 2K) = \nu f(u - \xi).$$

If we put $\xi + 2K$ for ξ in this last formula, the result is

$$f(u - \xi - 2K) = \frac{1}{\nu} f(u - \xi).$$

Also, since

$$F(\xi + 2K) = \nu F(\xi),$$

it follows that

$$\psi(\xi + 2K) = F(\xi) f(u - \xi),$$

or

$$\psi(\xi + 2K) = \psi(\xi),$$

and similarly

$$\psi(\xi + 2iK') = \psi(\xi).$$

It is thus seen that $\psi(\xi)$ is a doubly periodic function of the first sort.

For such a function we have proved that

$$\sum \operatorname{Res} \psi(\xi) = 0,$$

where the summation is to be taken over all the infinities within the period-parallelogram.

But $\psi(\xi)$ becomes infinite on the points where $F(\xi)$ is infinite and besides on the point $u - \xi = c$, where $f(u - \xi)$ is infinite. The points $\alpha_1, \alpha_2, \ldots, \alpha_n$ must be distinct from the point $u - c = \xi$.

The expansion of $f(u - \xi)$ in the neighborhood of the point c is of the form

$$f(u-\xi) = \frac{C}{u-\xi-c} + A_0 + A_1(u-\xi-c) + A_2(u-\xi-c)^2 + \cdots$$
$$= \frac{-C}{\xi-(u-c)} + A_0 - A_1[\xi-(u-c)] + \cdots.$$

(In the sequel we shall choose a fundamental function $f(u)$ such that the quantity C is unity.)

Next if we develop $F(\xi)$ in the neighborhood of $u - c$ by Taylor's Theorem, we have

$$F(\xi) = F(u-c) + F'(u-c)[\xi-(u-c)] + \cdots,$$

and since

$$\psi(\xi) = F(\xi)f(u-\xi),$$

we have

$$\operatorname*{Res}_{\xi=u-c} \psi(\xi) = -CF(u-c).$$

In the neighborhood of the infinity α_k, the expansion of $F(\xi)$ is of the form (cf. Art. 98)

$$\frac{A_{k,\lambda_k}}{(\xi-\alpha_k)^{\lambda_k}} + \frac{A_{k,\lambda_k-1}}{(\xi-\alpha_k)^{\lambda_k-1}} + \cdots + \frac{A_{k,1}}{(\xi-\alpha_k)^1} + B_0 + B_1(\xi-\alpha_k) + \cdots,$$

while the expansion of $f(u - \xi)$ in the neighborhood of this point is

$$f(u-\xi) = f(u-\alpha_k) - \frac{f'(u-\alpha_k)}{1!}(\xi-\alpha_k) + \frac{f''(u-\alpha_k)}{2!}(\xi-\alpha_k)^2 -$$
$$\cdots \pm \frac{f^{(\lambda_k-1)}(u-\alpha_k)}{(\lambda_k-1)!}(\xi-\alpha_k)^{\lambda_k-1} \mp \cdots.$$

Through the multiplication of these series it is seen that

$$\operatorname*{Res}_{\xi=\alpha_k} \psi(\xi) = A_{k,1}f(u-\alpha_k) - \frac{A_{k,2}}{1!}f'(u-\alpha_k) + \frac{A_{k,3}}{2!}f''(u-\alpha_k) -$$
$$\cdots \pm \frac{A_{k,\lambda_k}}{(\lambda_k-1)!}f^{(\lambda_k-1)}(u-\alpha_k).$$

Since

$$\sum \operatorname{Res} \psi(\xi) = 0,$$

we have

$$0 = -CF(u-c) + \sum_{k=1}^{k=n}\Big[A_{k,1}f(u-\alpha_k) - \frac{A_{k,2}}{1!}f'(u-\alpha_k) +$$
$$\cdots \pm \frac{A_{k,\lambda_k}}{(\lambda_k-1)!}f^{(\lambda_k-1)}(u-\alpha_k)\Big].$$

If next we write $u + c$ in the place of u, it follows that

$$CF(u) = \sum_{k=1}^{k=n}\left[A_{k,1} f(u + c - \alpha_k) - \frac{A_{k,2}}{1!} f'(u + c - \alpha_k) + \cdots \pm \frac{A_{k,\lambda_k}}{(\lambda_k - 1)!} f^{(\lambda_k - 1)}(u + c - \alpha_k)\right],$$

which is the expression of $F(u)$ in terms of the fundamental function $f(u)$.

ART. 235. *Formation of the fundamental function $f(u)$ which has prescribed factors (or multipliers) ν and ν', where ν and ν' are any constants different from zero.*

We had the formulas

$$\mathrm{H}(u + 2K) = -\mathrm{H}(u),$$
$$\mathrm{H}(u + 2iK') = -\mu \mathrm{H}(u),$$

where $\mu = \mu(u) = e^{-\frac{\pi i}{K}(u + iK')}$.

If we write

$$\phi(u) = \mathrm{H}(u + \beta),$$

it follows that

$$\phi(u + 2K) = \mathrm{H}(u + \beta + 2K) = -\mathrm{H}(u + \beta);$$

or,

$$\phi(u + 2K) = -\phi(u),$$

and similarly

$$\phi(u + 2iK') = -\mu e^{-\frac{\pi i \beta}{K}} \phi(u).$$

Consider next the function

$$\Psi(u) = \frac{\mathrm{H}(u + \beta)}{\mathrm{H}(u)} = \frac{\phi(u)}{\mathrm{H}(u)}.$$

We have immediately

$$\Psi(u + 2K) = \Psi(u),$$
$$\Psi(u + 2iK') = \Psi(u) e^{-\frac{\pi i \beta}{K}}.$$

The function $\Psi(u)$ is therefore a doubly periodic function of the second sort having as factors $+1$ and $e^{-\frac{\pi i \beta}{K}}$. Suppose that ν and ν' are the prescribed factors. To form a function having them, write

$$f(u) = e^{\alpha u} \Psi(u);$$

so that

$$f(u + 2K) = e^{\alpha(u + 2K)} \Psi(u + 2K) = e^{\alpha 2K} f(u)$$

and

$$f(u + 2iK') = e^{\alpha(u + 2iK')} \Psi(u + 2iK') = e^{\alpha 2iK'} e^{-\frac{\pi \beta i}{K}} f(u).$$

Hence $f(u)$ is a doubly periodic function of the second sort with the factors $e^{\alpha 2K}$ and $e^{\alpha 2iK' - \frac{\pi i \beta}{K}}$.

The arbitrary constants α and β may be so chosen that

$$(1)\qquad e^{\alpha 2K} = \nu,$$

$$(2)\qquad e^{\alpha 2iK' - \frac{\pi i\beta}{K}} = \nu'.$$

From (1) it follows that

$$\alpha = \frac{1}{2K}\log\nu;$$

and from (2)

$$\alpha\, 2iK' - \frac{\pi i\beta}{K} = \log\nu',$$

or

$$\beta = \frac{K'\log\nu + Ki\log\nu'}{\pi}.$$

The quantities α and β being thus determined we have

$$f(u + 2K) = \nu f(u),$$
$$f(u + 2iK') = \nu' f(u),$$
$$f(u) = e^{\alpha u}\frac{\mathrm{H}(u+\beta)}{\mathrm{H}(u)}.$$

The function $f(u)$ is infinite of the first order for $u = 0$ (see Art. 203) and for no other point in the period-parallelogram, since the other vertices of the parallelograms are counted as belonging to the following parallelograms.

ART. 236. There is one case * in which we cannot determine $f(u)$ in the above manner, viz., when the multipliers or factors ν and ν' have been so chosen that

$$\beta = 2mK + 2niK',$$

where m and n are integers.

We would then have

$$f(u) = e^{\alpha u}\frac{\mathrm{H}(u + 2mK + 2niK')}{\mathrm{H}(u)}$$
$$= (-1)^m e^{\alpha u}\frac{\mathrm{H}(u + 2niK')}{\mathrm{H}(u)}.$$

Further, since (cf. Art. 91)

$$\mathrm{H}(u + n\,2iK') = (-1)^n e^{-\frac{\pi i}{K}(nu + n^2 iK')}\mathrm{H}(u),$$

it follows that

$$f(u) = (-1)^{m+n} e^{-\frac{\pi i}{K}(nu + n^2 iK')} e^{\alpha u}\frac{\mathrm{H}(u)}{\mathrm{H}(u)},$$

so that $f(u)$ is an exponential function and no longer a doubly periodic function of the second sort.

* See Forsyth, *Theory of Functions*, p. 279.

ART. 237. We must proceed differently for this exceptional case. We had by hypothesis

$$\beta = 2\,mK + 2\,niK',$$

and consequently

$$2\,mK\pi + 2\,nK'i\pi = K'\log\nu + Ki\log\nu'.$$

Further, since $\log\nu = 2\,K\alpha$, it follows that

$$Ki\log\nu' = 2\,mK\pi + 2\,nK'i\pi - 2\,KK'\alpha,$$

or

$$\log\nu' = -\,2\,m\pi i + 2\,n\frac{K'}{K}\pi + 2\,K'\alpha i.$$

We thus have

$$\nu' = e^{-2m\pi i + 2n\pi\frac{K'}{K} + 2K'\alpha i} = e^{2iK'\left(\alpha - \frac{n\pi i}{K}\right)},$$

and

$$\nu = e^{2K\alpha} = e^{2K\left(\alpha - \frac{n\pi i}{K}\right)}.$$

If we put

$$\alpha - \frac{n\pi i}{K} = \gamma,$$

the above expressions become

$$\nu = e^{2K\gamma}, \quad \nu' = e^{2K'i\gamma}.$$

We have the exceptional case* when ν and ν' have this form. The quantity γ is arbitrary; but if the factors ν and ν' are given, then γ is known.

We now write

$$f(u) = \frac{\mathrm{H}'(u)}{\mathrm{H}(u)}e^{\gamma u},$$

where $\mathrm{H}'(u)$ is the derivative of $\mathrm{H}(u)$.

From the formulas

$$\mathrm{H}(u + 2\,K) = -\,\mathrm{H}(u), \quad \mathrm{H}(u + 2\,iK') = -\,e^{-\frac{\pi i}{K}(u+iK')}\mathrm{H}(u),$$

we have at once

$$\mathrm{H}'(u + 2\,K) = -\,\mathrm{H}'(u), \quad \mathrm{H}'(u+2\,iK') = e^{-\frac{\pi i}{K}(u+iK')}\left[\frac{\pi i}{K}\mathrm{H}(u) - \mathrm{H}'(u)\right].$$

It follows that

$$f(u + 2\,K) = f(u)e^{2K\gamma} = \nu f(u).$$

We further have

$$\frac{\mathrm{H}'(u + 2\,iK')}{\mathrm{H}(u + 2\,iK')} = -\,\frac{\pi i}{K} + \frac{\mathrm{H}'(u)}{\mathrm{H}(u)},$$

so that

$$f(u + 2\,iK') = \nu' f(u) - \nu'\frac{\pi i}{K}e^{\gamma u}.$$

* First noted by Mittag-Leffler, *Comptes Rendus*, t. 90, p. 178; see also Halphen, *Fonct. Ellipt.*, t. I, p. 232.

The function $f(u)$ is therefore *not* a doubly periodic function of the second sort. It will nevertheless serve for the formation of a doubly periodic function of the second sort with the factors $e^{2K\gamma}$ and $e^{2K'i\gamma}$, which function becomes infinite on an arbitrary number of points within the period-parallelogram.

Let $F(u)$ be the function required, so that

$$F(u + 2K) = \nu F(u), \quad \nu = e^{2K\gamma}$$

and

$$F(u + 2iK') = \nu' F(u), \quad \nu' = e^{2K'i\gamma}.$$

We shall express $F(u)$ in terms of $f(u) = \dfrac{\mathrm{H}'(u)}{\mathrm{H}(u)} e^{\gamma u}$.

The period-parallelogram is to be chosen so that $F(u)$ does not become infinite on its sides.

We again form the function

$$\psi(\xi) = F(\xi) f(u - \xi).$$

We shall see that $\psi(\xi)$ is here *not* a doubly periodic function of the first sort as was the case in Art. 234.

From the formulas

$$f(u + 2K) = \nu f(u),$$

$$f(u + 2iK') = \nu' \left\{ f(u) - \frac{\pi i}{K} e^{\gamma u} \right\},$$

it follows that

$$\psi(\xi + 2K) = \psi(\xi),$$

and further that

$$\psi(\xi + 2iK') = \psi(\xi) + e^{\gamma(u-\xi)} F(\xi) \frac{\pi i}{K}.$$

We again note that $2iK'$ is *not* a period of $\psi(\xi)$.

We compute next $\Sigma \operatorname{Res} \psi(\xi)$ for the interior of the parallelogram whose sides are $2K$ and $2iK'$. It is seen that $\xi = u$ is an infinity of $\psi(\xi)$; for $\mathrm{H}(0) = 0$, and as $\mathrm{H}(u)$ is an odd function, its expansion is

$$\mathrm{H}(u) = u(c_0 + c_1 u^2 + \cdots),$$

so that

$$\frac{\mathrm{H}'(u)}{\mathrm{H}(u)} = \frac{1}{u} + P(u),$$

where $P(u)$ is a power series in positive integral powers of u.

Similarly we have

$$\frac{\mathrm{H}'(u - \xi)}{\mathrm{H}(u - \xi)} = \frac{1}{u - \xi} + P(u - \xi).$$

Further, since $e^{\gamma u} = 1 + \dfrac{\gamma u}{1!} + \cdots$,

we have

$$f(u - \xi) = e^{\gamma(u-\xi)} \frac{\mathrm{H}'(u - \xi)}{\mathrm{H}(u - \xi)} = -\frac{1}{\xi - u} + P_1(u - \xi),$$

where $P_1(u - \xi)$ denotes a power series in positive integral powers of $u - \xi$.

The expansion of $F(\xi)$ in the neighborhood of $\xi = u$ is

$$F(\xi) = F(u) + F'(u)\frac{u-\xi}{1!} + \cdots.$$

We therefore have

$$\operatorname*{Res}_{\xi=u} \psi(\xi) = -F(u).$$

As in Art. 234,

$$\operatorname*{Res}_{\xi=\alpha_k} \psi(\xi) = A_{k,1}f(u-\alpha_k) - \frac{A_{k,2}}{1!}f'(u-\alpha_k) + \cdots \pm \frac{A_{k,\lambda_k}}{(\lambda_k-1)!}f^{(\lambda_k-1)}(u-\alpha_k).$$

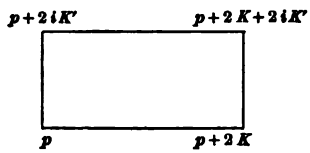
It follows that

$$\sum \text{Res } \psi(\xi) = -F(u) + \sum_{k=1}^{k=n}\left[A_{k,1}f(u-\alpha_k) - \frac{A_{k,2}}{1!}f'(u-\alpha_k) + \cdots \pm \frac{A_{k,\lambda_k}}{(\lambda_k-1)!}f^{(\lambda_k-1)}(u-\alpha_k)\right].$$

We cannot put $\Sigma \text{ Res } \psi(\xi) = 0$, as in Art. 234; but after Cauchy's Theorem

$$\sum \text{Res } \psi(\xi) = \frac{1}{2\pi i}\int \psi(\xi)d\xi,$$

where the integration is to be taken over the four sides of the parallelogram in the figure.



We have as in Art. 92

$$2\pi i \sum \text{Res } \psi(\xi) = \int_p^{p+2K}\psi(\xi)d\xi + \int_{p+2K}^{p+2K+2iK'}\psi(\xi)d\xi + \int_{p+2K+2iK'}^{p+2iK'}\psi(\xi)d\xi + \int_{p+2iK'}^{p}\psi(\xi)d\xi,$$

or by Art. 92,

$$= 2K\int_0^1 \psi(p+2Kt)dt + 2iK'\int_0^1 \psi(p+2K+2iK't)dt + 2K\int_1^0 \psi(p+2iK'+2Kt)dt + 2iK'\int_1^0 \psi(p+2K'it)dt.$$

But since $\psi(\xi)$ has the period $2K$, it follows that

$$2\pi i\sum \text{Res } \psi(\xi) = 2K\int_0^1 \left\{\psi(p+2Kt) - \psi(p+2iK'+2Kt)\right\}dt$$

$$= 2K\int_0^1\left\{-e^{\gamma(u-p-2Kt)}F(p+2Kt)\frac{\pi i}{K}\right\}dt;$$

or

$$\sum \text{Res } \psi(\xi) = -e^{\gamma u}\int_0^1 e^{-(p+2Kt)}F(p+2Kt)dt.$$

The definite integral is a quantity independent of u, which we may denote by A, so that therefore

$$\sum \operatorname{Res} \psi(\xi) = -Ae^{\gamma u}.$$

Equating the two expressions that have been found for $\Sigma \operatorname{Res} \psi(\xi)$, it is seen that

$$F(u) = Ae^{\gamma u} + \sum_{k=1}^{k=n} \left\{ A_{k,1} f(u-\alpha_k) - \frac{A_{k,2}}{1!} f'(u-\alpha_k) + \cdots \pm \frac{A_{k,\lambda_k}}{(\lambda_k - 1)!} f^{(\lambda_k - 1)}(u-\alpha_k) \right\}.$$

Further, since $F(u + 2iK') = \nu' F(u)$, we may write

$$F(u+2iK') = \nu' A e^{\gamma u} + \nu' \sum_{k=1}^{k=n} \left\{ A_{k,1} f(u-\alpha_k) - \cdots \pm \frac{A_{k,\lambda_k}}{(\lambda_k-1)!} f^{(\lambda_k-1)}(u-\alpha_k) \right\}.$$

On the other hand if we put $u + 2iK'$ for u in the expression above, we have

$$F(u+2iK') = \nu' A e^{\gamma u} + \nu' \sum_{k=1}^{k=n} \left\{ A_{k,1} f(u-\alpha_k) - \cdots \pm \frac{A_{k,\lambda_k-1}}{(\lambda_k-1)!} f(u-\alpha_k) \right\}$$
$$- \frac{\nu' \pi i}{K} \sum_{k=1}^{k=n} \left\{ A_{k,1} e^{\gamma(u-\alpha_k)} - \frac{A_{k,2}}{1!} \frac{d}{du} e^{\gamma(u-\alpha_k)} + \cdots \pm \frac{A_{k,\lambda_k-1}}{(\lambda_k-1)!} \frac{d^{\lambda_k-1}}{du^{\lambda_k-1}} e^{\gamma(u-\alpha_k)} \right\}.$$

Comparing the two results just derived, it is seen that

$$0 = \sum_{k=1}^{k=n} \left\{ e^{-\gamma \alpha_k} \left(A_{k,1} - \frac{A_{k,2}}{1!}\gamma + \frac{A_{k,3}}{2!}\gamma^2 - \cdots \pm \frac{A_{k,\lambda_k-1}}{(\lambda_k-1)!} \gamma^{\lambda_k-1} \right) \right\}.$$

This condition must be satisfied by the A's in the formation of the function $F(u)$.

Since γ is an arbitrary quantity, it may be made equal to zero. We then have

$$\Sigma A_{k,1} = 0.$$

But $A_{k,1}$ is the residue of $F(\xi)$ for $\xi = \alpha_k$.

We therefore have

$$\Sigma A_{k,1} = \Sigma \operatorname{Res} F(\xi);$$

and consequently

$$\Sigma \operatorname{Res} F(\xi) = 0, \quad \text{when } \gamma = 0.$$

But if $\gamma = 0$, then $F(u + 2K) = F(u)$

and $$F(u + 2iK') = F(u),$$

so that $F(u)$ is a doubly periodic function of the first sort.

We thus have another proof of the theorem* (see Art. 99) that *for a doubly periodic function of the first sort the sum of the residues with respect to all its infinities within a period-parallelogram is equal to zero.*

ART. 238. *A preliminary formula of addition.*† — By means of the above results, and as an illustration of them, we may compute the addition-theorem for $sn\, u$.

In the function $sn(u + v)$ we consider v as constant and u as the variable. This function becomes infinite on the points where $\Theta(u + v)$ is zero, viz.,

$$u + v = 2mK + (2n + 1)iK'.$$

It is seen that $\Theta(u + v)$ vanishes on the point $u + v = iK'$ or $u = iK' - v$ and on all congruent points (modd. $2K$, $2iK'$).

It is quite possible, when we consider the parallelogram of periods, that the point $iK' - v$ does not lie within it. There is, however, some congruent point which does lie within it, and we shall simply denote this point by $iK' - v$.

Consider the product

$$sn(u + v)\left\{sn\, u - sn\,(iK' - v)\right\}.$$

If $u = iK - v$, the expression within the braces becomes zero of the first order, while $sn(u + v)$ is infinite of the first order. The product therefore remains finite for $u = iK' - v$.

We form next the function

$$G(u) = sn(u + v)\left\{sn\, u - sn(iK' - v)\right\}\left\{sn\, u - sn\,(iK' + 2K - v)\right\}.$$

This product remains finite for $u = iK' - v$ and for $u = iK' + 2K - v$ and for all points congruent to these two points (modd. $2K$, $2iK'$).

We have

$$sn(iK' - v) = \frac{1}{k\, sn(-v)} = -\frac{1}{k\, sn\, v}.$$

It follows that

$$\begin{aligned} G(u) &= sn(u + v)\left\{sn\, u + \frac{1}{k\, sn\, v}\right\}\left\{sn\, u - \frac{1}{k\, sn\, v}\right\} \\ &= sn(u + v)\left\{sn^2 u - \frac{1}{k^2 sn^2 v}\right\}, \end{aligned}$$

or $$G(u)k^2 sn^2 v = sn(u + v)\left\{k^2 sn^2 u\, sn^2 v - 1\right\} = F(u), \text{ say.}$$

* See Forsyth, *Theory of Functions*, p. 280.

† Hermite's "*Cours*" (*Quatrième édition*, p. 242); see also Appell et Lacour, *Fonctions Elliptiques*, p. 129.

It follows at once that

$$F(u+2K) = -F(u), \text{ so that } \nu = -1$$

and

$$F(u+2iK') = F(u), \text{ or } \nu' = 1.$$

We note that $F(u)$ is a doubly periodic function of the second sort with the periods $2K$ and $2iK'$. Consider the parallelogram with the sides $2K$ and $2iK'$ in which the point iK' lies. The function $F(u)$ becomes infinite on this point but on no other point of the parallelogram.

To determine the order of the infinity of $F(u)$ for the point $u = iK'$, it is seen that

$$sn\, h = h + c_3 h^3 + c_5 h^5 + \cdots ;$$

and consequently if we put

$$u = iK' + h \quad \text{or} \quad h = u - iK',$$

we have

$$sn(iK'+h) = \frac{1}{k\, sn\, h} = \frac{1}{kh} \cdot \frac{1}{1 + e_3 h^2 + e_5 h^4 + \cdots}$$
$$= \frac{1}{kh}\left\{1 + e_2 h^2 + e_4 h^4 + \cdots\right\}.$$

It follows at once that

$$sn\, u = \frac{1}{k(u - iK')}\left\{1 + e_2(u - iK')^2 + \cdots\right\},$$

and consequently

$$k^2 sn^2 u\, sn^2 v - 1 = \frac{1}{(u-iK')^2} sn^2 v + 2 e_2 sn^2 v + \cdots - 1.$$

Noting that $\dfrac{d\, sn\, v}{dv} = cn\, v\, dn\, v$, it is seen that the expansion of $sn(u+v)$ in the neighborhood of $u = h + iK'$ is

$$sn(u+v) = sn(v + iK' + h) = \frac{1}{k\, sn(v+h)},$$

which by Taylor's Theorem

$$= \frac{1}{k\, sn\, v} - h\frac{cn\, v\, dn\, v}{k\, sn^2 v} + \cdots$$
$$= \frac{1}{k\, sn\, v} - \frac{cn\, v\, dn\, v}{k\, sn^2 v}(u - iK') + \cdots.$$

We therefore have

$$F(u) = \frac{1}{k}\frac{sn\, v}{(u-iK')^2} - \frac{cn\, v\, dn\, v}{k}\frac{1}{u - iK'} + P(u - iK').$$

Writing

$$-\frac{cn\, v\, dn\, v}{k} = A_0, \quad \frac{sn\, v}{k} = A_1,$$

we have

$$F(u) = \frac{A_0}{u - iK'} + \frac{A_1}{(u-iK')^2} + P(u - iK').$$

We shall next express $F(u)$ through a fundamental function $f(u)$. The function $f(u)$ must be a doubly periodic function of the second sort with the factors $+1$ and -1 and with the periods $2K$ and $2iK'$.

We may consequently choose $\frac{1}{sn\,u}$ for this fundamental function. We have

$$\frac{1}{sn\,u} = \frac{1}{u} + \text{positive powers of } u.$$

Consequently we have $\operatorname*{Res}_{u=0} f(u) = 1 = C$ (of Art. 234).

Hence (see the formula at the end of Art. 234) it follows that

$$F(u) = A_0 f(u - iK') - A_1 f'(u - iK').$$

We have further $\quad f(u - iK') = \frac{1}{sn\,(u - iK')} = k\,sn\,u,$

and also $f'(u - iK') = k\,cn\,u\,dn\,u$, so that

$$F(u) = -\frac{cn\,v\,dn\,v}{k} k\,sn\,u - \frac{sn\,v}{k} k\,cn\,u\,dn\,u.$$

Equating the two values of $F(u)$, it is seen that

$$sn(u + v)\,[k^2 sn^2 u\, sn^2 v - 1] = -sn\,u\,cn\,v\,dn\,v - sn\,v\,cn\,u\,dn\,u,$$

or finally $\quad sn(u + v) = \frac{sn\,u\,cn\,v\,dn\,v + sn\,v\,cn\,u\,dn\,u}{1 - k^2 sn^2 u\, sn^2 v},$

which is the *addition-theorem* for the modular sine.

When $k = 0$, we have $sn\,u = \sin u$, $cn\,u = \cos u$, $dn\,u = 1$, and consequently

$$\sin(u + v) = \sin u \cos v + \cos u \sin v.$$

The above addition-theorem may also be written in the form

$$sn(u + v) = \frac{sn\,u \frac{d\,sn\,v}{dv} + sn\,v \frac{d\,sn\,u}{du}}{1 - k^2 sn^2 u\, sn^2 v}.$$

As an exercise the student may derive the addition-theorems for $cn(u + v)$ and $dn(u + v)$ and compare the result with those given in Chapter XVI.

ART. 239. As a further application of the doubly periodic functions of the second sort we may develop in series of sines and cosines such expressions as

$$\frac{\Theta(u + a)}{\Theta(u)},\quad \frac{\mathrm{H}(u + a)}{\Theta(u)},\quad \frac{\Theta_1(u + a)}{\Theta(u)},\quad \frac{\mathrm{H}_1(u + a)}{\Theta(u)},$$

which appear in Jacobi's investigations relative to the *rotation of a body which is not subjected to an accelerating force.**

* Jacobi, Werke, II, pp. 292 *et seq.*

Consider with Hermite* the series

$$\sum \frac{e^{\frac{\pi i n a}{K}}}{\sin\frac{\pi}{2K}(u+2niK')},$$

where n takes all values from $-\infty$ to $+\infty$, a being a constant which will be represented by $\alpha + i\alpha'$.

We shall first show that this series is convergent, whatever be the value of u, provided that α' is less in absolute value than $2K'$.

Writing the general term in the form

$$\frac{2ie^{\frac{\pi i n a}{K}}}{e^{\frac{\pi i}{K}(u+2niK')} - e^{-\frac{\pi i}{K}(u+2niK')}},$$

it is seen that we may neglect the first or the second exponential term in the denominator according as n becomes positively or negatively indefinitely large.

We thus have either

$$-2ie^{\frac{ni\pi}{K}(a+2iK')+i\frac{\pi u}{2K}} \quad \text{or} \quad 2ie^{\frac{ni\pi}{K}(a-2iK')-\frac{i\pi u}{2K}}.$$

If we write $-n$ in the place of n in the second of these quantities and take the limit for n indefinitely large of the nth root of the moduli, we have after a has been replaced by $\alpha + i\alpha'$

$$\text{either} \quad e^{-\frac{\pi}{K}(\alpha'+2K')} \quad \text{or} \quad e^{\frac{\pi}{K}(\alpha'-2K')}.$$

If for the first $\alpha' + 2K' > 0$ and for the second $\alpha' - 2K < 0$, the two limits are less than unity and the series in question is convergent.

Consider next the function

$$\Phi(u) = \sum \frac{e^{\frac{\pi i n a}{K}}}{\sin\frac{\pi}{2K}(u+2niK')};$$

and noting that, since n varies from $-\infty$ to $+\infty$, we may change n into $n+1$, we have

$$\Phi(u) = \sum \frac{e^{\frac{(n+1)\pi i a}{K}}}{\sin\frac{\pi}{2K}[u+2(n+1)iK']} = e^{\frac{\pi i a}{K}} \sum \frac{e^{\frac{\pi i n a}{K}}}{\sin\frac{\pi}{2K}[u+2iK'+2niK']}.$$

It follows at once that

$$\Phi(u) = e^{\frac{i\pi a}{K}}\Phi(u+2iK'),$$

or

$$\Phi(u+2iK') = e^{-\frac{i\pi a}{K}}\Phi(u).$$

* Hermite, *Ann. de l'École Norm. Supér.*, 3e série, t. II (1885); see also Hermite, *Sur quelques applications des fonctions elliptiques*, p. 35.

On the other hand we have immediately

$$\Phi(u + 2K) = -\Phi(u),$$

so that $\Phi(u)$ is a doubly periodic function of the second sort with the multipliers -1 and $e^{-\frac{i\pi a}{K}}$.

The poles are obtained by writing

$$\sin\frac{\pi}{2K}(u + 2niK') = 0,$$

from which we have

$$u = 2mK - 2niK',$$

where m is an arbitrary integer.

We therefore see that on the interior of the rectangle of periods $2K$ and $2iK'$ there is only one pole $u = 0$, the corresponding residue being $\frac{2K}{\pi}$. We further note that the quantity

$$\frac{2K}{\pi}\frac{\mathrm{H}'(0)\Theta(u+a)}{\mathrm{H}(u)\Theta(a)}$$

has the same multipliers, the same pole, and the same residue.

We may therefore write (see Art. 83)

$$\frac{2K}{\pi}\frac{\mathrm{H}'(0)\Theta(u+a)}{\mathrm{H}(u)\Theta(a)} = \sum\frac{e^{\frac{\pi i n a}{K}}}{\sin\frac{\pi}{2K}(u + 2niK')}.$$

If a and u are permuted in this equation, we have

$$(1)\qquad \frac{2K}{\pi}\frac{\mathrm{H}'(0)\Theta(u+a)}{\mathrm{H}(a)\Theta(u)} = \sum\frac{e^{\frac{\pi i n u}{K}}}{\sin\frac{\pi}{2K}(a + 2niK')}.$$

We may deduce the others as follows:

If we change a into $a + iK'$, we have

$$\frac{2K}{\pi}\frac{\mathrm{H}'(0)\mathrm{H}(u+a)}{\Theta(u)\Theta(a)}e^{-\frac{\pi i u}{2K}} = \sum\frac{e^{\frac{\pi i n u}{K}}}{\sin\frac{\pi}{2K}[a + (2n+1)iK']},$$

or

$$(2)\qquad \frac{2K}{\pi}\frac{\mathrm{H}'(0)\mathrm{H}(u+a)}{\Theta(u)\Theta(a)} = \sum\frac{e^{\frac{(2n+1)\pi i u}{2K}}}{\sin\frac{\pi}{2K}[a + (2n+1)iK']}.$$

If further $a + K$ is written for a in (1) and (2), these formulas become

$$\frac{2K}{\pi}\frac{\mathrm{H}'(0)\Theta_1(u+a)}{\Theta(u)\mathrm{H}_1(a)} = \sum \frac{e^{\frac{\pi i n u}{K}}}{\cos\frac{\pi}{2K}[a + 2niK']}, \tag{3}$$

$$\frac{2K}{\pi}\frac{\mathrm{H}'(0)\mathrm{H}_1(u+a)}{\Theta(u)\Theta_1(a)} = \sum \frac{e^{\frac{(2n+1)\pi i u}{2K}}}{\cos\frac{\pi}{2K}[a + (2n+1)iK']}. \tag{4}$$

If $u + K$ is written for u in the four formulas above, we have the four following formulas, in which $\Theta_1(u)$ is found in the denominators:

$$\frac{2K}{\pi}\frac{\mathrm{H}'(0)\Theta_1(u+a)}{\Theta_1(u)\mathrm{H}(a)} = \sum \frac{(-1)^n e^{\frac{\pi i n u}{K}}}{\sin\frac{\pi}{2K}(a + 2niK')}, \tag{5}$$

$$\frac{2K}{\pi}\frac{\mathrm{H}'(0)\mathrm{H}_1(u+a)}{\Theta_1(u)\Theta(a)} = \sum \frac{i^{2n+1} e^{\frac{(2n+1)\pi i u}{2K}}}{\sin\frac{\pi}{2K}[a + (2n+1)iK']}, \tag{6}$$

$$\frac{2K}{\pi}\frac{\mathrm{H}'(0)\Theta(u+a)}{\Theta_1(u)\mathrm{H}_1(a)} = \sum \frac{(-1)^n e^{\frac{\pi i n u}{K}}}{\cos\frac{\pi}{2K}(a + 2niK')}, \tag{7}$$

$$\frac{2K}{\pi}\frac{\mathrm{H}'(0)\mathrm{H}(u+a)}{\Theta_1(u)\Theta_1(a)} = \sum \frac{(-i)^{2n+1} e^{\frac{(2n+1)\pi i u}{2K}}}{\cos\frac{\pi}{2K}[a + (2n+1)iK']}. \tag{8}$$

ART. 240. Hermite next formed a series entirely different from the one of the preceding Article which is represented as follows:

$$\cot\frac{\pi a}{2K} + \sum e^{\frac{\pi i n a}{2K}}\left[\cot\frac{\pi}{2K}(u + niK') + \epsilon i\right],$$

where n takes all even integral values from $-\infty$ to $+\infty$, while the quantity ϵ must be supposed *zero* for $n = 0$ and equal to unity positive or negative according as n is positive or negative.

If we allow n to take only the positive integers

$$n = 2, 4, 6, \cdots,$$

the series above may be decomposed into the two partial series

$$\cot\frac{\pi a}{2K} + \cot\frac{\pi u}{2K} + \sum e^{\frac{\pi i n a}{2K}}\left[\cot\frac{\pi}{2K}(u + niK') + i\right]$$
$$+ \sum e^{-\frac{\pi i n a}{2K}}\left[\cot\frac{\pi}{2K}(u - niK') - i\right],$$

which by an easy transformation becomes

$$\cot\frac{\pi a}{2K} + \cot\frac{\pi u}{2K} + \sum \frac{e^{\frac{\pi i}{2K}(na+niK'+u)}}{\cos\frac{\pi}{2K}(u+niK')}$$

$$+\sum \frac{e^{-\frac{\pi i}{2K}(na-niK'+u)}}{\cos\frac{\pi}{2K}(u-niK')}.$$

To prove the convergence of this series, note that for large values of n the two denominators

$$\cos\frac{\pi}{2K}(u+niK') \quad\text{and}\quad \cos\frac{\pi}{2K}(u-niK')$$

may be replaced by

$$\tfrac{1}{2}e^{-\frac{\pi i}{2K}(u+niK')} \quad\text{and}\quad \tfrac{1}{2}e^{\frac{\pi i}{2K}(u-niK')},$$

the general terms becoming

$$2e^{\frac{\pi i}{2K}(na+2niK'+2u)}, \quad 2e^{-\frac{\pi i}{2K}(na-2niK'+2u)}.$$

If we put $a = \alpha + i\alpha'$, we have for the limit of the nth root of their moduli as n becomes very large, the quantities

$$e^{-\frac{\pi}{2K}(\alpha'+2K')} \quad\text{and}\quad e^{\frac{\pi}{2K}(\alpha'-2K')},$$

and consequently the conditions

$$\alpha' + 2K' > 0, \qquad \alpha' - 2K' < 0.$$

It follows that the series in this Article is, as the one in the preceding Article, convergent when the coefficient of i in the constant a is in absolute value less than $2K'$. This series also defines a doubly periodic function of the second sort. For writing

$$\Psi(u) = \cot\frac{\pi a}{2K} + \sum e^{\frac{\pi i n a}{2K}}\left[\cot\frac{\pi}{2K}(u+niK') + \epsilon i\right],$$

we have the relations

$$\Psi(u+2K) = \Psi(u), \quad \Psi(u+2iK') = e^{-\frac{\pi i a}{K}}\Psi(u).$$

The second of these relations is evident from the expression of the product $e^{\frac{\pi i a}{K}}\Psi(u+2iK')$, viz.,

$$e^{\frac{\pi i a}{K}}\Psi(u+2iK') = e^{\frac{\pi i a}{K}}\cot\frac{\pi a}{2K} + e^{\frac{\pi i a}{K}}\sum e^{\frac{\pi i n a}{2K}}\left[\cot\frac{\pi}{2K}(u+2iK'+niK') + \epsilon i\right].$$

We have
$$e^{\frac{\pi i a}{K}}\cot\frac{\pi a}{2K} = \cot\frac{\pi a}{2K} + i\left(e^{\frac{\pi i a}{K}} + 1\right),$$
and if we change, as is permissible, n to $n - 2$ in the general term, it becomes
$$e^{\frac{\pi i a}{K}}\Psi(u+2iK') = \cot\frac{\pi a}{2K} + i\left(e^{\frac{\pi i a}{K}}+1\right) + \sum e^{\frac{\pi i n a}{2K}}\left[\cot\frac{\pi}{2K}(u+niK') + \epsilon i\right],$$
where now there is a modification regarding ϵ.

The quantity ϵ must be $= 1$ for $n = 4, 6, 8, \ldots$, while $\epsilon = 0$ for $n = 2$ and $\epsilon = -1$ for $n = 0, -2, -4, \cdots$. We note that in adding to the terms corresponding to $n = 2$ and $n = 0$ on the one hand $ie^{\frac{\pi i a}{K}}$ and on the other i, and consequently in causing the quantity $i\left(e^{\frac{\pi i a}{K}} + 1\right)$ to enter the summation, we find for ϵ precisely the significance which was accorded it in the function $\Psi(u)$. We further note that within the rectangle of periods there exists the one pole $u = 0$, to which corresponds the residue $\frac{2K}{\pi}$. We may therefore represent the function $\Psi(u)$ by
$$\frac{2K}{\pi}\frac{\mathrm{H}'(0)\,\mathrm{H}(u+a)}{\mathrm{H}(u)\,\mathrm{H}(a)}.$$

If we interchange u and a we have finally
$$(9)\quad \frac{2K}{\pi}\frac{\mathrm{H}'(0)\,\mathrm{H}(u+a)}{\mathrm{H}(u)\,\mathrm{H}(a)} = \cot\frac{\pi u}{2K} + \sum e^{\frac{\pi i n u}{2K}}\left[\cot\frac{\pi}{2K}(a+niK') + \epsilon i\right],$$
where n represents all even integers and the unity ϵ must be taken positive when n is positive and negative when n is negative.

Next changing a to $a + iK'$, we have, after having multiplied by $e^{\frac{\pi i u}{2K}}$, the formula
$$\frac{2K}{\pi}\frac{\mathrm{H}'(0)\Theta(u+a)}{\mathrm{H}(u)\Theta(a)} = e^{\frac{\pi i u}{2K}}\cot\frac{\pi u}{2K} + \sum e^{\frac{\pi i m u}{2K}}\left[\cot\frac{\pi}{2K}(a+miK') + \epsilon i\right],$$
where m denotes the odd integer $n + 1$.

Since
$$e^{\frac{\pi i u}{2K}}\cot\frac{\pi u}{2K} = \frac{1}{\sin\frac{\pi u}{2K}} + ie^{\frac{\pi i u}{2K}},$$
we have, if the term $ie^{\frac{\pi i u}{2K}}$ is introduced under the summation sign,
$$(10)\quad \frac{2K}{\pi}\frac{\mathrm{H}'(0)\Theta(u+a)}{\mathrm{H}(u)\Theta(a)} = \frac{1}{\sin\frac{\pi u}{2K}} + \sum e^{\frac{\pi i m u}{2K}}\left[\cot\frac{\pi}{2K}(a+miK') + \epsilon i\right],$$
where m represents all odd integers and ϵ must be taken $+1$ or -1 according as m is positive or negative. Changing a to $a + K$ we have

the formulas (11) and (12) below, and by replacing u by $u + K$ in the formulas (9), (10), (11), (12) we have the formulas below, (13), (14), (15), (16).

$$(11)\quad \frac{2K}{\pi}\frac{H'(0)H_1(u+a)}{H(u)H_1(a)}=\cot\frac{\pi u}{2K}-\sum e^{\frac{\pi i n u}{2K}}\left[\tan\frac{\pi}{2K}(a+niK')-\epsilon i\right],$$

$$(12)\quad \frac{2K}{\pi}\frac{H'(0)\Theta_1(u+a)}{H(u)\Theta_1(a)}=\operatorname{cosec}\frac{\pi u}{2K}-\sum e^{\frac{\pi i m u}{2K}}\left[\tan\frac{\pi}{2K}(a+miK')-\epsilon i\right],$$

$$(13)\quad \frac{2K}{\pi}\frac{H'(0)H_1(u+a)}{H_1(u)H(a)}=-\tan\frac{\pi u}{2K}+\sum(-1)^{\frac{n}{2}}e^{\frac{\pi i n u}{2K}}\left[\cot\frac{\pi}{2K}(a+niK')+\epsilon i\right],$$

$$(14)\quad \frac{2K}{\pi}\frac{H'(0)\Theta_1(u+a)}{H_1(u)\Theta(a)}=\sec\frac{\pi u}{2K}+\sum i^m e^{\frac{\pi i m u}{2K}}\left[\cot\frac{\pi}{2K}(a+miK')+\epsilon i\right],$$

$$(15)\quad \frac{2K}{\pi}\frac{H'(0)H_1(u+a)}{H_1(u)H_1(a)}=-\tan\frac{\pi u}{2K}+\sum(-1)^{\frac{n}{2}}e^{\frac{\pi i n u}{2K}}\left[\tan\frac{\pi}{2K}(a+niK')-\epsilon i\right],$$

$$(16)\quad \frac{2K}{\pi}\frac{H'(0)\Theta(u+a)}{H_1(u)\Theta_1(a)}=\sec\frac{\pi u}{2K}-\sum i^m e^{\frac{\pi i m u}{2K}}\left[\tan\frac{\pi}{2K}(a+miK')-\epsilon i\right],$$

the quantities m, n, and ϵ being defined as above.

EXAMPLES

1. If $m = 1, 3, 5, \ldots$; $n = 2, 4, 6, \ldots$, show that

$$\frac{2K}{\pi}\frac{H'(0)\Theta(u+a)}{\Theta(u)H(a)}=\operatorname{cosec}\frac{\pi a}{2K}+4\sum q^{\frac{mn}{2}}\sin\frac{\pi}{2K}(ma-nu).$$

2. Further if $m' = 1, 3\ 5, \ldots$, prove that

$$\frac{2K}{\pi}\frac{H'(0)H(u+a)}{\Theta(u)\Theta(a)}=4\sum q^{\frac{mm'}{2}}\sin\frac{\pi}{2K}(ma-m'u).$$

3. Show that

$$\frac{2K}{\pi}\frac{H'(0)\Theta_1(u+a)}{\Theta(u)H_1(a)}=\sec\frac{\pi a}{2K}+4\sum(-1)^{\frac{m-1}{2}}q^{\frac{mn}{2}}\cos\frac{\pi}{2K}(ma-nu).$$

4. Prove that

$$\frac{2K}{\pi}\frac{H'(0)H_1(u+a)}{\Theta(u)\Theta_1(a)}=4\sum(-1)^{\frac{m-1}{2}}q^{\frac{mm'}{2}}\cos\frac{\pi}{2K}(ma-m'u).$$

[Kronecker.]

CHAPTER XIII

ELLIPTIC INTEGRALS OF THE SECOND KIND

ARTICLE 241. From the investigations relative to the integrals of the first kind in Legendre's normal form (see Chapter VII) it is seen that the elliptic integral of the second kind

$$\int \frac{z^2 dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}$$

is finite and continuous on the finite portion of the Riemann surface.

In the neighborhood of the point $z = \infty$, we have

$$\frac{1}{\sqrt{(1 - z^2)(1 - k^2 z^2)}} = \frac{1}{z^2} + \frac{a_1}{z^4} + \frac{a_2}{z^6} + \cdots,$$

so that

$$\int \frac{z^2 dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}} = z + \frac{b_1}{z} + \frac{b_2}{z^3} + \cdots,$$

where the a's and b's are constants.

It follows that the elliptic integral of the second kind is algebraically infinite of the first order for the value $z = \infty$ in both the upper and the lower leaves.

In the Weierstrassian normal form

$$\int \frac{t dt}{\sqrt{S(t)}}, \quad S(t) = 4t^3 - g_2 t - g_3,$$

the expansion in the neighborhood of the point $t = \infty$, which is a branch-point, is

$$t^{\frac{1}{2}} + \frac{c_1}{t^{\frac{3}{2}}} + \frac{c_2}{t^{\frac{5}{2}}} + \cdots,$$

the limits of integration being so chosen that no constant term appears in this development. The question naturally arises whether it is possible to form a one-valued function of position on the Riemann surface which is algebraically infinite at only one point.

To investigate this question, consider the integral

$$\int \frac{C dt}{(t - \alpha)^2},$$

where C is a constant.

This is the simplest integral which is algebraically infinite of the first order at the two points*

$$\alpha, \sqrt{S(\alpha)} \quad \text{and} \quad \alpha, -\sqrt{S(\alpha)}.$$

We note also that the integral

$$\int \frac{At + B}{(t-\alpha)^2\sqrt{S(t)}}\, dt,$$

where A and B are constants, becomes infinite in the same manner at the same two points as the integral above. Neither of these integrals is infinite for $t = \infty$.

We shall so choose the constants A and B that the latter integral becomes infinite on the point $\alpha, -\sqrt{S(\alpha)}$ in the same manner as does the first integral.

By Taylor's Theorem we have in the neighborhood of the point $t = \alpha$

$$\frac{At + B}{\sqrt{S(t)}} = \frac{A\alpha + B}{\sqrt{S(\alpha)}} + \frac{A\sqrt{S(\alpha)} - \frac{1}{2}(A\alpha + B)\frac{S'(\alpha)}{\sqrt{S(\alpha)}}}{S(\alpha)}(t - \alpha) + \cdots.$$

It follows, if we put

(1)
$$A\sqrt{S(\alpha)} - \frac{1}{2}(A\alpha + B)\frac{S'(\alpha)}{\sqrt{S(\alpha)}} = \lambda = 0,$$

that

$$\int \frac{(At + B)dt}{(t-\alpha)^2\sqrt{S(t)}} = -\frac{A\alpha + B}{(t-\alpha)\sqrt{S(\alpha)}} + \lambda\frac{\log(t-\alpha)}{S(\alpha)} + P(t - \alpha)$$

will not contain a logarithmic term in the expansion according to ascending powers of $t - \alpha$.

Further, since

$$\int \frac{Cdt}{(t-\alpha)^2} = -\frac{C}{t-\alpha},$$

it is seen that the two integrals become infinite alike on the point $\alpha, -\sqrt{S(\alpha)}$, if

(2)
$$-C = \frac{A\alpha + B}{\sqrt{S(\alpha)}}.$$

It follows from equations (1) and (2) that

$$A = -\frac{1}{2}C\frac{S'(\alpha)}{\sqrt{S(\alpha)}},$$

$$B = -C\sqrt{S(\alpha)} - A\alpha = C\left[\frac{-S(\alpha) + \frac{1}{2}\alpha S'(\alpha)}{\sqrt{S(\alpha)}}\right],$$

* The following results are true not only when $S(t)$ is of the third degree in t, but also when this degree is n, where n is any positive integer.

and consequently that the integral

$$\int\left(\frac{C}{(t-\alpha)^2}-\frac{At+B}{(t-\alpha)^2\sqrt{S(t)}}\right)dt$$
$$=C\int\frac{\frac{1}{2}\frac{S'(\alpha)}{\sqrt{S(\alpha)}}(t-\alpha)+\sqrt{S(t)}+\sqrt{S(\alpha)}}{(t-\alpha)^2}\frac{dt}{\sqrt{S(t)}}$$

is an integral of the second kind, which is infinite of the first order* at only the one position $(\alpha, \sqrt{S(\alpha)})$. Write $C = \frac{1}{2}$ and put

$$\mathsf{E}_0(t, \sqrt{S(t)}) = \int\frac{\frac{1}{2}\frac{S'(\alpha)}{\sqrt{S(\alpha)}}(t-\alpha)+\sqrt{S(t)}+\sqrt{S(\alpha)}}{2(t-\alpha)^2}\frac{dt}{\sqrt{S(t)}}$$
$$= -\frac{1}{t-\alpha}+P(t-\alpha).$$

We may regard this integral as the *fundamental* integral of the second kind.

ART. 242. We next raise the question: Is there another integral $\mathsf{E}_1(t, \sqrt{S(t)})$ of the second kind which becomes algebraically infinite of the first order on the point $\alpha, \sqrt{S(\alpha)}$? If such an integral exists, its development in the neighborhood of $t = \alpha$ is of the form

$$\mathsf{E}_1(t, \sqrt{S(t)}) = \frac{-b}{t-\alpha}+P_1(t-\alpha).$$

Writing
$$\frac{1}{b}\mathsf{E}_1(t, \sqrt{S(t)}) = \mathsf{E}(t, \sqrt{S(t)}),$$

it is seen that

$$\mathsf{E}(t, \sqrt{S(t)}) - \mathsf{E}_0(t, \sqrt{S(t)})$$

does not become infinite for any point on the Riemann surface. It is therefore an integral of the first kind, $= F(t, \sqrt{S(t)})$, say. It follows that

$$\mathsf{E}(t, \sqrt{S(t)}) = \mathsf{E}_0(t, \sqrt{S(t)}) + F(t, \sqrt{S(t)}).$$

Hence, if we add to an integral of the second kind an integral of the first kind, we have an integral of the second kind which is infinite only at the point $(\alpha, \sqrt{R(\alpha)})$ provided the original integral of the second kind is infinite only at this point. There are consequently an infinite number of integrals of the second kind which are algebraically infinite of the first order on the one point $(\alpha, \sqrt{R(\alpha)})$.

* Cf. Koenigsberger, *Elliptische Functionen*, p. 250.

ART. 243. If we put

$$\Psi(t) = \frac{1}{2}\frac{S'(\alpha)}{\sqrt{S(\alpha)}}(t-\alpha) + \sqrt{S(t)} + \sqrt{S(\alpha)}$$

and
$$\psi(t) = \frac{\sqrt{S(t)} + \sqrt{S(\alpha)}}{2(t-\alpha)},$$

then
$$\frac{\partial\psi(t)}{\partial\alpha} = \frac{\Psi(t)}{2(t-\alpha)^2}.$$

We further write (see Art. 287)

$$\zeta(t, \sqrt{S(t)}) = \int \frac{tdt}{\sqrt{S(t)}},$$

which integral, as we saw above, becomes algebraically infinite at infinity. It is then evident that the expression

$$\sqrt{S(\alpha)} \cdot \mathsf{E}_0(t, \sqrt{S(t)}) + \psi(t) - \zeta(t, \sqrt{S(t)})$$

remains finite and continuous in both the finite and the infinite portion of the Riemann surface. It is therefore an integral of the first kind.

Similar results hold when *mutatis mutandis* $S(t)$ is of the fourth degree in t. It is thus seen that the elliptic integral of the second kind, which becomes algebraically infinite at the point infinity, may be replaced by one which is algebraically infinite at only one position on the Riemann surface, the latter position being a definitely prescribed one.

ART. 244. If in the integral of the first kind

$$u = \int_{0,1}^{\mathbf{z},s} \frac{dz}{\sqrt{(1-\mathbf{z}^2)(1-k^2\mathbf{z}^2)}},$$

we put $\mathbf{z} = \sin\phi$, we have in Legendre's notation

$$F(\phi, k) = \int_0^\phi \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = \int_0^\phi \frac{d\phi}{\Delta\phi}, \quad \text{where } \Delta\phi = \sqrt{1-k^2\sin^2\phi}.$$

The *complete* integrals of the first kind are therefore

$$F\left(\frac{\pi}{2}, k\right) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} = F,$$

$$F\left(\frac{\pi}{2}, k'\right) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k'^2\sin^2\phi}} = F(k') = F'.$$

In Legendre's notation (*Fonct. Elliptiques*, t. I, p. 15) the integral of the second kind is

$$E(k, \phi) = \int_0^\phi \sqrt{1-k^2\sin^2\phi}\, d\phi = \int_0^\phi \Delta\phi\, d\phi.$$

The complete integrals are (see also Art. 249):

$$E\left(\frac{\pi}{2}, k\right) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2\phi}\, d\phi = E,$$

$$E\left(\frac{\pi}{2}, k'\right) = \int_0^{\frac{\pi}{2}} \sqrt{1 - k'^2 \sin^2\phi}\, d\phi = E' = E(k');$$

or
$$E = \int_0^1 \frac{\sqrt{1 - k^2 z^2}}{\sqrt{1 - z^2}} dz; \quad E' = \int_0^1 \frac{\sqrt{1 - k'^2 z^2}}{\sqrt{1 - z^2}} dz.$$

If we put $d\phi = dn u\, du$, $\Delta\phi = dn u$, we have

$$E(k, \phi) = E(u) = \int_0^u dn^2 u\, du = \int_0^u (1 - k^2 sn^2 u)\, du.$$

(Jacobi, Werke, I, p. 299.)

ART. 245. To study the integral of the second kind

$$\int_{0,1}^{z,s} \frac{z^2 dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}$$

as a function of u, where

$$u = \int_{0,1}^{z,s} \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}},$$

we may with Hermite * multiply this integral by k^2 and put

$$I_2(u) = \int_0^u k^2 sn^2 u\, du.$$

We note that the function $sn^2 u$ has the periods $2K$ and $2iK'$; and from the developments above it is seen that $I_2(u)$ is a one-valued function of z. But z considered as a function of I_2 is *not* one-valued, and consequently the *problem of inversion* for these integrals, which is effected with difficulty, does not lead to unique results (see Casorati, *Acta Math.*, Bd. 8).

ART. 246. We saw (Art. 217) that $sn u$ became infinite on the points

$$2mK + (2n + 1)iK' = \alpha, \text{ say.}$$

Writing $u - \alpha = h$ or $u = \alpha + h$, we must develop $sn^2 u = sn^2 [2mK + (2n + 1)iK' + h]$ in powers of h. Since $sn^2 u$ has the periods $2K$ and $2iK'$, we have

$$sn^2 [2mK + (2n + 1)iK' + h] = sn^2 (iK' + h) = \frac{1}{k^2 sn^2 h},$$

* Hermite, Serret's *Calcul*, t. II, p. 828; Œuvres, II, p. 195; *Crelle's Journ.*, Bd. 84. This integral Hermite denotes by $Z(u)$. We shall, however, reserve this symbol for the integral employed by Jacobi (Art. 250).

so that

$$sn^2u = \frac{1}{k^2sn^2h}, \text{ or } k^2sn^2u = \frac{1}{sn^2h} = \frac{1}{h^2 + ch^4 + \cdots} = \frac{1}{h^2} + e_0 + e_1h^2 + \cdots.$$

It follows that

$$k^2sn^2u = \frac{1}{(u-\alpha)^2} + e_0 + e_1(u-\alpha)^2 + \cdots;$$

and consequently, since the integrand does not contain the term $(u-\alpha)^{-1}$, *the integral*

$$I_2(u) = \int_0^u k^2sn^2u\, du$$

is a one-valued function of u.

ART. 247. *The analytic expression for $I_2(u)$.*—The function k^2sn^2u is doubly periodic of the first sort, having the periods $2K$ and $2iK'$. The only infinity within the period-parallelogram having the sides $2K$ and $2iK'$ is iK'.

We may, however, consider k^2sn^2u as a doubly periodic function of the second sort with the factors $\nu = 1$ and $\nu' = 1$; or $\nu = e^{2K\gamma}$, $\nu' = e^{2iK'\gamma}$, where $\gamma = 0$.

We have here the exceptional case of Art. 237 where

$$F(u) = Ce^{\gamma u} + \sum_{k=1}^{k=n}\left[A_{k,1}f(u-\alpha) - \frac{A_{k,1}}{1!}f'(u-\alpha) + \cdots\right],$$

the function $F(u)$ being k^2sn^2u and $f(u) = \frac{\mathrm{H}'(u)}{\mathrm{H}(u)}e^{\gamma u}$ being $f(u) = \frac{\mathrm{H}'(u)}{\mathrm{H}(u)}$, since $\gamma = 0$.

The development of k^2sn^2u in the neighborhood of the infinity iK' is

$$k^2sn^2u = \frac{1}{(u-iK')^2} + e_0 + e_1(u-iK')^2 + \cdots.$$

Hence in the formula above, $A_{k,1}$, the coefficient of $(u-iK')^{-1}$, is zero; and $A_{k,2}$, the coefficient of $\frac{d}{du}(u-iK')^{-1}$, is -1.

We consequently have

$$k^2sn^2u = C - f'(u-iK').$$

It follows that

$$\int_0^u k^2sn^2u\, du = [Cu - f(u-iK')]_0^u,$$

or

$$I_2(u) = Cu - \frac{\mathrm{H}'(u-iK')}{\mathrm{H}(u-iK')} + \frac{\mathrm{H}'(-iK')}{\mathrm{H}(-iK')}.$$

It is thus seen again that $I_2(u)$ is a one-valued function of u.

Since

$$\mathrm{H}(u+iK')=ie^{\frac{\pi}{4}\frac{K'}{K}-\frac{\pi i u}{2K}}\Theta(u),$$

we have

$$\mathrm{H}(u-iK')=-ie^{\frac{\pi}{4}\frac{K'}{K}+\frac{\pi i u}{2K}}\Theta(u)$$

$$=-ie^{-\frac{\pi i}{4K}(-2u+iK')}\Theta(u).$$

It follows that

$$\mathrm{H}'(u-iK')=\frac{\pi i}{2K}\mathrm{H}(u-iK')-ie^{-\frac{\pi i}{4K}(-2u+iK')}\Theta'(u).$$

We therefore have

$$\frac{\mathrm{H}'(u-iK')}{\mathrm{H}(u-iK')}=\frac{\pi i}{2K}+\frac{\Theta'(u)}{\Theta(u)}$$

and

$$\frac{\mathrm{H}'(-iK')}{\mathrm{H}(-iK')}=\frac{\pi i}{2K}+\frac{\Theta'(0)}{\Theta(0)}=\frac{\pi i}{2K},$$

since $\Theta'(0)=0$.

It has thus been shown that*

$$I_2(u)=Cu-\frac{\Theta'(u)}{\Theta(u)}.$$

To determine C, we have from above

$$k^2sn^2u=C-\frac{d}{du}\frac{\Theta'(u)}{\Theta(u)}.$$

Equating powers of u on either side of this equation, we have

$$C=\frac{\Theta''(0)}{\Theta(0)}.$$

It follows that

$$\int_0^u I_2(u)du=\tfrac{1}{2}Cu^2-\log\Theta(u)+C',$$

where C' is the constant of integration.

From this it is seen that

$$\log\Theta(u)=C'+\tfrac{1}{2}Cu^2-\int_0^u I_2(u)\,du,$$

or

$$\Theta(u)=e^{C'+\frac{1}{2}Cu^2-\int_0^u I_2(u)\,du}.$$

Finally we may write †

$$\Theta(u)=C''e^{\frac{1}{2}Cu^2-\int_0^u I_2(u)\,du},\quad \text{where}\quad C''=\Theta(0).$$

* Hermite, Serret's *Calcul*, t. 2, p. 829.

† Jacobi (*Crelle*, Bd. 26, pp. 86–88; Werke, II, pp. 161–170) defines the Θ-function by this formula and therefrom derives directly the series through which this transcendent may be expressed and its other characteristic properties.

ART. 248. We may next consider the integral of the second kind

$$I_2(\mathbf{z}, \mathbf{s}) = \int_{0,1}^{\mathbf{z},\mathbf{s}} \frac{k^2\mathbf{z}^2 d\mathbf{z}}{\sqrt{(1-\mathbf{z}^2)(1-k^2\mathbf{z}^2)}},$$

regarded as a function of $\mathbf{z}$, $\mathbf{s}$ on its associated Riemann surface.

In the simply connected Riemann surface T', we saw that $\overline{u}(\mathbf{z}, \mathbf{s})$ was a one-valued function of $\mathbf{z}, \mathbf{s}$. If $\mathbf{z}, \mathbf{s}$ are given, then $\overline{u}(\mathbf{z}, \mathbf{s})$ is uniquely determined, and if $\overline{u}$ is known, then also $I_2(\overline{u})$ is known. Hence in T' not only the elliptic integral of the first kind but also the elliptic integral of the second kind is a one-valued function of $\mathbf{z}, \mathbf{s}$. Since $\overline{I}_2(\mathbf{z}, \mathbf{s})$, that is, the elliptic integral of the second kind in T', is a one-valued function of $\mathbf{z}, \mathbf{s}$, it is independent of the path of integration. This, however, is *not* true of $I_2(\mathbf{z}, \mathbf{s})$, that is, of the integral of the second kind in the Riemann surface T which does not contain the canals $\boldsymbol{a}$ and $\boldsymbol{b}$.

For the elliptic integral of the first kind $\overline{u}(\mathbf{z}, \mathbf{s})$ we had

$$(1) \qquad \begin{cases} \overline{u}(\lambda) - \overline{u}(\rho) = A(k) = 2iK' \text{ on the canal } \boldsymbol{a}, \\ \overline{u}(\rho) - \overline{u}(\lambda) = B(k) = 4K \text{ on the canal } \boldsymbol{b}. \end{cases}$$

In a corresponding manner we shall represent the constant differences of the integral of the second kind at opposite points of the banks as follows: *

$$(2) \qquad \begin{cases} \overline{I}_2(\lambda) - \overline{I}_2(\rho) = 2iJ' \text{ on the canal } \boldsymbol{a}, \\ \overline{I}_2(\rho) - \overline{I}_2(\lambda) = 4J \text{ on the canal } \boldsymbol{b}. \end{cases}$$

We had (Art. 193)

$$K = \int_0^1 \frac{d\mathbf{z}}{\sqrt{(1-\mathbf{z}^2)(1-k^2\mathbf{z}^2)}},$$

$$K' = \int_0^1 \frac{d\mathbf{z}}{\sqrt{(1-\mathbf{z}^2)(1-k'^2\mathbf{z}^2)}} = \int_1^{\frac{1}{k}} \frac{dt}{\sqrt{(t^2-1)(1-k^2t^2)}}.$$

In a corresponding manner we may write with Weierstrass (Werke, I, pp. 117, 118)

$$J = \int_0^1 \frac{k^2\mathbf{z}^2 d\mathbf{z}}{\sqrt{(1-\mathbf{z}^2)(1-k^2\mathbf{z}^2)}}, \quad J' = \int_1^{\frac{1}{k}} \frac{k^2t^2 dt}{\sqrt{(t^2-1)(1-k^2t^2)}}.$$

We note that J' is not deduced from J by changing k to k'.

From these definitions of J and J', it is seen in the remark at the end of Art. 249 that the formulas (2) above follow.

* Hermite, *loc. cit.*, p. 828; Fuchs, *Crelle*, Bd. 83, pp. 13–38.

ART. 249. We had above

$$I_2(u) = Cu - \frac{\Theta'(u)}{\Theta(u)}.$$

If in this formula we write $u = K$, we have

$$I_2(K) = CK - \frac{\Theta'(K)}{\Theta(K)}.$$

From the formulas

$$\Theta(u + K) = \Theta_1(u), \quad \Theta'(u + K) = \Theta_1'(u),$$

it is seen that for $u = 0$

$$\Theta'(K) = \Theta_1'(0) = 0,$$

and consequently $\qquad I_2(K) = CK.$

To compute $I_2(K)$ we put $u = K$ in $\mathbf{z} = sn\, u$, and if $\mathbf{z}_0$ is the value of $\mathbf{z}$ that corresponds to $u = K$, we have

$$\mathbf{z}_0 = sn\, K = 1 \text{ (Art. 218)}.$$

It follows that

$$I_2(K) = \int_{0,1}^{1} \frac{k^2\mathbf{z}^2 d\mathbf{z}}{\sqrt{(1 - \mathbf{z}^2)(1 - k^2\mathbf{z}^2)}} = J.$$

We therefore have

$$J = CK, \quad \text{or} \quad C = \frac{J}{K};$$

and finally

$$I_2(u) = \frac{J}{K}u - \frac{\Theta'(u)}{\Theta(u)}.$$

We may next compute the constant C in a different manner. If in the equation

$$I_2(u) = Cu - \frac{\Theta'(u)}{\Theta(u)},$$

we write $K + iK'$ for u, it becomes

$$I_2(K + iK') = C(K + iK') - \frac{\Theta'(K + iK')}{\Theta(K + iK')},$$

or

$$I_2(K + iK') = C(K + iK') + \frac{\pi i}{2K}.$$

To compute $I_2(K + iK')$ we put $u = K + iK'$ in $sn\, u$.

If $\mathbf{z}_1$ is the corresponding value of $\mathbf{z}$, we have $\mathbf{z}_1 = \frac{1}{k}$. Further, since

$$iJ' = \int_1^{\frac{1}{k}} \frac{k^2\mathbf{z}^2 d\mathbf{z}}{\sqrt{(1 - \mathbf{z}^2)(1 - k^2\mathbf{z}^2)}} = \int_{0,1}^{\frac{1}{k}} \frac{k^2\mathbf{z}^2 d\mathbf{z}}{\sqrt{(1 - \mathbf{z}^2)(1 - k^2\mathbf{z}^2)}}$$
$$- \int_{0,1}^{1} \frac{k^2\mathbf{z}^2 d\mathbf{z}}{\sqrt{(1 - \mathbf{z}^2)(1 - k^2\mathbf{z}^2)}},$$

we have

$$iJ' = I_2(K + iK') - J,$$

or

$$I_2(K + iK') = J + iJ';$$

and consequently

$$J + iJ' = C(K + iK') + \frac{\pi i}{2K}.$$

Eliminating C from this formula and the formula $CK = J$, it is seen that

$$J'K - K'J = \frac{\pi}{2}.$$

We note that

$$K - J = \int_0^1 \frac{\sqrt{1 - k^2 z^2}}{\sqrt{1 - z^2}} dz = E \text{ (Legendre)};$$

and making the transformation

$$t = \frac{1}{k}\sqrt{1 - k'^2 w^2},$$

it is seen that

$$J' = \int_0^1 \frac{\sqrt{1 - k'^2 w^2}}{\sqrt{1 - w^2}} dw = E'.$$

It follows that

$$KE' + K'E - KK' = \frac{\pi}{2},$$

which is the celebrated formula of Legendre (*Fonct. Ellipt.*, I, p. 60).

Remark.—The characteristic properties of $I_2(u)$ are expressed through the formulas

$$I_2(u + 2K) = I_2(u) + 2J,$$
$$I_2(u + 2iK') = I_2(u) + 2iJ'.$$

These formulas follow at once, when we note that

$$\Theta(u + 2K) = \Theta(u),$$
$$\Theta(u + 2iK') = -e^{-\frac{i\pi}{K}(u + iK')}\Theta(u).$$

Change u to $u + 2K$ and $u + 2iK'$ respectively in the equation

$$I_2(u) = \frac{J}{K}u - \frac{\Theta'(u)}{\Theta(u)},$$

and use the relation

$$KJ' - JK' = \frac{\pi}{2}.$$

ART. 250. We note that

$$\frac{J}{K}u - \int_0^u k^2 sn^2 u\, du = \frac{\Theta'(u)}{\Theta(u)},$$

or

$$\left(1 - \frac{E}{K}\right)u - \int_0^u k^2 sn^2 u\, du = \frac{\Theta'(u)}{\Theta(u)}.$$

With Jacobi (Werke, I, p. 189) we define the *zeta*-function by the relation

$$Z(u) = \left(1 - \frac{E}{K}\right)u - \int_0^u k^2 sn^2 u\, du,$$

which is Jacobi's elliptic integral of the second kind. It follows * also that

$$\Theta(u) = \Theta(0)e^{\int_0^u Z(u)\,du}, \quad \text{where} \quad \Theta(0) = \sqrt{\frac{2k'K}{\pi}} \quad \text{(Art. 341)}.$$

The Θ-function may thus be considered as originating from the function $Z(u)$ [see Cayley, *Elliptic Functions*, p. 143].

From the formula

$$Z(u) = \int_0^u \left(dn^2 u - \frac{E}{K}\right) du$$

we have $dn^2 u = \dfrac{E}{K} + Z'(u)$ and consequently $Z'(0) = 1 - \dfrac{E}{K}$.

It follows at once that

$$k^2 sn^2 u = Z'(0) - Z'(u),$$

and

$$k^2 cn^2 u = k^2 - Z'(0) + Z'(u); \quad Z'(K) = Z'(0) - k^2.$$

It is further seen, since

$$Z(u) = \frac{\Theta'(u)}{\Theta(u)},$$

that

$$Z(u+K) = \frac{\Theta'(u+K)}{\Theta(u+K)} = \frac{\Theta_1'(u)}{\Theta_1(u)}.$$

As $\Theta_1(u)$ is an even function, its derivative is odd, so that

$$Z(K) = 0.$$

ART. 251. With Jacobi (*Fund. Nova*, § 56; Werke, I, p. 214) we shall derive other properties of the Z-function and at the same time we may note the connection with the Θ-function. We emphasize the following results because the properties of the Θ-function are again derived independently and at the same time we have an *à priori* insight into the Weierstrassian functions. In Art. 220 we made the imaginary substitution

$$\sin\phi = i\tan\psi, \quad \frac{d\phi}{\Delta\phi} = \frac{i\,d\psi}{\Delta(\psi, k')}, \quad F(\phi) = iF(\psi, k').$$

It follows at once that

$$\Delta\phi\, d\phi = \frac{i(1 + k^2\tan^2\psi)d\psi}{\Delta(\psi, k')} = \frac{i\Delta(\psi, k')}{\cos^2\psi}d\psi.$$

* Jacobi, Werke, I, pp. 198, 224, 226, 231.

This expression, when integrated, becomes

$$\int_0^\phi \Delta\phi\, d\phi = i\left\{\tan\psi\Delta(\psi, k') + \int_0^\psi \frac{k'^2 \sin^2\psi}{\Delta(\psi, k')}\, d\psi\right\},$$

or

(1) $$E(\phi) = i\{\tan\psi\Delta(\psi, k') + F(\psi, k') - E(\psi, k')\}.$$

It follows that

(2) $$\frac{FE(\phi) - EF(\phi)}{i} = F\tan\psi\Delta(\psi, k') - \{FE(\psi, k') + (E-F)F(\psi, k')\}.$$

From the formula (Art. 249)

$$FE(k') + F(k')E - FF(k') = \frac{\pi}{2}$$

we have at once

$$FE(\psi, k') + (E - F)F(\psi, k') = \frac{F}{F(k')}[F(k')E(\psi, k') - E(k')F(\psi, k')] + \pi\frac{F(\psi, k')}{2\,F(k')}.$$

Equation (2) becomes through this substitution

(3) $$\frac{FE(\phi) - EF(\phi)}{iF} = \tan\psi\Delta(\psi, k') - \frac{F(k')E(\psi, k') - E(k')F(\psi, k)}{F(k')} - \frac{\pi F(\psi, k')}{2\,FF(k')}.$$

Using the Jacobi notation

$$\phi = \text{am}\, iu, \quad \psi = \text{am}\,(u, k'), \quad F(\phi) = iu, \quad F(\psi, k') = u,$$

we have

$$\frac{FE(\phi) - EF(\phi)}{F} = \mathrm{Z}(iu),$$

$$\frac{F(k')E(\psi, k') - E(k')F(\psi, k')}{F(k')} = \mathrm{Z}(u, k');$$

and consequently from (3) we have

(4) $$i\mathrm{Z}(iu, k) = -\, tn(u, k')\, dn(u, k') + \frac{\pi u}{2\,K'K} + \mathrm{Z}(u, k').$$

Multiplying (4) by du and integrating, this equation becomes

$$\int_0^u i\mathrm{Z}(iu, k)du = \log cn(u, k') + \frac{\pi u^2}{4\,KK'} + \int_0^u \mathrm{Z}(u, k')du.$$

Further, since

$$\int_0^u \mathrm{Z}(u)du = \log\frac{\Theta(u)}{\Theta(0)},$$

it follows that

(5) $$\frac{\Theta(iu, k)}{\Theta(0, k)} = e^{\frac{\pi u^2}{4\,KK'}}\, cn(u, k')\,\frac{\Theta(u, k')}{\Theta(0, k')} \qquad \text{(cf. Art. 204).}$$

Formulas (4) and (5) reduce the functions $Z(iu)$ and $\Theta(iu)$ to real arguments.

If in (5) we change u into $u + 2K'$, that formula becomes

$$\frac{\Theta(iu+2iK')}{\Theta(0)} = -e^{\frac{\pi(u+2K')^2}{4KK'}}\,cn(u,k')\frac{\Theta(u,k')}{\Theta(0,k')} = -e^{\frac{\pi(K'+u)}{K}}\frac{\Theta(iu)}{\Theta(0)}.$$

In this formula change iu to u and we have

$$(6)\qquad \Theta(u+2iK') = -e^{\frac{\pi(K'-iu)}{K}}\Theta(u) \qquad \text{(cf. Art. 202).}$$

Again write $u + K'$ for u in (5) and note that

$$cn(u+K',k') = -k\frac{sn(u,k')}{dn(u,k')},$$

$$\Theta(u+K',k') = \frac{dn(u,k')}{\sqrt{k}}\Theta(u,k').$$

It follows that

$$\frac{\Theta(iu+iK')}{\Theta(0)} = -e^{\frac{\pi(u+K')^2}{4KK'}}\sqrt{k}\,sn(u,k')\frac{\Theta(u,k')}{\Theta(0,k')}$$

$$= -e^{\frac{\pi(2u+K')}{4K}}\sqrt{k}\,tn(u,k')\frac{\Theta(iu)}{\Theta(0)}.$$

Write iu for u in this formula and it is seen that

$$(7)\qquad \Theta(u+iK') = ie^{\frac{\pi(K'-2iu)}{4K}}\sqrt{k}\,sn\,u\,\Theta(u),$$

which is a verification of formulas (V), Art. 202, and (VIII), Art. 217.

By taking the logarithmic derivatives of (6) and (7), we have

$$(8)\qquad Z(u+2iK') = -\frac{i\pi}{K} + Z(u),$$

$$(9)\qquad Z(u+iK') = -\frac{i\pi}{2K} + cotn\,u\,dn\,u + Z(u).$$

Write $u = 0$ in formulas (6), (7), (8), (9) and we have

$$\Theta(2iK') = -e^{\frac{\pi K'}{K}}\Theta(0), \quad \Theta(iK') = 0 \quad \text{(cf. Art. 203),}$$

$$Z(2iK') = -\frac{i\pi}{K}, \quad Z(iK') = \infty.$$

ART. 252. In Art. 227 we saw that

$$\frac{1}{2}\log\Theta\left(\frac{2Ku}{\pi}\right) = \text{const.} - \frac{q\cos 2u}{1-q^2} - \frac{q^2\cos 4u}{2(1-q^4)} - \frac{q^3\cos 6u}{3(1-q^6)} - \frac{q^4\cos 8u}{4(1-q^8)} - \cdots.$$

From the relation

$$Z(u) = \frac{\Theta'(u)}{\Theta(u)}$$

it follows that

$$(1) \qquad Z(u) = \frac{2\pi}{K}\sum_{m=1}^{m=\infty}\frac{q^m \sin\dfrac{m\pi u}{K}}{1-q^{2m}}.$$

[Jacobi, Werke, I, p. 187.]

We also have

$$(2) \quad Z(u) = \frac{\Theta'(u)}{\Theta(u)} = \frac{2\pi}{K}\,\frac{q\sin\dfrac{\pi u}{K} - 2q^4\sin\dfrac{2\pi u}{K} + 3q^9\sin\dfrac{3\pi u}{K} - \cdots}{1 - 2q\cos\dfrac{\pi u}{K} + 2q^4\cos\dfrac{2\pi u}{K} - 2q^9\cos\dfrac{3\pi u}{K} + \cdots}.$$

To be noted is the equality of the right-hand sides of (1) and (2). We further note that

$$\frac{k^2K^2}{2\pi^2}\,sn^2\frac{2Ku}{\pi} = \frac{JK}{2\pi^2} - \left[\frac{q\cos 2u}{1-q^2} + \frac{2q^2\cos 4u}{1-q^4} + \frac{3q^3\cos 6u}{1-q^6} + \cdots\right].$$

ART. 253. Thomae * introduced the notation

$$Z_{00}(u) = \frac{d}{du}\log\Theta_1(u),$$

$$Z_{01}(u) = \frac{d}{du}\log\Theta(u),$$

$$Z_{10}(u) = \frac{d}{du}\log H_1(u),$$

$$Z_{11}(u) = \frac{d}{du}\log H(u).$$

Differentiate logarithmically

$$\sqrt{k'}\,\frac{\Theta_1(u)}{\Theta(u)} = dn\,u,$$

and we have

$$Z_{00}(u) - Z_{01}(u) = -\frac{k^2 sn\,u\,cn\,u}{dn\,u}.$$

[Jacobi, Werke, I, p. 188, formula (6).]

Similarly we have

$$Z_{11}(u) - Z_{01}(u) = \frac{cn\,u\,dn\,u}{sn\,u},$$

$$Z_{10}(u) - Z_{01}(u) = -\frac{sn\,u\,dn\,u}{cn\,u}.$$

* Thomae, *Functionen einer complexen Veränderlichen*, pp. 123 *et seq.*; *Sammlung von Formeln*, etc., p. 15.

ART. 254. The derivatives of the Z-functions are one-valued doubly periodic functions; for differentiating

$$I_2(u) = \int_0^u k^2 sn^2 u\, du = \frac{J}{K}u - \frac{\Theta'(u)}{\Theta(u)},$$

it is seen that

$$\frac{d^2}{du^2}\log\Theta(u) = \frac{J}{K} - k^2 sn^2 u = \frac{J}{K} - k^2\mathbf{z}^2.$$

Further, since

$$\Theta_1(u) = \Theta(u + K),$$

it follows that

$$\frac{d^2}{du^2}\log\Theta_1(u) = \frac{J}{K} - k^2 sn^2(u + K) = \frac{J}{K} - k^2\frac{cn^2 u}{dn^2 u}$$

$$= \frac{J}{K} - k^2\frac{1 - \mathbf{z}^2}{1 - k^2\mathbf{z}^2}.$$

Similar results may be derived for $\mathrm{H}(u)$ and $\mathrm{H}_1(u)$.

The functions $\Theta(u), \Theta_1(u)$, etc., when for u is written the integral of the first kind $u(\mathbf{z}, \mathbf{s})$, are functions of $\mathbf{z}$, $\mathbf{s}$, but *not* one-valued, since $u(\mathbf{z}, \mathbf{s})$ is *not* one-valued in $\mathbf{z}, \mathbf{s}$. But from the formulas just written it is seen that the second logarithmic derivatives of these functions are rational, and consequently *one-valued* in $\mathbf{z}$ alone (*i.e.*, the $\mathbf{s}$ does not appear).

This is fundamental in the derivation of the Weierstrassian theory, which we shall consider in the next Chapter.

EXAMPLES

1. Show that
$$E\left(k, \frac{\pi}{2}\right) = E = \int_0^K dn^2(u, k)\, du,$$
$$E' = \int_0^{K'} dn^2(u, k')\, du.$$

2. Through the definitions of the zeta-functions of Art. 253 derive independently the formulas given in Chapter X for $\Theta_1(u)$, $\mathrm{H}_1(u)$ and $\mathrm{H}(u)$.

3. Prove that
$$iZ_{00}(iu, k) = Z_{00}(u, k') + \frac{\pi u}{2KK'}$$
and
$$iZ_{01}(iu, k) = Z_{10}(u, k') + \frac{\pi u}{2KK'}.$$

4. Prove that
$$Z_{00} = \frac{2\pi}{K}\sum_{m=1}^{m=\infty}\frac{(-q)^m \sin\frac{m\pi u}{K}}{1 - q^{2m}}$$
$$= -\frac{2\pi}{K}\,\frac{q\sin\frac{\pi u}{K} + 2q^4\sin\frac{2\pi u}{K} + 3q^9\sin\frac{3\pi u}{K} + \cdots}{1 + 2q\cos\frac{\pi u}{K} + 2q^4\cos\frac{2\pi u}{K} + 2q^9\cos\frac{3\pi u}{K} + \cdots}.$$

Derive similar expressions for $Z_{10}(u)$ and $Z_{11}(u)$.

(Thomae, *Sammlung*, etc., p. 16.)

5. Verify the results indicated in the table:

	$Z_{00}+\dfrac{i\pi}{2K}$	0	∞
iK'	$Z_{01}+\dfrac{i\pi}{2K}$	∞	0
	$Z_{10}+\dfrac{i\pi}{2K}$	0	0
$=$	$Z_{11}+\dfrac{i\pi}{2K}$	0	0
	Z_{00}	0	0
	Z_{01}	0	0
0	Z_{10}	0	∞
$=$	Z_{11}	∞	0
u	$=$	0	$=K$

6. Show that

$$\int_0^{\frac{\pi}{2}} \frac{\log \sin\phi}{\sqrt{1-k^2\sin^2\phi}}\,d\phi = \tfrac{1}{2}K\log\frac{1}{k} - \frac{\pi}{4}K'.$$

Roberts (*Liouville's Journ.* (1), Vol. 19),
Wangerin (*Schlömilch's Zeit.*, Bd. 34, p. 119).

7. Prove that

$$Z(\tfrac{1}{2}K) = \tfrac{1}{2}(1-k'),$$

$$Z(\tfrac{1}{2}iK') = \tfrac{1}{2}i(1+k) - \frac{i\pi}{4K},$$

$$Z(\tfrac{1}{2}K + \tfrac{1}{2}iK') = \tfrac{1}{2}(k+ik') - \frac{i\pi}{4K}.$$

8. Complete the table of Ex. 5 by letting u take values $\tfrac{1}{2}K$, $K+\tfrac{1}{2}iK'$, $\tfrac{1}{2}iK'$, $\tfrac{1}{2}K+\tfrac{1}{2}iK'$, $\tfrac{3}{2}K+\tfrac{3}{2}iK'$, etc.

CHAPTER XIV

INTRODUCTION TO WEIERSTRASS'S THEORY

ARTICLE 255. In the previous study we have followed the historical order of the development of the elliptic functions and have made fundamental Legendre's normal form. We may just as well use the one adopted by Weierstrass,

$$u(t, \sqrt{S(t)}) = u = -\int_{\infty}^{t, \sqrt{S(t)}} \frac{dt}{\sqrt{4(t-e_1)(t-e_2)(t-e_3)}},$$

where $\sqrt{4\,(t-e_1)(t-e_2)(t-e_3)} = \sqrt{S(t)}$ (see Chapter VIII).

We have taken infinity as the lower limit, because this value of t, as we shall later see, corresponds to the value $u = 0$. We saw, Art. 185, that this integral could be transformed by a simple substitution into the normal form of Legendre. Consequently in the derivation of the new formulas we need *not* always return to the consideration of the Riemann surface, but in this respect we may rely upon our former developments.

ART. 256. If in the above integral we write (see Art. 195)

$$t = \wp u,$$

it follows immediately that

$$-\sqrt{S(t)} = \frac{dt}{du} = \wp' u.$$

In Art. 185 we saw that the transformation of Weierstrass's normal form to that of Legendre is effected through the substitution

$$t = e_3 + \frac{1}{\epsilon z}, \quad \text{where} \quad \epsilon = \frac{1}{e_1 - e_3}$$

We therefore have

$$\wp u = e_3 + \frac{1}{\epsilon\, sn^2\left(\frac{u}{\sqrt{\epsilon}}\right)}.$$

Since $sn\left(\frac{u}{\sqrt{\epsilon}}\right)$ is a one-valued function, the function $\wp u$ must also be one-valued; and since $sn^2(u\sqrt{e_1 - e_3})$ has the periods $2\sqrt{\epsilon}K$ and $2\sqrt{\epsilon}iK'$, these are also the periods of $\wp u$.

We put (Art. 196)

$$2\sqrt{\epsilon}K = 2\,\omega, \quad 2\sqrt{\epsilon}iK' = 2\,\omega',$$

so that the function $\wp u$ has the periods $2\,\omega$ and $2\,\omega'$. We further note that $sn^2 u$ being an even function, the same is true also of $\wp u$.

ART. 257. As we have introduced the new function $\wp u$ in the place of $sn\, u$, following Weierstrass we shall introduce new functions for the Θ-functions, which new functions are, however, closely connected with the Θ-functions.

If in the formula of Art. 254

$$k^2 sn^2 u = \frac{J}{K} - \frac{d^2}{du^2}\log\Theta(u)$$

we put $u + iK'$ in the place of u, we have

$$\frac{1}{sn^2 u} = \frac{J}{K} - \frac{d^2}{du^2}\log \mathrm{H}(u).$$

Since $$\wp(v\sqrt{\varepsilon}) = e_3 + \frac{1}{\varepsilon\, sn^2 v},$$

it follows that $$\wp(v\sqrt{\varepsilon}) = e_3 + \frac{1}{\varepsilon}\frac{J}{K} - \frac{1}{\varepsilon}\frac{d^2 \log \mathrm{H}(v)}{dv^2}.$$

Noting the identity

$$e_3 + \frac{1}{\varepsilon}\frac{J}{K} \equiv \frac{d^2}{dv^2}\left\{\frac{1}{2}\left(e_3 + \frac{1}{\varepsilon}\frac{J}{K}\right)v^2\right\},$$

or

$$-\left(e_3 + \frac{1}{\varepsilon}\frac{J}{K}\right) \equiv \frac{d^2}{dv^2}\left\{\log e^{-\frac{1}{2}\left(e_3+\frac{1}{\varepsilon}\frac{J}{K}\right)v^2}\right\},$$

it is clear that

$$-\wp(v\sqrt{\varepsilon}) = \frac{d^2}{dv^2}\log\left\{e^{-\frac{1}{2}\left(e_3+\frac{1}{\varepsilon}\frac{J}{K}\right)v^2}\mathrm{H}(v)^{\frac{1}{\varepsilon}}\right\}.$$

Writing $v\sqrt{\varepsilon} = u$, this formula becomes

$$-\wp u = \frac{d^2 \log}{d\left(\frac{u}{\sqrt{\varepsilon}}\right)^2}\left\{e^{-\frac{1}{2}\left(e_3+\frac{1}{\varepsilon}\frac{J}{K}\right)\frac{u^2}{\varepsilon}}\mathrm{H}\left(\frac{u}{\sqrt{\varepsilon}}\right)^{\frac{1}{\varepsilon}}\right\},$$

which is a one-valued function of $\mathbf{z}$ (see Art. 254).

We thus have

$$-\wp u = \frac{d^2}{du^2}\log\left\{e^{-\frac{1}{2}\left(e_3+\frac{1}{\varepsilon}\frac{J}{K}\right)u^2}\mathrm{H}\left(\frac{u}{\sqrt{\varepsilon}}\right)\right\},$$

or, if we put

$$\sigma u = \beta e^{-\frac{1}{2}\left(e_3+\frac{1}{\varepsilon}\frac{J}{K}\right)u^2}\mathrm{H}\left(\frac{u}{\sqrt{\varepsilon}}\right),$$

where β is a constant, then is

$$-\wp u = \frac{d^2}{du^2}\log \sigma u.$$

The arbitrary constant β we may so choose that in the development of σu, the coefficient of the first power of u is unity.

By Maclaurin's Theorem this development is

$$\sigma u = \sigma(0) + u\sigma'(0) + \cdot \cdot \cdot \cdot .$$

Since $H(0) = 0$, we also have $\sigma(0) = 0$; and noting that

$$\sigma'(u) = -\beta\left[\left(e_3 + \frac{1}{\epsilon}\frac{J}{K}\right)u\right]e^{-\frac{1}{2}\left(e_3+\frac{1}{\epsilon}\frac{J}{K}\right)u^2}H\left(\frac{u}{\sqrt{\epsilon}}\right)$$

$$+ \beta e^{-\frac{1}{2}\left(e_3+\frac{1}{\epsilon}\frac{J}{K}\right)u^2}H'\left(\frac{u}{\sqrt{\epsilon}}\right)\frac{1}{\sqrt{\epsilon}},$$

we have

$$\sigma'(0) = 1 = \frac{\beta}{\sqrt{\epsilon}}H'(0).$$

It is thus shown that

$$\beta = \frac{\sqrt{\epsilon}}{H'(0)},$$

and consequently

$$\sigma u = \frac{\sqrt{\epsilon}}{H'(0)}e^{-\frac{1}{2}\left(e_3+\frac{1}{\epsilon}\frac{J}{K}\right)u^2}H\left(\frac{u}{\sqrt{\epsilon}}\right).$$

If we differentiate the expression

$$sn\, u = \frac{1}{\sqrt{k}}\frac{H(u)}{\Theta(u)},$$

we have

$$cn\, u\, dn\, u = \frac{1}{\sqrt{k}}\frac{\Theta(u)H'(u) - \Theta'(u)H(u)}{\Theta^2(u)}.$$

Writing $u = 0$, it is seen that

$$1 = \frac{1}{\sqrt{k}}\frac{\Theta(0)H'(0)}{\Theta^2(0)}, \quad \text{or} \quad \frac{1}{H'(0)} = \frac{1}{\sqrt{k}\,\Theta(0)}.$$

It follows from above that

$$\sigma u = \sqrt{\frac{\epsilon}{k}}\,\frac{1}{\Theta(0)}e^{-\frac{1}{2}\left(e_3+\frac{1}{\epsilon}\frac{J}{K}\right)u^2}H\left(\frac{u}{\sqrt{\epsilon}}\right).$$

ART. 258. The expression

$$\frac{d^2 \log \sigma v}{dv^2} = -\wp v,$$

becomes when integrated

$$\frac{d \log \sigma v}{dv} = -\int_\omega^z \wp(v)dv + \eta,$$

where the lower limit ω and the constant η are connected as follows:

If we define the small zeta-function by

$$\zeta v = \frac{\sigma' v}{\sigma v} \quad \text{(see Art. 277)},$$

we may write

$$\zeta v = \frac{\sigma' v}{\sigma v} = -\int_\omega^v \wp(v)dv + \eta.$$

Putting $v = \omega$ in this formula, we have at once

$$\zeta\omega = \frac{\sigma'}{\sigma}(\omega) = \eta = -\int_{\omega}^{\omega}\wp(v)dv + \eta.$$

We may similarly introduce the new quantities

$$\zeta\omega'' = \frac{\sigma'}{\sigma}(\omega'') = \eta'' = -\int_{\omega}^{\omega''}\wp(v)dv + \eta,$$

$$\zeta\omega' = \frac{\sigma'}{\sigma}(\omega') = \eta' = -\int_{\omega}^{\omega'}\wp(v)dv + \eta.$$

If we put (see Arts. 195 and 256)

$$\wp v = t,\quad dv = \frac{-dt}{\sqrt{S(t)}},\quad \wp\omega = e_1,\quad \wp\omega'' = e_2,\quad \wp\omega' = e_3,$$

it follows that

$$\eta'' = \eta + \int_{e_1}^{e_2}\frac{t\,dt}{\sqrt{S(t)}}$$

and

$$\eta' = \eta + \int_{e_1}^{e_3}\frac{t\,dt}{\sqrt{S(t)}},$$

or

$$\eta' = \eta + \int_{e_1}^{e_2}\frac{t\,dt}{\sqrt{S(t)}} + \int_{e_2}^{e_3}\frac{t\,dt}{\sqrt{S(t)}}.$$

In a similar manner as in Art. 194, it is seen that

$$2\int_{e_1}^{e_2}\frac{t\,dt}{\sqrt{S(t)}} = \overline{B}$$ along the upper bank of the canal e_1e_2 in the upper leaf; and

$$2\int_{e_2}^{e_3}\frac{t\,dt}{\sqrt{S(t)}} = \overline{A}$$ (upper leaf),

where $\overline{B}$ denotes the difference in the values of the integral

$$\int_{t_0,\sqrt{S(t_0)}}^{t,\sqrt{S(t)}}\frac{t\,dt}{\sqrt{S(t)}}$$

on the right and the left bank of the canal $\boldsymbol{b}$, and $\overline{A}$ the corresponding difference on the left and the right bank of the canal $\boldsymbol{a}$.

If any arbitrary path of integration is taken, we have*

$$\int_{e_1}^{e_2}\frac{t\,dt}{\sqrt{S(t)}} = \frac{\overline{B}}{2} + m\overline{A} + l\overline{B},$$

$$\int_{e_2}^{e_3}\frac{t\,dt}{\sqrt{S(t)}} = \frac{\overline{A}}{2} + m'\overline{A} + l'\overline{B},$$

where m, l, m', l' are integers.

* See Bruns, *Ueber die Perioden der elliptischen Integrale erster und zweiter Gattung*, *Math. Ann.*, Bd. 27, p. 234.

It follows from above that

$$\eta'' \equiv \eta + \frac{\bar{B}}{2}$$

and

$$\eta' \equiv \eta + \frac{\bar{B} + \bar{A}}{2},$$

the congruences being taken with regard to integral multiples of $\bar{A}$ and $\bar{B}$.

ART. 259. By definition of Art. 257 we have

$$\frac{\sigma'(v)}{\sigma(v)} = -\left\{e_3 + \frac{1}{\epsilon}\frac{J}{K}\right\}v + \frac{1}{\sqrt{\epsilon}}\frac{\mathrm{H}'\left(\frac{v}{\sqrt{\epsilon}}\right)}{\mathrm{H}\left(\frac{v}{\sqrt{\epsilon}}\right)}.$$

It follows that

$$\eta = \frac{\sigma'(\omega)}{\sigma(\omega)} = -\left\{e_3 + \frac{1}{\epsilon}\frac{J}{K}\right\}\omega + \frac{1}{\sqrt{\epsilon}}\frac{\mathrm{H}'(K)}{\mathrm{H}(K)}.$$

From the formulas $\mathrm{H}(u+K) = \mathrm{H}_1(u)$

and $\mathrm{H}'(u+K) = \mathrm{H}_1'(u)$

we have at once $\mathrm{H}(K) = \mathrm{H}_1(0)$

and $\mathrm{H}'(K) = \mathrm{H}_1'(0) = 0.$

It is seen that

(1) $$\eta = \frac{\sigma'\omega}{\sigma\omega} = -\left(e_3 + \frac{1}{\epsilon}\frac{J}{K}\right)\omega.$$

Further, since $J = K - E$ and $\omega = \sqrt{\epsilon}K$,

we may write *

(1′) $$\eta = \sqrt{e_1 - e_3}\left\{E - \frac{e_1}{e_1 - e_3}K\right\}.$$

Further, since $\sqrt{\epsilon}iK' = \omega'$,

we have $$\eta' = \frac{\sigma'\omega'}{\sigma\omega'} = -\left(e_3 + \frac{1}{\epsilon}\frac{J}{K}\right)\omega' + \frac{1}{\sqrt{\epsilon}}\frac{\mathrm{H}'(iK')}{\mathrm{H}(iK')};$$

or, since (see Art. 247)

$$\frac{\mathrm{H}'(iK')}{\mathrm{H}(iK')} = -\frac{\pi i}{2K},$$

we have

(2) $$\eta' = -\left(e_3 + \frac{1}{\epsilon}\frac{J}{K}\right)\omega' - \frac{\pi i}{2\omega}:$$

or,

(2′) $$\eta' = -i\sqrt{e_1 - e_3}\left\{E' + \frac{e_3}{e_1 - e_3}K'\right\}.$$

* See Schwarz, *loc. cit.*, p. 34.

It follows at once from (2′) and (1′) that

$$E = \frac{1}{\sqrt{e_1 - e_3}}(\eta + e_1\omega),$$

$$E' = \frac{i}{\sqrt{e_1 - e_3}}(\eta' + e_3\omega').$$

From the formulas

$$\omega'' = \omega + \omega' \quad \text{or} \quad \omega'' = \sqrt{\varepsilon}(K + iK')$$

we have

$$\eta'' = \frac{\sigma'\omega''}{\sigma\omega''} = -\left(e_3 + \frac{1}{\varepsilon}\frac{J}{K}\right)\omega'' + \frac{1}{\sqrt{\varepsilon}}\frac{\mathrm{H}'(K + iK')}{\mathrm{H}(K + iK')}.$$

Further, since (Art. 247)

$$\frac{\mathrm{H}'(K + iK')}{\mathrm{H}(K + iK')} = -\frac{\pi i}{2K},$$

it follows that

(3) $$\eta'' = \frac{\sigma'\omega''}{\sigma\omega''} = -\left(e_3 + \frac{1}{\varepsilon}\frac{J}{K}\right)\omega'' - \frac{\pi i}{2\omega}.$$

From the formulas (1), (2) and (3) it is evident that

$$\eta + \eta' = \eta''.$$

It is seen from the preceding article that

$$\eta' = \eta'' - \eta = \int_{e_1}^{e_2} \frac{t\,dt}{\sqrt{S(t)}} \equiv \frac{\overline{B}}{2};$$

and since

$$\eta = \eta' - \frac{\overline{B}}{2} - \frac{\overline{A}}{2},$$

we further have

$$\eta \equiv \frac{\overline{A}}{2}$$

and

$$\eta'' \equiv \frac{\overline{A} + \overline{B}}{2},$$

the congruences being taken with respect to the moduli of periodicity of the integral of the second kind.

We also have the relation corresponding to Legendre's formula of Art. 249,

$$\eta\omega' - \eta'\omega = \frac{\pi i}{2}.$$

We may note that

$$\zeta(u + 2\omega) = \zeta u + 2\eta,$$

$$\zeta(u + 2\omega') = \zeta u + 2\eta';$$

for $\wp u$ being an *even* function, its integral ζu is *odd*, and writing $u = -\omega$ and $-\omega'$ respectively in the two formulas just written, we establish their existence.

ART. 260. We have already derived the formulas

$$\sigma u = \beta e^{-\frac{1}{2}\left(e_3 + \frac{1}{\varepsilon}\frac{J}{K}\right)u^2} \mathrm{H}\left(\frac{u}{\sqrt{\varepsilon}}\right)$$

and

$$\eta = -\left[e_3 + \frac{1}{\varepsilon}\frac{J}{K}\right]\omega.$$

If we put

$$u = 2\,\omega v,$$

then is

$$\sigma u = \beta e^{2\eta\omega v^2} \mathrm{H}(2\,Kv),$$

from which it is seen that σu is an *odd* function, the function H being odd.

It follows immediately that

$$\begin{aligned}\sigma(u+\omega) &= \beta e^{2\eta\omega(v+\frac{1}{2})^2} \mathrm{H}(2\,Kv + K)\\ &= \beta e^{2\eta\omega(v+\frac{1}{2})^2} \mathrm{H}_1(2\,Kv).\end{aligned}$$

The following new notation is suggested:

$$\begin{aligned}\sigma_1 u &= \beta_1 e^{2\eta\omega v^2} \mathrm{H}_1(2\,Kv),\\ \sigma_2 u &= \beta_2 e^{2\eta\omega v^2} \Theta_1(2\,Kv),\\ \sigma_3 u &= \beta_3 e^{2\eta\omega v^2} \Theta(2\,Kv),\end{aligned}$$

where β_1, β_2 and β_3 are constants.*

It is seen that $\sigma_1 u$, $\sigma_2 u$ and $\sigma_3 u$ are *even* functions. We shall so determine β_1 that for $2\,\omega v = u = 0$ we have $\sigma_1(0) = 1$. We thus have

$$1 = \beta_1 \mathrm{H}_1(0), \quad \text{or} \quad \beta_1 = \frac{1}{\mathrm{H}_1(0)},$$

and similarly

$$\beta_2 = \frac{1}{\Theta_1(0)} \quad \text{and} \quad \beta_3 = \frac{1}{\Theta(0)}.$$

ART. 261. It is evident from the previous Article that

$$\sigma(u+\omega)e^{-2\eta\omega v} = C_1\sigma_1 u,$$

where C_1 is a constant. For $v = 0$, it is seen that $C_1 = \sigma\omega$, and consequently

$$\sigma_1 u = e^{-\eta u}\frac{\sigma(u+\omega)}{\sigma\omega}.$$

We further have

$$\begin{aligned}\sigma(u+\omega') &= \beta e^{2\eta\omega\left(v+\frac{1}{2}\frac{\omega'}{\omega}\right)^2} \mathrm{H}(2\,Kv + iK')\\ &= \beta i e^{2\eta\omega\left(v+\frac{1}{2}\frac{\omega'}{\omega}\right)^2 + \frac{\pi}{4}\frac{\omega'}{\omega} - \pi i\frac{2\,Kv}{2\,K}} \Theta(2\,Kv),\end{aligned}$$

or

$$\sigma(u+\omega')e^{-(2\eta\omega' - \pi i)v} = C_3\sigma_3 u.$$

* These constants are expressed through Weierstrassian transcendents in Art. 345.

Writing $u = 0 = v$ and noting that $2\,\eta\omega' - \pi i = 2\,\eta'\omega$, it is seen that

$$\sigma_3 u = e^{-\eta' u}\frac{\sigma(u+\omega')}{\sigma\omega'}.$$

It also follows without difficulty that

$$\sigma_2 u = e^{-\eta'' u}\frac{\sigma(u+\omega'')}{\sigma\omega''}.$$

The functions $\sigma_1 u$, $\sigma_2 u$, $\sigma_3 u$ are like σu, one-valued functions of u, that have everywhere in the finite portion of the plane the character of integral functions.

ART. 262. From the formulas above we have

$$\left(\frac{\sigma_1 u}{\sigma u}\right)^2 = a\left(\frac{\mathrm{H}_1(2\,Kv)}{\mathrm{H}(2\,Kv)}\right)^2 = a\left(\frac{\dfrac{\mathrm{H}_1(2\,Kv)}{\Theta(2\,Kv)}}{\dfrac{\mathrm{H}(2\,Kv)}{\Theta(2\,Kv)}}\right)^2,$$

or

$$\left(\frac{\sigma_1 u}{\sigma u}\right)^2 = a'\frac{cn^2 2\,Kv}{sn^2 2\,Kv};$$

and similarly

$$\left(\frac{\sigma_2 u}{\sigma u}\right)^2 = b'\frac{dn^2 2\,Kv}{sn^2 2\,Kv},$$

$$\left(\frac{\sigma_3 u}{\sigma u}\right)^2 = c'\frac{1}{sn^2 2\,Kv},$$

where a, a', b', c' are constants.

Since

$$\wp u - e_3 = \frac{e_1 - e_3}{sn^2 2\,Kv},$$

it is evident that

$$\left(\frac{\sigma_3 u}{\sigma u}\right)^2 = c_3(\wp u - e_3),$$

where c_3 is a constant.

If we put

$$\frac{\wp u - e_3}{e_1 - e_3} = \xi = \frac{1}{sn^2 2\,Kv},$$

we have

$$\xi - 1 = \frac{cn^2 2\,Kv}{sn^2 2\,Kv},$$

so that

$$\left(\frac{\sigma_1 u}{\sigma u}\right)^2 = a'(\xi - 1) = a'\frac{\wp u - e_1}{e_1 - e_3},$$

or

$$\left(\frac{\sigma_1 u}{\sigma u}\right)^2 = c_1(\wp u - e_1), \text{ where } c_1 \text{ is a constant.}$$

In the same manner we have

$$\xi - k^2 = \frac{dn^2 2\,Kv}{sn^2 2\,Kv},$$

or

$$\left(\frac{\sigma_2 u}{\sigma u}\right)^2 = c_2(\wp u - e_2), \text{ where } c_2 \text{ is a constant.}$$

We have accordingly

$$\sqrt{\wp u - e_1} = d_1 \frac{\sigma_1 u}{\sigma u},$$

$$\sqrt{\wp u - e_2} = d_2 \frac{\sigma_2 u}{\sigma u},$$

$$\sqrt{\wp u - e_3} = d_3 \frac{\sigma_3 u}{\sigma u},$$

where d_1, d_2, and d_3 are constants.

To determine the constants we note that σu may be developed in the form

$$\sigma u = u + b_3 u^3 + b_5 u^5 + \ldots,$$

and also that

$$\sigma_k u = 1 + b_{2,k} u^2 + b_{4,k} u^4 + \ldots \quad (k = 1, 2, 3),$$

where the b's are definite constants.

We therefore have

$$\frac{\sigma_k u}{\sigma u} = \frac{1}{u} \cdot \frac{1 + b_{2,k} u^2 + \ldots}{1 + b_3 u^2 + \ldots} = \frac{1}{u} + d_{1,k} u + d_{3,k} u^3 + \ldots.$$

In the neighborhood of the point $u = 0$ we also have

$$sn\, v = v + e_3 v^3 + \ldots,$$

so that

$$sn \frac{u}{\sqrt{\varepsilon}} = \frac{u}{\sqrt{\varepsilon}} + e_3 \left(\frac{u}{\sqrt{\varepsilon}}\right)^3 + \ldots$$

and

$$\frac{1}{sn^2 \frac{u}{\sqrt{\varepsilon}}} = \frac{\varepsilon}{u^2} + g_0 + g_2 \left(\frac{u}{\sqrt{\varepsilon}}\right)^2 + \ldots.$$

Since

$$\wp u - e_3 = \frac{e_1 - e_3}{sn^2 \frac{u}{\sqrt{\varepsilon}}},$$

it follows that

$$\wp u - e_3 = \frac{1}{u^2} + h_0 + h_2 u^2 + \ldots,$$

$$\wp u - e_2 = \frac{1}{u^2} + h_0 + e_3 - e_2 + h_2 u^2 + \ldots,$$

$$\wp u - e_1 = \frac{1}{u^2} + h_0 + e_3 - e_1 + h_2 u^2 + \ldots.$$

On the other hand we had

$$\wp u - e_k = d_k^2 \left(\frac{\sigma_k u}{\sigma u}\right)^2 = d_k^2 \left\{\frac{1}{u} + d_{1,k} u + \ldots\right\}^2 \quad (k = 1, 2, 3).$$

It follows that $d_k^2 = 1$ or $d_k = \pm 1$, and consequently

$$\sqrt{\wp u - e_k} = \pm \frac{\sigma_k u}{\sigma u}.$$

Since the quotient $\frac{\sigma_k u}{\sigma u}$ is a one-valued function, we may take the *positive* sign (see Schwarz, *loc. cit.*, p. 21).

We further have

$$sn\left(\frac{u}{\sqrt{\varepsilon}}\right)=\frac{\sqrt{e_1-e_3}}{\sqrt{\wp u-e_3}}=\frac{1}{\sqrt{\varepsilon}}\frac{\sigma u}{\sigma_3 u}.$$

Similarly it is seen that

$$cn^2\left(\frac{u}{\sqrt{\varepsilon}}\right)=\frac{\wp u-e_1}{\wp u-e_3}=\left(\frac{\frac{\sigma_1 u}{\sigma u}}{\frac{\sigma_3 u}{\sigma u}}\right)^2,$$

or

$$cn\frac{u}{\sqrt{\varepsilon}}=\frac{\sigma_1 u}{\sigma_3 u},$$

and also that

$$dn\frac{u}{\sqrt{\varepsilon}}=\frac{\sigma_2 u}{\sigma_3 u}.$$

ART. 263. It follows from the formulas

$$sn^2(\sqrt{e_1-e_3}\cdot u)=\frac{e_1-e_3}{\wp u-e_3},\quad k^2=\frac{e_2-e_3}{e_1-e_3},$$

that

$$dn^2(\sqrt{e_1-e_3}\cdot u)=1-\frac{e_2-e_3}{\wp u-e_3}.$$

Further, since

$$\sigma_3 u=e^{-\eta' u}\frac{\sigma(u+\omega')}{\sigma\omega'},$$

we have

$$\frac{d^2}{du^2}\log\sigma_3 u=\frac{d^2}{du^2}\log\sigma(u+\omega')=-\wp(u+\omega')=-\wp(u-\omega').$$

Admitting the relation (see Art. 316)

$$\wp(u-\omega')=e_3+\frac{(e_1-e_3)(e_2-e_3)}{\wp u-e_3},$$

we have

$$\frac{d}{du}\left(\frac{\sigma_3' u}{\sigma_3 u}+e_1 u\right)=e_1-e_3-\frac{(e_1-e_3)(e_2-e_3)}{\wp u-e_3}=(e_1-e_3)dn^2(\sqrt{e_1-e_3}\cdot u).$$

Since

$$E(u)=\int_0^u dn^2 u\,du,$$

it is seen that *

$$E(\sqrt{e_1-e_3}\cdot u)=\frac{1}{\sqrt{e_1-e_3}}\left(\frac{\sigma_3' u}{\sigma_3 u}+e_1 u\right).$$

* See Schwarz, *loc. cit.*, p. 52.

Further, since (Art. 259)

$$E = \frac{\eta + e_1\omega}{\sqrt{e_1 - e_3}} \quad \text{and} \quad K = \sqrt{e_1 - e_3}\cdot\omega,$$

it follows from

$$\mathrm{Z}(u) = E(u) - u\frac{E}{K}$$

that

$$\mathrm{Z}(\sqrt{e_1 - e_3}\cdot u) = \frac{1}{\sqrt{e_1 - e_3}}\left(\frac{\sigma_3' u}{\sigma_3 u} + e_1 u\right) - \frac{1}{\sqrt{e_1 - e_3}}\left(\frac{\eta}{\omega} + e_1\right)u$$

$$= \frac{1}{\sqrt{e_1 - e_3}}\left(\frac{\sigma_3' u}{\sigma_3 u} - \frac{\eta u}{\omega}\right).$$

The last formula may be written *

$$\mathrm{Z}(\sqrt{e_1 - e_3}\cdot u) = \frac{1}{\sqrt{e_1 - e_3}}\left[\zeta(u + \omega') - \frac{\eta}{\omega}u - \eta'\right].$$

EXAMPLES

1. Jacobi, Werke, I, p. 527, wrote

$$\mathfrak{Z}(x) = \frac{d\log\vartheta(x)}{dx} = \frac{\vartheta'(x)}{\vartheta(x)}.$$

If
$$\phi = \operatorname{am}\frac{2Kx}{\pi},$$

show that
$$x\mathfrak{Z}'(0) - \mathfrak{Z}(x) = \frac{2K}{\pi}[F(\phi) - E(\phi)].$$

2. Prove that

$$\frac{\pi}{2}\mathfrak{Z}(x) = KE(\phi) - EF(\phi).$$

3. Let
$$P(u) = \frac{1}{sn^2 u} - \frac{1 + k^2}{3}.$$

Show that

$$P(u) = \frac{1}{u^2} + \frac{1 - k^2 + k^4}{15}u^2 + \frac{2 - 3k^2 - 3k^4 + 2k^6}{189}u^4 + \cdots.$$

4. If $F(k^2)$ is the coefficient of u^{2n-2} in the preceding example, show that

$$k^n F\left(\frac{1}{k^2}\right) = F(k^2) \text{ and that } F(1 - k^2) = (-1)^n F(k^2).$$

5. Prove that the function $P(u)$ of Example 3 satisfies the relation

$$P'(u)^2 = 4P(u)^3 - \tfrac{4}{3}(1 - k^2 + k^4)P(u) - \tfrac{4}{27}(1 + k^2)(1 - 2k^2)(2 - k^2);$$

or $$P'(u)^2 = 4P(u)^3 - g_2 P(u) - g_3.$$

(Hermite, Serret's *Calcul*, t. II, p. 856.)

* See Enneper, *Elliptische Functionen*, p. 221.

CHAPTER XV

THE WEIERSTRASSIAN FUNCTIONS $\wp u$, ζu, σu

ARTICLE 264. We saw in Chapter V that the doubly periodic functions of the second order or degree are the simplest doubly periodic functions. These functions are either infinite of the first order at two distinct points of the period-parallelogram, or they are infinite of the second order at one point of the period-parallelogram. The first case has been considered in Chapter XI. We shall now consider the latter case. Among this group of functions we shall take the simplest, viz., those which become infinite of the second order at the origin.

Such a function may be expressed in the form

$$\phi(u) = \frac{b}{u^2} + \frac{a}{u} + P(u),$$

where $b \neq 0$, and where $P(u)$ is a power series in integral ascending powers of u.

It is shown below that the constant $a = 0$. We therefore have

$$\frac{\phi(u)}{b} = \frac{1}{u^2} + \frac{P(u)}{b}.$$

The constant term that occurs in the power series $P(u)$ is put on the left-hand side of the equation, and the function which we thus have was called by Weierstrass the *Pe-function* and denoted by

$$\wp(u) \text{ or more simply } \wp u.$$

This function is of the form

$$\wp u = \frac{1}{u^2} + * + ((u)).$$

The "*star*" indicates that no constant term appears on the right-hand side of the equation, since it has been put on the left-hand side, and the symbol $((u))$ denotes that all the following terms are infinitesimally small when u is taken infinitesimally small and are of the first or higher orders.

If the point at which the function becomes infinite is not the origin but the point v, we may transform the origin to this point and consequently have to write everywhere u in the place of $u - v$.

We may show as follows that the constant a is zero: We had

$$\phi(u)=\frac{b}{u^2}+\frac{a}{u}+c+c_1u+c_2u^2+c_3u^3+\cdots.$$

Consider also the function $\phi(-u)$. It is doubly periodic, having the same pair of primitive periods as has $\phi(u)$, and consequently like $\phi(u)$ is infinite of the second order on all points congruent to the origin. It may be written

$$\phi(-u)=\frac{b}{u^2}-\frac{a}{u}+c-c_1u+c_2u^2-\cdots.$$

We therefore have

$$\phi(u)-\phi(-u)=2\frac{a}{u}+2c_1u+\cdots.$$

It follows also that $\phi(u)-\phi(-u)$ is a doubly periodic function with the same pair of primitive periods as $\phi(u)$, and consequently can become infinite only where $\phi(u)$ and $\phi(-u)$ become infinite and therefore only on the points congruent to the origin. But, as seen from the last equation, $\phi(u)-\phi(-u)$ becomes infinite at the origin only of the first order. We thus have a doubly periodic function which becomes infinite at only one point within the period-parallelogram and at this point of the first order. We have seen in Art. 101 that there does not exist such a function. It follows that $a=0$; and we further conclude that

$$\phi(u)-\phi(-u)=\text{Constant},$$

otherwise we would have a doubly periodic function which is an integral transcendent contrary to Art. 83. As there appeared no constant term on the right-hand side in the development in series of the function $\phi(u)-\phi(-u)$, we conclude that

$$\phi(u)-\phi(-u)=0,$$

or

$$\phi(u)=\phi(-u).$$

It is thus seen that the elliptic function of the second degree which becomes infinite of the second order at only one point of the period-parallelogram must be an *even* function.

It follows that

$$\phi(u)=\frac{b}{u^2}+c+c_2u^2+c_4u^4+\cdots,$$

or

$$\frac{1}{b}[\phi(u)-c]=\frac{1}{u^2}+\frac{c_2}{b}u^2+\frac{c_4}{b}u^4+\cdots.$$

This function we denote by $\wp u$ and we require that *$\wp u$ be a one-valued doubly periodic function of the unrestricted variable u which has the character of an integral rational function at all points that are not congruent to the origin. At the origin and the congruent points $\wp u$ must be infinite of the second order and is to be an even function.*

ART. 265. We may next show that in reality there exists a function which has the properties required of $\wp u$.

Let
$$w = 2\mu\omega + 2\mu'\omega',$$
where $\mu = 0, \pm 1, \pm 2, \ldots$; $\mu' = 0, \pm 1, \pm 2, \ldots$; $w = 0$ excluded.

Form the function
$$\frac{1}{u^2} + \sum_w \frac{1}{(u-w)^2}.$$

This function does not have the properties desired of $\wp u$, since the series $\sum_w \frac{1}{(u-w)^2}$ is not convergent. For if we give to u the value zero, we have $\sum_w \frac{1}{w^2}$, which is not convergent (see next Article).

But if we form the series
$$\frac{1}{u^2} + \sum \left\{ \frac{1}{(u-w)^2} - \frac{1}{w^2} \right\}$$
and impose the condition that the minuend and the subtrahend which appear in the difference under the summation sign cannot be separated, then this series is absolutely convergent (Art. 266).

If we put an accent on the summation sign to indicate that the value $w = 0$ is excluded from the summation, we may write
$$\wp u = \frac{1}{u^2} + \sum{}' \left\{ \frac{1}{(u-w)^2} - \frac{1}{w^2} \right\}.$$

ART. 266. We must show that the series
$$\sum{}' \left\{ \frac{1}{(u-w)^2} - \frac{1}{w^2} \right\}$$
is absolutely convergent.

Let the shortest distance from the origin to any point on the periphery of the parallelogram passing through the points 2ω, -2ω, $2\omega'$, $-2\omega'$ be d_1, and let d_2 be the longest distance from the origin to any point on the periphery.

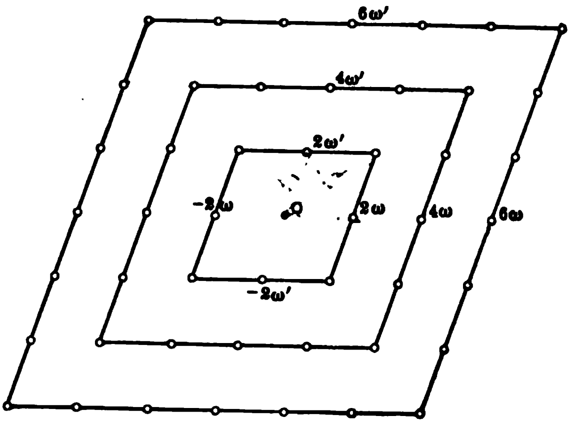


Fig. 70.

On the periphery of this parallelogram there lie $8 = 3^2 - 1^2$ period-points. For these points we have
$$d_1 \leqq |w| \leqq d_2.$$

On the second parallelogram passing through the points 4ω, -4ω, $4\omega'$, $-4\omega'$ there are $5^2 - 3^2 = 8 \cdot 2$ period-points, and for these we have

$$2d_1 \leqq |w| \leqq 2d_2.$$

On the third parallelogram indicated in the figure there are $7^2 - 5^2 = 8 \cdot 3$ period-points, and for them there exists the inequality $3d_1 \leqq |w| \leqq 3d_2$, and for the $n + 1$st parallelogram there are $(2n + 3)^2 - (2n + 1)^2 = 8(n + 1)$ period-points, and for them we have

$$(n + 1)d_1 \leqq |w| \leqq (n + 1)d_2.$$

In the *first* parallelogram we have $\left|\frac{1}{w^2}\right| \geqq \frac{1}{d_2^2}$,

in the *second* parallelogram we have $\left|\frac{1}{w^2}\right| \geqq \frac{1}{(2d_2)^2}$,

in the *third* parallelogram we have $\left|\frac{1}{w^2}\right| \geqq \frac{1}{(3d_2)^2}$,

. . . .

It follows that

$$\sum\left|\frac{1}{w^2}\right| \geqq \frac{8 \cdot 1}{d_2^2} \text{ for the \textit{first} parallelogram,}$$

$$\sum\left|\frac{1}{w^2}\right| \geqq \frac{8 \cdot 2}{(2d_2)^2} \text{ for the \textit{second} parallelogram,}$$

$$\sum\left|\frac{1}{w^2}\right| \geqq \frac{8 \cdot 3}{(3d_2)^2} \text{ for the \textit{third} parallelogram,}$$

. . . .

and consequently

$$\sum{}'\left|\frac{1}{w^2}\right| \geqq \frac{8}{d_2^2}\left\{\frac{1}{1^2} + \frac{2}{2^2} + \frac{3}{3^2} + \cdots\right\}.$$

The series on the right is the well-known divergent harmonic series.

We have further

$$\sum\left|\frac{1}{w}\right|^3 \leqq \frac{8 \cdot 1}{d_1^3} \text{ for the \textit{first} parallelogram,}$$

$$\sum\left|\frac{1}{w}\right|^3 \leqq \frac{8 \cdot 2}{(2d_1)^3} \text{ for the \textit{second} parallelogram,}$$

$$\sum\left|\frac{1}{w}\right|^3 \leqq \frac{8 \cdot 3}{(3d_1)^3} \text{ for the \textit{third} parallelogram,}$$

. . . .

and consequently

$$\sum{}'\left|\frac{1}{w}\right|^3 \leqq \frac{8}{d_1^3}\left\{\frac{1}{1^3} + \frac{2}{2^3} + \frac{3}{3^3} + \cdots\right\},$$

which is absolutely convergent.*

* Eisenstein, *Genaue Untersuchung*, etc., *Crelle*, Bd. 35, p. 156; Vivanti-Gutzmer, *Eindeutige Analytische Functionen*, pp. 168 *et seq.;* Osgood, *Lehrbuch der Funktionentheorie*, p. 444.

ART. 267. We may next show that

$$\sum{}'\left\{\frac{1}{(u-w)^2}-\frac{1}{w^2}\right\}$$

is absolutely convergent.

We limit u to the interior of a circle with radius R, where R is arbitrarily large, but finite. With $2R$ as a radius a circle is described about the origin. Within this circle there is only a finite number of points w. Any of these quantities w situated within or on the circumference of the circle with radius $2R$ is denoted by w', so that

$$|w'| \leqq 2R.$$

We denote any of the points w without the circle by w'' so that

$$|w''| > 2R.$$

It is clear that

$$\sum_w{}' w = \sum_{w'}{}' w + \sum_{w''}{}' w.$$

The series

$$\sum_{w'}{}'\left\{\frac{1}{(u-w)^2}-\frac{1}{w^2}\right\}$$

is composed of a finite number of terms and has a finite value if u does not coincide with any of the values w.

It is seen that this series has the character of an integral rational function and is continuous for all points except $u = w'$ which are situated within the circle with radius $2R$.

We consider next the series

$$\sum_{w''}\left\{\frac{1}{(u-w)^2}-\frac{1}{w^2}\right\}$$

and limit u to the interior of the circle with radius R about the origin as center.

We then have

$$\left|\frac{u}{w''}\right| < \frac{1}{2}.$$

We also have

$$\frac{1}{(w''-u)^2} = \frac{1}{w''^2}\left\{\frac{1}{\left(1-\frac{u}{w''}\right)^2}\right\},$$

and since

$$\left|\frac{u}{w''}\right| < 1,$$

the expression may be developed in the series

$$\frac{1}{(w''-u)^2} = \frac{1}{w''^2}\left\{1+\frac{2u}{w''}+3\left(\frac{u}{w''}\right)^2+4\left(\frac{u}{w''}\right)^3+\cdots\right\},$$

or

$$\frac{1}{(w''-u)^2}-\frac{1}{w''^2} = \frac{u}{w''^3}\left\{2+\frac{3u}{w''}+4\frac{u^2}{w''^2}+\cdots\right\}.$$

By reducing all the terms to their absolute values we have

$$\left|\frac{1}{(w''-u)^2}-\frac{1}{w''^2}\right|<\frac{R}{|w''|^3}\left\{2+\frac{3}{2}+4\frac{1}{4}+5\frac{1}{8}+\cdots\right\}.$$

The expression in the braces converges towards a definite limit, G, say.

It follows that

$$\sum\left|\frac{1}{(w''-u)^2}-\frac{1}{w''^2}\right|<GR\sum_{w''}\frac{1}{(w'')^3},$$

which we saw above is an absolutely convergent series. It follows that

$$\sum_{w''}\left|\frac{1}{(u-w'')^2}-\frac{1}{w''^2}\right|$$

is a finite quantity, and since

$$\sum_{w'}{}'\left|\frac{1}{(u-w')^2}-\frac{1}{w'^2}\right|$$

is a finite quantity, it is seen that

$$\sum_{w}{}'\left\{\frac{1}{(u-w)^2}-\frac{1}{w^2}\right\}$$

is *absolutely* convergent within any finite interval that is free from period-points. The series is also seen to be *uniformly* convergent within (Art. 7) the same interval.

We have thus shown that the function

$$\frac{1}{u^2}+\sum_{w}{}'\left\{\frac{1}{(u-w)^2}-\frac{1}{w^2}\right\}$$

$$\left(w=2\mu\omega+2\mu'\omega';\ \begin{matrix}\mu\\ \mu'\end{matrix}\right\}=0,\pm1,\pm2,\ldots;\ w=0\text{ excluded}\right)$$

has only at the points $u=w$ (including $w=0$) the character of a rational (fractional) function; at all other points it has the character of an integral (rational) function. At the points $u=w$ the function becomes infinite of the second order.

Art. 268. In order to show that the function

$$\frac{1}{u^2}+\sum_{w}{}'\left\{\frac{1}{(u-w)^2}-\frac{1}{w^2}\right\}$$

corresponds completely with the function $\wp u$ defined in Art. 264 we must first show that it is doubly periodic.

Since the expression is uniformly convergent,* we may differentiate term by term and have

$$\wp' u = -\frac{2}{u^3} - 2\sum_{w'}{}' \frac{1}{(u-w)^3} - 2\sum_{w''}\frac{1}{(u-w)^3},$$

or

$$\wp' u = -2\sum_{w}\frac{1}{(u-w)^3} \quad (w = 0 \text{ inclusive}).$$

It follows that

$$\wp'(u+2\omega) = -2\sum_{w}\frac{1}{[u-(w-2\omega)]^3}.$$

From this it is seen that the totality of values on the right-hand side is not altered provided the series is absolutely convergent, and consequently

$$\wp'(u+2\omega) = \wp' u.$$

In a similar manner we have

$$\wp'(u+2\omega') = \wp' u.$$

We have thus shown that the function $\wp' u$ is a doubly periodic function which is infinite of the third order for $u = 0$ and for the congruent points. For all other points this function has the character of an integral function.

We may prove that the series $\sum_{w}\frac{1}{(u-w)^3}$ is absolutely convergent as follows: As above write

$$\sum\frac{1}{(u-w)^3} = \sum_{w'}\frac{1}{(u-w')^3} + \sum_{w''}\frac{1}{(u-w'')^3}.$$

The series $\sum_{w'}\frac{1}{(u-w')^3}$ has a finite value if u does not take one of the values w'. To show that $\sum_{w''}\frac{1}{(u-w'')^3}$ is convergent, we note that

$$\frac{-1}{(u-w)^3} = \frac{1}{w^3}\frac{1}{\left(1-\frac{u}{w}\right)^3},$$

and since

$$\left|\frac{u}{w''}\right| < \frac{1}{2},$$

we have

$$\left|\frac{1}{(u-w'')^3}\right| < \frac{8}{|w''|^3};$$

also, since $\sum_{w''}\frac{1}{(w'')^3}$, as shown above, is convergent for all values of w'' except $w'' = 0$, it follows that

$$\sum_{w''}\frac{1}{(u-w'')^3} \text{ is absolutely convergent.}$$

* Osgood, *Lehrbuch der Funktionentheorie*, pp. 83, 258.

ART. 269. We have at once from the formulas above

$$\wp'(u + 2\omega)du = \wp' u\, du,$$

and consequently also

$$\wp(u + 2\omega) = \wp u + c.$$

Similarly it is seen that

$$\wp(u + 2\omega') = \wp u + c'.$$

Since $\wp u$ is an even function, its first derivative $\wp' u$ is necessarily odd, so that

$$\wp'(-u) = -\wp'(u).$$

If then we write $-\omega$ for u in the formula * above, we have

$$\wp\omega = \wp(-\omega) + c, \quad \text{so that } c = 0.$$

Similary it is seen that $c' = 0$.

ART. 270. We may derive as follows another proof that $\wp u$ is doubly periodic without making use of its first derivative.

The formula

$$\wp u = \frac{1}{u^2} + \sum_w{}' \left\{ \frac{1}{(u - w)^2} - \frac{1}{w^2} \right\}$$

becomes, if w is changed into $-w$,

$$\wp u = \frac{1}{u^2} + \sum_w{}' \left\{ \frac{1}{(u + w)^2} - \frac{1}{w^2} \right\}.$$

The term which corresponds to $w = -2\omega$ is taken without the summation sign. The sum taken over all the values of w except $w = 0$ and $w = -2\omega$ is denoted by Σ *.

We thus have

$$\wp u = \frac{1}{u^2} + \left\{ \frac{1}{(u - 2\omega)^2} - \frac{1}{(2\omega)^2} \right\} + \sum_w{}^* \left\{ \frac{1}{(u + w)^2} - \frac{1}{w^2} \right\}.$$

The totality of the values of w under the summation sign is not changed if we write $-w - 2\omega$ instead of w.

It follows then that

$$\wp u = \frac{1}{u^2} + \left\{ \frac{1}{(u - 2\omega)^2} - \frac{1}{(2\omega)^2} \right\} + \sum_w{}^* \left\{ \frac{1}{(u - w - 2\omega)^2} - \frac{1}{(w + 2\omega)^2} \right\}.$$

Adding these two expressions and dividing by 2, we have

$$\text{(I)} \qquad \wp u = \frac{1}{u^2} + \left\{ \frac{1}{(u - 2\omega)^2} - \frac{1}{(2\omega)^2} \right\} +$$
$$\frac{1}{2} \sum{}^* \left\{ \frac{1}{(u + w)^2} + \frac{1}{(u - w - 2\omega)^2} - \frac{1}{w^2} - \frac{1}{(w + 2\omega)^2} \right\}.$$

* See Osgood, *loc. cit.*, p. 444; Humbert, *Cours d'Analyse*, t. II, p. 194.

In this formula write $-u$ for u; then since $\wp u$ is an even function, it is seen that

$$\wp u = \frac{1}{u^2} + \left\{\frac{1}{(u+2\omega)^2} - \frac{1}{(2\omega)^2}\right\} +$$

$$\frac{1}{2}\sum{}^{*}\left\{\frac{1}{(u-w)^2} + \frac{1}{(u+w+2\omega)^2} - \frac{1}{w^2} - \frac{1}{(w+2\omega)^2}\right\}.$$

Finally, changing u into $u - 2\omega$, we have

$$\text{(II)} \qquad \wp(u-2\omega) = \frac{1}{(u-2\omega)^2} + \left\{\frac{1}{u^2} - \frac{1}{(2\omega)^2}\right\} +$$

$$\frac{1}{2}\sum{}^{*}\left\{\frac{1}{(u-w-2\omega)^2} + \frac{1}{(u+w)^2} - \frac{1}{w^2} - \frac{1}{(w+2\omega)^2}\right\}.$$

Comparing the formulas (II) and (I), it follows that

$$\wp(u - 2\omega) = \wp u,$$

or writing $u + 2\omega$ for u,

$$\wp u = \wp(u + 2\omega).$$

In a similar manner it may be shown that

$$\wp u = \wp(u + 2\omega').$$

ART. 271. It is evident from the formulas above that 2ω, $2\omega'$ form a primitive pair of periods of the argument of the function $\wp u$. The parallelogram with the vertices 0, 2ω, $2\omega'$, $2\omega + 2\omega'$ is free from periods, since all the quantities w represent points that are congruent to these four points. If we select the pair of periods 2ω, $2\omega'$, we may bring them into prominence by writing $\wp u$ in the form

$$\wp(u; \omega, \omega').$$

If a transition is made to an equivalent pair of periods, we write

$$2\tilde{\omega} = 2p\omega + 2q\omega', \quad 2\tilde{\omega} = 2p'\omega + 2q'\omega',$$

where $pq' - qp' \sim 1$ (p, q, p', q' being integers).

It is clear (Art. 80) that the totality of values w remains unaltered by this transformation and consequently we have

$$\wp(u;\ \omega, \omega') = \wp(u; \tilde{\omega}, \tilde{\omega}').$$

It is thus seen that $\wp u$ remains unchanged by a transition to an equivalent pair of primitive periods.

THE SIGMA-FUNCTION.

ART. 272. By integrating twice the $\wp u$-function we may derive another important function.

It is clear that

$$-\int \wp u\, du = -\int \frac{1}{u^2} du - \int {\sum_w}' \left\{ \frac{1}{(u-w)^2} - \frac{1}{w^2} \right\} du,$$

or

$$-\int \wp u\, du = \frac{1}{u} + {\sum}' \left\{ \frac{1}{u-w} + \frac{u}{w^2} \right\} + \text{Constant}.$$

The sum of the terms on the right-hand side is not convergent, but it may be made convergent by a proper choice of the arbitrary constant. For writing

$$-\int \wp u\, du = \frac{1}{u} + {\sum}' \left\{ \frac{1}{u-w} + \frac{1}{w} + \frac{u}{w^2} \right\},$$

we shall show that this expression is absolutely convergent and becomes infinite of the first order only at the points $u = 0$ and $u = w$.

It is seen that

$$\frac{1}{u-w} = -\frac{1}{w}\left\{\frac{1}{1-\dfrac{u}{w}}\right\} = -\frac{1}{w}\left\{1 + \frac{u}{w} + \left(\frac{u}{w}\right)^2 + \left(\frac{u}{w}\right)^3 + \cdots\right\},$$

or

$$\frac{1}{u-w} + \frac{1}{w} + \frac{u}{w^2} = -\frac{u^2}{w^3}\left\{1 + \frac{u}{w} + \frac{u^2}{w^2} + \cdots\right\}.$$

As in Art. 268, it may be shown that the series is convergent, so that the above development of $-\int \wp u\, du$ is convergent.

It is also seen that the above series is infinite only of the first degree at the origin and its congruent points. It follows that $-\int \wp u\, du$ cannot be doubly periodic.

Integrating again the above expression we have

$$-\int du \int \wp u\, du = \log u + {\sum_w}' \left\{ \log\left(1 - \frac{u}{w}\right) + \frac{u}{w} + \frac{1}{2}\frac{u^2}{w^2} \right\},$$

where we have introduced the constant of integration under the logarithm which comes after the summation sign.

We shall next show that this expression is also absolutely convergent if u does not coincide with one of the periods of $\wp u$.

To do this we limit u to the interior of a circle with radius R, where R is arbitrarily large but finite.

The quantities w we again, Art. 267, distribute into two groups, so that

$$\left|w'\right| \leqq 2R, \quad \left|w''\right| > 2R, \quad \left|\frac{u}{w''}\right| < \frac{1}{2}.$$

We then have

$$\sum_w{}' \left\{\log\left(1-\frac{u}{w}\right)+\frac{u}{w}+\frac{1}{2}\frac{u^2}{w^2}\right\} = \sum_{w'}{}' \left\{\log\left(1-\frac{u}{w}\right)+\frac{u}{w}+\frac{1}{2}\frac{u^2}{w^2}\right\}$$
$$+\sum_{w''} \left\{\log\left(1-\frac{u}{w}\right)+\frac{u}{w}+\frac{1}{2}\frac{u^2}{w^2}\right\},$$

where the first summation on the right consists of a finite number of terms, and is consequently finite so long as none of the logarithmic terms which appear is infinite, that is, so long as u does not coincide with one of the quantities w'.

Noting that

$$\log\left(1-\frac{u}{w''}\right) = -\frac{u}{w''}-\frac{1}{2}\left(\frac{u}{w''}\right)^2-\frac{1}{3}\left(\frac{u}{w''}\right)^3-\cdots,$$

it is seen that

$$\sum_{w''}\left\{\log\left(1-\frac{u}{w''}\right)+\frac{u}{w''}+\frac{1}{2}\frac{u^2}{w''^2}\right\} = \sum_{w''}\left\{-\frac{1}{3}\left(\frac{u}{w''}\right)^3-\frac{1}{4}\left(\frac{u}{w''}\right)^4-\cdots\right\},$$

which is an absolutely convergent series (Art. 268).

It follows that

$$-\int du \int \wp u\, du$$

is absolutely convergent for all values of u other than $u = 0$ and $u = w$.

Since the logarithmic function is many-valued, the above integral function is many-valued. To avoid this difficulty we no longer consider this function but the one-valued function

$$\sigma u = e^{-\int du \int \wp u\, du} = u \prod_w{}' \left\{\left(1-\frac{u}{w}\right)e^{\frac{u}{w}+\frac{1}{2}\frac{u^2}{w^2}}\right\}.$$

This *sigma-function* is therefore expressed as a product of an infinite number of factors. As shown in a following Article this product is absolutely convergent if the two factors that occur under the product sign are not separated. The agreement of this function with the function defined in Art. 257 follows in the sequel.

The function σu is one-valued and becomes zero at the origin and at the points congruent to the origin. The accent on the product sign denotes that the factor which corresponds to $w = 0$ is excluded. The sign σ is chosen on account of the similarity of this function with the sine-function.

It is seen at once that

$$\wp u = -\frac{d^2}{du^2}\log \sigma u.$$

The function σu is *not* doubly periodic. It has like the theta-functions for all finite values of u the character of an integral function and may be expressed as an absolutely convergent power-series with integral positive exponents (Arts. 276, 336). Like the function $\wp u$ it is not changed when a transition is made from one pair of primitive periods of the function $\wp u$ to an equivalent pair.

ART. 273. *Historical.* — Eisenstein (in *Crelle's Journal*, Bd. 27, p. 285, 1844) formed the product

$$\prod\left(1 - \frac{z}{\mu A + \mu' A'}\right),$$

where A and A' are quantities such that

$$\frac{A}{A'} = \alpha + i\beta \quad (\beta \neq 0),$$

while μ and μ' take all values $\pm 1, \pm 3, \pm 5, \cdots$; and on page 287 he formed the products

$$z\prod\left(1 - \frac{z}{\lambda A + \lambda' A'}\right), \quad \prod\left(1 - \frac{z}{\lambda A + \mu A'}\right)$$

$$(\lambda, \lambda') = \pm 2, \pm 4, \pm 6, \cdots,$$

$$\mu = \pm 1, \pm 3, \pm 5, \cdots.$$

On page 288 Eisenstein says that the quotient of any two such products gives rise to the doubly periodic functions and he closes the article with the remark:

"*Die hier angestellte Untersuchung ist übrigens so elementar Natur, dass sie sich wohl eignen möchte, den Anfanger in die Theorie der elliptischen Functionen einzuführen.*"

In *Crelle's Journal*, Bd. 30, p. 184, Jacobi called attention to the fact that Eisenstein had formed defective Θ-functions owing to the fact that the above products are not absolutely convergent. Jacobi at the end of this article claims that the "exact formulas" are given (by Jacobi) in *Crelle's Journal*, Bd. 4, p. 382; Werke, Bd. I, p. 297 (see also Werke, Bd. I, p. 372).

Cayley (*Elliptic Functions*, p. 101) remarks that such products as the above "in the absence of further definition as to the limits are wholly meaningless;" but Cayley, *loc. cit.*, pp. 301–303, fixed these limits (see also Cayley, *Camb. and Dublin Math. Journ.*, Vol. IV (1845), pp. 257–277, and *Liouville's Journal*, t. X (1845), pp. 385–420), and illustrated them by means of a "bounding curve."

It may be observed that the above remarks are applicable also to the infinite products of Abel (*Recherches sur les fonctions elliptiques, Crelle*, Bd. 2, p 154; Œuvres, t. I, p. 226) and of Jacobi, *Fund. nova*, § 35; Werke, I, p. 141.

Professor Klein, *Theorie der elliptischen Modulfunctionen*, Bd. I, p. 150, calls attention to the fact that the quantities $\wp u$, $\wp' u$, g_2, g_3, e_1, e_2, e_3 are defined by Eisenstein, *Genaue Untersuchung der unendlichen Doppelprodukte, aus welchen die elliptischen Funktionen als Quotienten zusammen gesetzt sind, Crelle*, Bd. 35 (1847), pp. 153–274, and *Mathematische Abhandlungen*, pp. 213–334.

We also note that the relation

$$(\wp' u)^2 = 4(\wp u)^3 - g_2 \wp u - g_3$$

is the identical relation given by Eisenstein, *Crelle*, 35, p. 225, formula (5).

On page 226, Eisenstein derives the normal integrals of the first and second kinds in the forms

$$\int \frac{dy}{2\sqrt{(y-a)(y-a')(y-a'')}} \text{ and } -\int \frac{y\,dy}{2\sqrt{(y-a)(y-a')(y-a'')}}.$$

It also appears from this paper that Eisenstein had some idea of the nature of the quantities g_2 and g_3 whose invariantive properties were discovered by Cayley and Boole in 1845.

Weierstrass, recognizing the true nature of these invariants, was the first (cf. Klein, *loc. cit.*, p. 24) to make the Theory of Elliptic Functions from the standpoint of the infinite products and series as given in this Chapter (and developed by him) of consequence, and so he is to be considered the founder of this theory.

In his last lectures Professor Kronecker, *Theorie der elliptischen Functionen zweier Paare reeller Argumente* (W. S., 1891), especially emphasized the Eisenstein theory and made paramount a certain function *En* (denoting Eisenstein's name) which is a generalized $\wp$-function.

ART. 274. We saw in Chapter I that the infinite product

$$\prod_{\nu=1}^{\nu=\infty} (1 + a_\nu) \text{ is absolutely convergent if } \sum_{\nu=1}^{\nu=\infty} |a_\nu|$$

is absolutely convergent.

To prove the absolute convergence of the infinite product through which the sigma-function is expressed let $|u| < R$, $|w'| \leqq 2R$, $|w''| > 2R$ as above. We omit from the infinite product all those factors which correspond to the quantities w'. Such factors being finite in number exercise no influence upon the question of convergence.

The factors remaining in the product are of the form

$$\left(1-\frac{u}{w''}\right)e^{\frac{u}{w''}+\frac{1}{2}\frac{u^2}{w''^2}}=e^{\log\left(1-\frac{u}{w''}\right)+\frac{u}{w''}+\frac{1}{2}\frac{u^2}{w''^2}}.$$

Since $\left|\frac{u}{w''}\right|<\frac{1}{2}$, we may develop the logarithm in a power series and have

$$e^{-\frac{u}{w''}-\frac{1}{2}\frac{u^2}{w''^2}-\frac{1}{3}\frac{u^3}{w''^3}-\cdots+\frac{u}{w''}+\frac{1}{2}\frac{u^2}{w''^2}},$$

or finally

$$e^{-\frac{1}{3}\frac{u^3}{w''^3}\left\{1+\frac{3}{4}\frac{u}{w''}+\frac{3}{5}\frac{u^2}{w''^2}+\cdots\right\}}.$$

Since $\left|\frac{u}{w''}\right|<\frac{1}{2}$, this expression is

$$<e^{\frac{1}{3}\left|\frac{u}{w''}\right|^3\left\{1+\frac{1}{2}+\frac{1}{4}+\cdots\right\}},$$

and consequently

$$<e^{\left|\frac{u}{w''}\right|^3}\left(=1+\left|\frac{u}{w''}\right|^3+\frac{1}{1.2}\left|\frac{u}{w''}\right|^6+\cdots\right).$$

It is thus seen that the quantities in the sigma-function corresponding to a_ν above are such that

$$\left|a_\nu\right|<\left|\frac{u}{w''}\right|^3\left\{1+\frac{1}{2!}\cdot\frac{1}{8}+\frac{1}{3!}\cdot\frac{1}{8^2}+\cdots\right\}$$

$$<\left|\frac{u}{w''}\right|^3\left\{1+\frac{1}{16}+\frac{1}{16^2}+\cdots\right\}<\left|\frac{u}{w''}\right|^3\frac{1}{1-\frac{1}{16}};$$

or finally

$$\left|a_\nu\right|<\frac{16}{15}\left|\frac{u}{w''}\right|^3.$$

It follows that

$$\sum\left|a_\nu\right|<\frac{16}{15}\left|u\right|^3\sum\left|\frac{1}{w''}\right|^3,$$

which we saw above was absolutely convergent. To the $\Sigma\ |a_\nu|$ we must add the quantities $|a_\nu|$ which correspond to the quantities w'; but the convergence is unchanged by the addition of these terms. It follows that *the product through which the sigma-function is expressed is absolutely convergent.* Since an absolutely convergent infinite product is only zero when at least one of its factors becomes zero, it is seen that σu vanishes only at the points $u = 0$ and $u = w$ and at these points σu is zero of the first order.

ART. 275. Other properties of the sigma-function may be developed as follows:

We have

$$\sigma(-u)=(-u)\prod_w{}'\left\{\left(1+\frac{u}{w}\right)e^{-\frac{u}{w}+\frac{1}{2}\frac{u^2}{w^2}}\right\}.$$

If w is changed into $-w$ the product is not altered, and we have

$$\sigma(-u)=-u\prod_w{}'\left\{\left(1-\frac{u}{w}\right)e^{\frac{u}{w}+\frac{1}{2}\frac{u^2}{w^2}}\right\}.$$

It follows that

$$\sigma(-u)=-\sigma u,$$

and consequently *the function σu is an odd function.*

ART. 276. We shall consider next more closely the form of the development of σu. In the product

$$\sigma u=u\prod_w{}'\left\{\left(1-\frac{u}{w}\right)e^{\frac{u}{w}+\frac{1}{2}\frac{u^2}{w^2}}\right\}$$

we join any two factors that correspond to opposite values of w and thus have *

$$\sigma u=u\prod_w{}^*\left\{\left(1-\frac{u^2}{w^2}\right)e^{\frac{u^2}{w^2}}\right\},$$

the *star* denoting that of every pair of values w and $-w$ only one value is to be taken.

It follows that

$$\sigma u=u\prod_w{}^* e^{\frac{u^2}{w^2}+\log\left(1-\frac{u^2}{w^2}\right)}.$$

If u is chosen smaller than any of the values w, we may write

$$\begin{aligned}\sigma u&=u\prod_w{}^* e^{-\frac{1}{2}\frac{u^4}{w^4}-\frac{1}{3}\frac{u^6}{w^6}-\cdots}\\&=u\prod_w{}^*\left\{1-\frac{1}{2}\frac{u^4}{w^4}-\frac{1}{3}\frac{u^6}{w^6}-\cdots\right\};\end{aligned}$$

and consequently

$$\sigma u=u\left\{1-\frac{1}{2}\sum_w{}^*\frac{u^4}{w^4}-\frac{1}{3}\sum_w{}^*\frac{u^6}{w^6}-\cdots\right\},$$

or

$$\sigma u=u-\frac{1}{2}\sum_w{}^*\frac{1}{w^4}u^5-\frac{1}{3}\sum{}^*\frac{1}{w^6}u^7-\cdots.$$

* Cf. Daniels, *Amer. Journ. Math.*, Vol. 6, p. 178.

We may write
$$2^2 \cdot 3 \cdot 5 \sum_w{}' \frac{1}{w^4} = g_2,$$
$$2^2 \cdot 5 \cdot 7 \sum_w{}' \frac{1}{w^6} = g_3,$$
where, as will be evident from the sequel, the quantities g_2, g_3 are the invariants introduced in Art. 184. It is also evident that g_2 and g_3 remain unaltered when we pass from one pair of equivalent primitive periods to another pair.

It is seen that
$$\sigma u = u + * - \frac{g_2}{2^4 \cdot 3 \cdot 5} u^5 - \frac{g_3}{2^3 \cdot 3 \cdot 5 \cdot 7} u^7 - \cdots,$$
the star indicating that the term with u^3 is wanting. The function σu is an integral function that is regular in the whole plane and may be expressed through a series that is everywhere convergent (Art. 13).

The ζu-Function.

ART. 277. From the formula just written it follows that
$$\log \sigma u = \log \left\{ u - \frac{g_2}{2^4 \cdot 3 \cdot 5} u^5 - \frac{g_3}{2^3 \cdot 3 \cdot 5 \cdot 7} u^7 - \cdots \right\}$$
$$= \log u + \log \left\{ 1 - \frac{g_2}{2^4 \cdot 3 \cdot 5} u^4 - \frac{g_3}{2^3 \cdot 3 \cdot 5 \cdot 7} u^6 - \cdots \right\}$$
$$= \log u - \frac{g_2}{2^4 \cdot 3 \cdot 5} u^4 - \frac{g_3}{2^3 \cdot 3 \cdot 5 \cdot 7} u^6 - \cdots.$$
It is evident from the consideration of the product through which σu is defined that this series is convergent within a circle with the origin as center and a radius that passes through the nearest period-point.

If this expression is differentiated with respect to u, it follows that
$$\frac{\sigma' u}{\sigma u} = \frac{1}{u} + * - \frac{g_2}{2^2 \cdot 3 \cdot 5} u^3 - \frac{g_3}{2^2 \cdot 5 \cdot 7} u^5 - \cdots.$$
The quotient $\dfrac{\sigma'(u)}{\sigma u}$ is often denoted by ζu (Art. 258, see Halphen, *Fonct. Elliptiques*, t. I, Chap. V).

Differentiating this expression again and multiplying by -1, we have
$$\wp u = -\frac{d^2}{du^2} \log \sigma u = -\frac{d}{du}\left\{\frac{\sigma'(u)}{\sigma u}\right\} = \frac{1}{u^2} + * + \frac{g_2}{2^2 \cdot 5} u^2 + \frac{g_3}{2^2 \cdot 7} u^4 + \cdots$$
and also
$$\wp' u = -\frac{2}{u^3} + * + \frac{g_2}{10} u + \frac{g_3}{7} u^3 + \cdots.$$
The series through which $\wp u$, $\wp' u$ and ζu are expressed are convergent within a circle which has the origin as center and which does not contain any period-point.

The functions $\wp u$ and $\wp' u$ are, as we have already seen, doubly periodic, $\wp u$ being an *even* and $\wp u'$ an *odd* function. The function $(\wp' u)^2$ is an *even* doubly periodic function of the sixth degree and is infinite of the sixth order at the origin and all congruent points.

ART. 278. We may next prove that $\wp u$ satisfies the differential equation of the first order *

$$(\wp' u)^2 = 4(\wp u)^3 - g_2 \wp u - g_3.$$

We have

$$\wp' u)^2 = \frac{4}{u^6} + * - \frac{2}{5} g_2 \frac{1}{u^2} - \frac{4}{7} g_3 + ((u^2))$$

and

$$(\wp u)^3 = \frac{1}{u^6} + * + \frac{3}{2^2 \cdot 5} g_2 \frac{1}{u^2} + \frac{3}{2^2 \cdot 7} g_3 + ((u^2)).$$

It follows that

$$(\wp' u)^2 - 4(\wp u)^3 = - g_2 \frac{1}{u^2} - g_3 + ((u^2)),$$

and also that

$$(\wp' u)^2 - \left\{4(\wp u)^3 - g_2 \wp u - g_3\right\} = ((u^2)).$$

We note that the left-hand side of this expression is doubly periodic, while the right-hand side has everywhere the character of an integral function. By the theorem of Art. 83, such a doubly periodic function must be a constant, and as there is present no constant term, the right-hand side is zero. We therefore have as our *eliminant equation*

$$(\wp' u)^2 = 4 \wp^3 u - g_2 \wp u - g_3.$$

ART. 279. If in the above equation we use Weierstrass's notation and put $\wp u = s$, and $\wp' u = - \frac{ds}{du}$, Art. 256, we have

$$\left(\frac{ds}{du}\right)^2 = 4 s^3 - g_2 s - g_3;$$

or

$$u = \pm \int \frac{ds}{\sqrt{4 s^3 - g_2 s - g_3}},$$

agreeing with the results of Chapters VIII and XIV. No confusion can arise from the fact that here we have written s for the variable t before used. The double sign is accounted for by means of the Riemann surface of Art. 143.

Since $s = \infty$ for $u = 0$, we may write this integral in the form

$$u = \int_s^\infty \frac{ds}{2 s^{\frac{3}{2}} \sqrt{1 - \frac{g_2}{4 s^2} - \frac{g_3}{4 s^3}}} = \int_s^\infty \frac{ds}{2 s^{\frac{3}{2}}} \left(1 - \frac{g_2}{4 s^2} - \frac{g_3}{4 s^3}\right)^{-\frac{1}{2}}.$$

* See for example, Humbert, *loc. cit.*, p. 204.

If we consider values of s lying in the neighborhood of infinity so that $1 - \frac{g_2}{4\,s^2} - \frac{g_3}{4\,s^3} > 0$, we may expand the integrand in a power series and then integrate term by term. We thus have

$$u = \int_s^\infty \frac{ds}{s^{\frac{3}{2}}}\left[1 + \frac{g_2}{8\,s^2} + \frac{g_3}{8\,s^3} + \left(\left(\frac{1}{s^4}\right)\right)\right],$$

or

$$u = \frac{1}{\sqrt{s}}\left\{1 + \frac{1}{40} g_2 \frac{1}{s^2} + \frac{1}{56} g_3 \frac{1}{s^3} + \left(\left(\frac{1}{s^4}\right)\right)\right\}.$$

It follows that

$$u = P_1\left(\frac{1}{s^{\frac{1}{2}}}\right).$$

All the coefficients of this power series are clearly functions of g_2 and g_3 with rational numerical coefficients.

When this series is reverted, it is seen that $\frac{1}{s^{\frac{1}{2}}}$ may in the neighborhood of the origin be expanded in powers of u; and it is also evident that $s = \wp u$ may be expanded in the neighborhood of the origin in a power-series whose coefficients are integral functions of g_2 and g_3 with rational numerical coefficients. The functions $\zeta u = \frac{\sigma'(u)}{\sigma(u)}$ and $\log_e \sigma u$ have the same properties, and by passing from the logarithm to the exponential function, it is found that the same is also true of the function σu, so that the development of σu in the neighborhood of the origin is such that all the coefficients are integral functions of g_2 and g_3 with rational numerical coefficients. The sigma-function is therefore a function of u, g_2, g_3. A method of determining the coefficients of σu by means of a partial differential equation is found in Art. 336.

ART. 280. It follows from the equation above that

$$u^2 = \frac{1}{s}\left\{1 + \frac{g_2}{20}\frac{1}{s^2} + \frac{g_3}{28}\frac{1}{s^3} + \left(\left(\frac{1}{s^4}\right)\right)\right\},$$

or

$$s = \frac{1}{u^2}\left\{1 + \frac{g_2}{20}\frac{1}{s^2} + \frac{g_3}{28}\frac{1}{s^3} + \left(\left(\frac{1}{s^4}\right)\right)\right\}.$$

Hence as an approximation (up to terms of the order u^6) we have

$$\frac{1}{s} = u^2.$$

If then on the right-hand side of the last equation we write $\frac{1}{u^2}$ for s, we have

$$s = \wp u = \frac{1}{u^2} + * + \frac{g^2}{20} u^2 + \frac{g_3}{28} u^4 + ((u^6)).$$

Writing $\wp u = \frac{1}{u^2} + * + c_2 u^2 + c_3 u^4 + c_4 u^6 + \cdots + c_\lambda u^{2\lambda - 2} + \cdots$, it follows that

$$c_2 = \tfrac{1}{20} g_2 \quad \text{and} \quad c_3 = \tfrac{1}{28} g_3.$$

We shall express the other constants c_4, c_5, ... through these two quantities.

From the relation

$$(\wp' u)^2 = 4\wp^3 u - g_2 \wp u - g_3$$

we have through differentiation

$$2\wp' u\, \wp'' u = 12\wp^2 u\, \wp' u - g_2 \wp' u,$$

or, if we give to u such values that $\wp' u \neq 0$,

$$\wp'' u = 6\wp^2 u - \frac{g_2}{2}\cdot \qquad \text{(Eisenstein, \textit{Crelle}, Bd. 35, p. 195.)}$$

Multiplying through by u^4 we have

$$\text{(A)} \qquad u^4 \wp'' u = 6u^4 \wp^2 u - \tfrac{1}{2} g_2 u^4.$$

From the equation

$$\wp u = \frac{1}{u^2} + * + c_2 u^2 + c_3 u^4 + \cdots + c_\lambda u^{2\lambda - 2} + \cdots$$

it follows that

$$\wp' u = -\frac{2}{u^3} + * + 2c_2 u + 4c_3 u^3 + \cdots + (2\lambda - 2)c_\lambda u^{2\lambda - 3} + \cdots,$$

$$\wp'' u = \frac{6}{u^4} + * + 2c_2 + 3\cdot 4c_3 u^2 + \cdots + (2\lambda - 2)(2\lambda - 3)c_\lambda u^{2\lambda - 4} + \cdots,$$

or

$$u^4 \wp'' u = 6 + * + 2c_2 u^4 + 3\cdot 4c_3 u^6 + \cdots + (2\lambda - 2)(2\lambda - 3)c_\lambda u^{2\lambda} + \cdots.$$

We also have

$$u^2 \wp u = 1 + * + c_2 u^4 + c_3 u^6 + \cdots + c_\lambda u^{2\lambda} + \cdots,$$

or $\qquad u^2 \wp u = \cdots + c_\lambda u^{2\lambda} + c_{\lambda-1} u^{2\lambda - 2} + \cdots + c_{\lambda-\nu} u^{2\lambda - 2\nu} + \cdots + 1;$

and consequently

$$6[u^2 \wp u]^2 = \cdots + 6\cdot 2c_\lambda u^{2\lambda} + 6\sum_{\nu=2}^{\nu=\lambda-2} c_\nu c_{\lambda-\nu} u^{2\lambda} + \cdots,$$

where we have written down only the terms that contain $u^{2\lambda}$.

Writing these values in the equation (A) above and equating the coefficients of $u^{2\lambda}$, we have *

$$12c_\lambda + 6\sum_{\nu=2}^{\nu=\lambda-2} c_\nu c_{\lambda-\nu} = (2\lambda - 3)(2\lambda - 2)c_\lambda,$$

or

$$c_\lambda = \frac{3}{(2\lambda + 1)(\lambda - 3)} \sum_{\nu=2}^{\nu=\lambda-2} c_\nu c_{\lambda-\nu} \qquad (\lambda > 3).$$

This is a recursion formula by means of which each of the coefficients c_λ in the development of $\wp u$ may be expressed through coefficients with smaller indices.

* Cf. Schwarz, *Formeln und Lehrsätze*, etc., p. 11; the Berlin lectures of Prof. Schwarz have been freely used in the preparation of this Chapter.

We have, for example,

$$c_4 = \frac{3}{9 \cdot 1} c_2^2,$$

or, since $c_2 = \frac{1}{20} g_2$, it follows that

$$c_4 = \frac{1}{2^4 \cdot 3 \cdot 5^2} g_2^2;$$

and similarly

$$c_5 = \frac{3\, g_2 g_3}{2^4 \cdot 5 \cdot 7 \cdot 11},$$

$$c_6 = \frac{1}{2^4 \cdot 13}\left(\frac{g_3^2}{7} + \frac{g_2^3}{2 \cdot 3 \cdot 5^3}\right),$$

$$c_7 = \frac{g_2^2 g_3}{2^5 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11}, \text{ etc.}$$

We may therefore write

$$\wp u = \frac{1}{u^2} + * + \frac{g_2}{2^2 \cdot 5} u^2 + \frac{g_3}{2^2 \cdot 7} u^4 + \frac{g_2^2}{2^4 \cdot 3 \cdot 5^2} u^6 + \cdots,$$

$$\zeta u = \frac{1}{u} + * - \frac{g_2}{2^2 \cdot 3 \cdot 5} u^3 - \frac{g_3}{2^2 \cdot 5 \cdot 7} u^5 - \frac{g_2^2}{2^4 \cdot 3 \cdot 5^2 \cdot 7} u^7 - \cdots,$$

$$\sigma u = u + * - \frac{g_2 u^5}{2^4 \cdot 3 \cdot 5} - \frac{g_3 u^7}{2^3 \cdot 3 \cdot 5 \cdot 7} u^7 - \cdots.$$

ART. 281. We saw in Art. 268 that

$$\wp u = \frac{1}{u^2} + \sum_w{}' \left\{\frac{1}{(u - w)^2} - \frac{1}{w^2}\right\}.$$

If we make the condition that $|u| < w$, we may write

$$\frac{1}{w - u} = \frac{1}{w} + \frac{u}{w^2} + \frac{u^2}{w^3} + \cdots + \frac{u^n}{w^{n+1}} + \cdots.$$

This equation differentiated with respect to u becomes

$$\frac{1}{(w - u)^2} = \frac{1}{w^2} + \frac{2\,u}{w^3} + \cdots + \frac{nu^{n-1}}{w^{n+1}} + \cdots.$$

It follows at once that

$$\sum_w{}' \left\{\frac{1}{(w - u)^2} - \frac{1}{w^2}\right\} = \sum_w{}' \frac{2\,u}{w^3} + \sum_w{}' \frac{3\,u^2}{w^4} + \cdots + \sum_w{}' \frac{nu^{n-1}}{w^{n+1}} + \cdots.$$

We note that all terms in which w appears with an odd exponent vanish, since a value $-w$ belongs to every value $+w$.

If then we write $n - 1 = 2\lambda - 2$, or $n = 2\lambda - 1$, and compare the above expression with

$$\wp u = \frac{1}{u^2} + * + c_2 u^2 + \cdots + c_\lambda u^{2\lambda - 2} + \cdots,$$

it is seen that

$$c_\lambda = (2\lambda - 1)\sum{}' \frac{1}{w^{2\lambda}}.$$

It follows from the results of the preceding Article that $\sum{}' \frac{1}{w^{2\lambda}}$ may be integrally expressed in terms of g_2 and g_3. This is a very remarkable fact (cf. Halphen, *Fonct. Ellip.*, t. I, p. 366).

In Art. 272 we saw that

$$\wp u = -\frac{d^2 \log \sigma u}{du^2} = \frac{d}{du}\left(-\frac{\sigma' u}{\sigma u}\right),$$

or

$$\wp u = \frac{\sigma u \sigma'' u - (\sigma' u)^2}{(\sigma u)^2}.$$

The function σu is uniformly convergent for all values of u in the finite portion of the plane. The same is true of $\sigma' u$ and $\sigma'' u$. Hence it is seen that $\wp u$ may be expressed as the quotient of two power-series that are uniformly convergent for all values of u in the finite portion of the plane. We saw in Chapter XI that the functions $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$ have the same property. In Arts. 262, 324–326 we consider the analogues of these three functions in Weierstrass's Theory.

ART. 282. *Another expression for the function $\wp u$.* — We write (cf. Art. 60)

$$t = e^{\frac{u\pi i}{\omega}},$$

and we shall first derive a function of t which behaves at the origin in the same manner as $\wp u$. The development of t in the neighborhood of $u = 0$ is

$$t = 1 + \frac{u\pi i}{\omega} + \frac{1}{1 \cdot 2}\left(\frac{u\pi i}{\omega}\right)^2 + \cdots,$$

or

$$t - 1 = \frac{u\pi i}{\omega}\left\{1 + \frac{1}{1 \cdot 2}\frac{u\pi i}{\omega} + \frac{1}{1 \cdot 2 \cdot 3}\left(\frac{u\pi i}{\omega}\right)^2 + \cdots\right\}.$$

We note that $t - 1$ becomes zero of the first order at the point $u = 0$ and at all other points where t has the value 1. The totality of all these points is expressed through

$$u = 2\mu\omega (\mu = 0, \pm 1, \pm 2, \cdots).$$

The function

$$(t - 1)^2 = -\frac{u^2\pi^2}{\omega^2}\left\{1 + \frac{u\pi i}{\omega} + \frac{7}{12}\left(\frac{u\pi i}{\omega}\right)^2 + \cdots\right\}$$

becomes zero of the second order at all the points $u = 2\mu\omega$.

Let $g(t)$ be an integral function of t which does not vanish for $t=1$. The function

$$\frac{g(t)}{(t-1)^2}$$

will therefore be infinite of the second order for the value $u=0$ and for all the values $u=2\mu\omega$. Hence this function behaves at these points in the same way as does the function $\wp u$.

We may write $g(t)=a+bt+ct^2$, a, b and c being constants. It follows that

$$\frac{g(t)}{(t-1)^2}=\frac{a+bt+ct^2}{(t-1)^2}.$$

Since $t=e^{\frac{u\pi i}{\omega}}$, it is seen that t^2 may be derived from t by writing $2u$ in the place of u in the expansion of t.

Accordingly we have

$$\frac{g(t)}{(t-1)^2}=$$

$$\frac{a+b\left[1+\frac{u\pi i}{\omega}+\frac{1}{1\cdot 2}\left(\frac{u\pi i}{\omega}\right)^2+\cdots\right]+c\left[1+\frac{2u\pi i}{\omega}+\frac{1}{1\cdot 2}\left(\frac{2u\pi i}{\omega}\right)^2+\cdots\right]}{-\frac{u^2\pi^2}{\omega^2}\left[1+\frac{u\pi i}{\omega}+\frac{1}{3}\left(\frac{u\pi i}{\omega}\right)^2+\cdots\right]}$$

$$=-\frac{\omega^2}{\pi^2}\frac{\left\{a+b\left[1+\frac{u\pi i}{\omega}+\frac{1}{2}\left(\frac{u\pi i}{\omega}\right)^2+\cdots\right]+c\left[1+\frac{2u\pi i}{\omega}+\cdots\right]\right\}}{u^2}\left[1-\frac{u\pi i}{\omega}+((u^2))\right].$$

We wish that the following conditions be satisfied:

First. The term which becomes infinite of the second order must be of the form $\frac{1}{u^2}$.

Second. The term which becomes infinite of the first order must not be present.

Third. The constant term in the development of the function in powers of u must be zero.

To fulfill the first condition we must have

$$-\frac{\omega^2}{\pi^2}\frac{a+b+c}{u^2}=\frac{1}{u^2};$$

for the second condition, we must put

$$-\frac{\omega^2}{\pi^2}\left[b\frac{\pi i}{\omega}\frac{1}{u}+\frac{2c\pi i}{\omega}\frac{1}{u}-(a+b+c)\frac{\pi i}{\omega}\frac{1}{u}\right]=0,$$

or

$$c-a=0.$$

From the first condition it follows that

$$b = -\frac{\pi^2}{\omega^2} - 2a.$$

These values substituted in $\frac{g(t)}{(1-t)^2}$ cause this function to become

$$\frac{a(t^2 - 2t + 1) - \frac{\pi^2}{\omega^2}t}{(t-1)^2} = -\frac{\pi^2}{\omega^2}\frac{t}{(t-1)^2} + a.$$

The constant a must be so chosen that the third condition above may be satisfied.

We note that

$$t^{\frac{1}{2}} - t^{-\frac{1}{2}} = \frac{u\pi i}{\omega}\left[1 + \frac{1}{24}\left(\frac{u\pi i}{\omega}\right)^2 + \cdots\right];$$

and since

$$\frac{t}{(t-1)^2} = \frac{1}{(t^{\frac{1}{2}} - t^{-\frac{1}{2}})^2},$$

it follows that

$$\frac{t}{(t-1)^2} = \frac{-1}{\frac{u^2\pi^2}{\omega^2}\left[1 + \frac{1}{12}\left(\frac{u\pi i}{\omega}\right)^2 + \cdots\right]}$$

$$= \frac{1 - \frac{1}{12}\left(\frac{u\pi i}{\omega}\right)^2 - \cdots}{-\frac{u^2\pi^2}{\omega^2}};$$

and consequently that the third condition may be satisfied, we must have

$$a = -\frac{1}{12}\frac{\pi^2}{\omega^2}.$$

Noting that

$$\frac{t^{\frac{1}{2}} - t^{-\frac{1}{2}}}{2i} = \sin\frac{\pi u}{2\omega},$$

it is seen that

$$\frac{g(t)}{(t-1)^2} = a - \frac{\frac{\pi^2}{\omega^2}}{\left(2i\sin\frac{u\pi}{2\omega}\right)^2} = a + \frac{\pi^2}{4\omega^2\sin^2\frac{u\pi}{2\omega}}.$$

We have thus shown that the function

$$\frac{g(t)}{(t-1)^2} = \left(\frac{\pi}{2\omega}\right)^2\left[\frac{1}{\sin^2\frac{u\pi}{2\omega}} - \frac{1}{3}\right]$$

corresponds in its initial terms with the development of $\wp u$, so that it differs from $\wp u$ only in quantities which become infinitesimally small of the first order when u becomes indefinitely small.

ART. 283. We had

$$\wp u = \frac{1}{u^2} + \sum_w \left\{ \frac{1}{(u-w)^2} - \frac{1}{w^2} \right\}.$$

The quantities w may be distributed into two groups. The first group contains all values w for which $\mu' = 0$, so that $w = 2\mu\omega$. The second group contains those w's for which $\mu' \gtrless 0$, so that $w = 2\mu\omega + 2\mu'\omega'$. If then we let ω' become infinite, the values w of the second group become infinite, and we have

$$\wp(u, \omega, \infty) = \frac{1}{u^2} + \sum_\mu{}' \left\{ \frac{1}{(u-2\mu\omega)^2} - \frac{1}{(2\mu\omega)^2} \right\}.$$

It is seen from Art. 22 that this expression is none other than the function

$$\frac{g(t)}{(t-1)^2}.$$

If then the period $2\omega'$ becomes infinite, the function $\wp u$ is represented by

$$\frac{g(t)}{(t-1)^2} = \left(\frac{\pi}{2\omega}\right)^2 \left\{ \frac{1}{\sin^2 \dfrac{u\pi}{2\omega}} - \frac{1}{3} \right\}.$$

ART. 284. We shall next write (cf. Eisenstein, *loc. cit.*, p. 216)

$$F(t) = -\frac{\pi^2}{\omega^2} \frac{t}{(t-1)^2}$$

and we shall seek to express* $\wp u$ through t even when the second period $2\omega'$ is finite. $F(t)$ being a rational function of $t = e^{\frac{\pi i u}{\omega}}$ remains unchanged when u is increased by $u + 2\omega$; but when u is increased by $2\omega'$ then $e^{(u+2\omega')\frac{\pi i}{\omega}} = e^{\frac{2\omega'\pi i}{\omega}} t$. Weierstrass used the letter h to denote the quantity $e^{\frac{\pi i \omega'}{\omega}}$, which Jacobi denoted by q. In Art. 86 we wrote $\frac{\omega'}{\omega} = \alpha + i\beta$, where $\beta > 0$. From this it is seen that

$$q = h = e^{(\alpha+i\beta)\pi i} = e^{-\beta\pi} e^{\alpha\pi i},$$

and consequently

$$|h| = e^{-\beta\pi}.$$

Noting Art. 81, it is evident that we may always choose a pair of primitive periods so that

$$|h| < 1.$$

Since t becomes $h^2 t$ when u is increased by $2\omega'$, it follows that when u becomes $u + 2\omega'$

$$F(t) \text{ becomes } F(h^2 t),$$
$$F(h^2 t) \text{ becomes } F(h^4 t),$$
$$F(h^4 t) \text{ becomes } F(h^6 t),$$
$$\cdots\cdots\cdots$$

* See also Halphen, *Fonct. Ellip.*, t. I, Chap. XIII.

If we consider the infinite series

(S′) $$F(t)+F(h^2t)+F(h^4t)+\cdots+F(h^{2n}t)+\cdots,$$

then, if u is increased by $2\omega'$, each term becomes the following term.

Hence the series

$$F(t)\begin{array}{l}+F(h^2t)+F(h^4t)+\cdots+F(h^{2n}t)+\cdots\\ +F(h^{-2}t)+F(h^{-4}t)+\cdots+F(h^{-2n}t)+\cdots\end{array}$$

is a doubly periodic function having the two periods 2ω, $2\omega'$. At the point $u=0$ and all its congruent points this function becomes infinite of the second order; for then t equals unity or some even power of h.

ART. 285. We shall next show that this series is absolutely convergent for all points except the origin and the points congruent to it.

We limit u to a region in which $|u|<R$, where R may be arbitrarily large, but finite. The quantity t has everywhere within this region the nature of an integral function and is different from zero.

Further, since

$$t=e^{\frac{u\pi i}{\omega}}=e^{\mu+i\nu}=e^{\mu}e^{i\nu},$$

it is seen that

$$|t|=e^{\mu},$$

so that $|t|$ becomes a maximum with μ, that is, with $\mathsf{R}\left(\frac{u\pi i}{\omega}\right)$.

If we put $u=\omega'$, then is

$$\mathsf{R}\left(\frac{u\pi i}{\omega}\right)=-\beta\pi.$$

If M is the greatest value that $\mathsf{R}\left(\frac{u\pi i}{\omega}\right)$ can take for values of u within the region in question and m the smallest, it is always possible to find an integer n_0, say, such that

$$-n_0\beta\pi<m$$

and

$$M<n_0\beta\pi.$$

Hence for this region there exists the inequality

$$-n_0\beta\pi<\mathsf{R}\left(\frac{u\pi i}{\omega}\right)<n_0\beta\pi,$$

and consequently, since $|h|=e^{-\beta\pi}$, it follows that

$$|h^{n_0}|<|t|<|h^{-n_0}|.$$

Since $F(t)=-\frac{\pi^2}{\omega^2}\frac{t}{(t-1)^2}$, it is seen that in the first term of the infinite series (S′) there appears $(1-t)^2$ in the denominator: in the second

term there appears $(1 - h^2t)^2$ in the denominator; in the third term there appears $(1 - h^4t)^2$ in the denominator; $\cdots$.

The greatest absolute value that t can take within the fixed region being $< | h^{-n_0} |$, the greatest absolute value that $h^{2n}t$ can take in the same region is $< | h^{2n-n_0} |$. If then we choose $2n \geqq n_0$, then is $| h^{2n}t | < 1$. In the series (S′) we separate from the remaining series those terms (finite in number) in which h occurs to a power less than n_0.

The denominator in any of the remaining terms is

$$(1 - h^{2\lambda}t)^2,$$

where

$$| h^{2\lambda}t | < | h^{2n_0-n_0} |,$$

and consequently

$$1 - | h^{2\lambda}t | > 1 - | h^{n_0} |.$$

We therefore diminish the denominator of the terms in question if instead of $(1 - h^{2\lambda}t)^2$ we write $(1 - | h^{n_0} |)^2$, and consequently we increase the value of the term $F(h^{2\lambda}t)$.

The numerators of the terms which have been separated from the first n_0 terms are

$$h^{2n_0}t, \qquad h^{2n_0+2}t, \qquad h^{2n_0+4}t, \; \cdots ,$$

which is a geometrical series whose common ratio is less than unity. It follows that the series (S′) is absolutely convergent for the region in question. It follows also (see Osgood's *Lehrbuch der Funktionentheorie*, pp. 72, 259) that this series is uniformly convergent and represents an analytic function. The terms

$$F(t) + F(h^2t) + F(h^4t) + \cdots ,$$

which also belong to the series (S′) but which were not taken into consideration above, do not affect the question of convergence, since they constitute a finite number of finite terms.

We shall next establish the convergence of the series

$$\text{(S}''\text{)} \qquad F(t) + F(h^{-2}t) + F(h^{-4}t) + \cdots .$$

We may write

$$F(t) = -\frac{\pi^2}{\omega^2}\frac{t^{-1}}{(1 - t^{-1})^2},$$

$$F(h^{-2}t) = -\frac{\pi^2}{\omega^2}\frac{h^2t^{-1}}{(1 - h^2t^{-1})^2},$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

By separating a finite number of these terms from the series (S″) it may be shown as above that the remaining terms are less than the corresponding terms of a decreasing geometrical series.

It follows that the series

$$\text{(S)} \qquad \begin{matrix} & + F(h^2 t) + F(h^4 t) + \cdots \\ F(t) & \\ & + F(h^{-2} t) + F(h^{-4} t) + \cdots \end{matrix}$$

is absolutely and uniformly convergent in any interval that is free from the points $u = 0$, $u = w$.

This series therefore represents a one-valued doubly periodic function of u which for all finite values of u has the character of an integral or (fractional) rational function. At the points $u = 0$ and the congruent points this function becomes infinite of the second order.

ART. 286. We note that $F(0) = F(\infty) = 0$. It is also seen that the series (S) has the same periods and becomes infinite of the same order at the same points as the function $\wp u$. Two doubly periodic functions which in the finite portion of the plane have everywhere the character of an integral or (fractional) rational function and which become infinite of the same order at the same points can differ from each other only by a constant (Art. 83). Hence the above series can differ from $\wp u$ only by a constant, which constant it will appear later is $-\frac{\eta}{\omega}$.

Further, put z^2 for t, retaining the notation of Weierstrass, as no confusion can arise between the z used here and the z formerly employed.

It follows,* since $z = e^{\frac{u\pi i}{2\omega}}$, that

$$\wp u = -\frac{\eta}{\omega} - \frac{\pi^2}{\omega^2}\left\{\frac{1}{(z - z^{-1})^2} + \sum_{n=1}^{n=\infty}\frac{h^{2n}z^{-2}}{(1 - h^{2n}z^{-2})^2} + \sum_{n=1}^{n=\infty}\frac{h^{2n}z^2}{(1 - h^{2n}z^2)^2}\right\},$$

where $h = e^{\frac{\omega' \pi i}{\omega}} = q$.

In order to determine the constant η, it follows, when we expand

$$z = e^{\frac{u\pi i}{2\omega}} \quad \text{and} \quad z^{-1} = e^{-\frac{u\pi i}{2\omega}},$$

that
$$z - z^{-1} = \frac{u\pi i}{\omega}\left\{1 + \frac{1}{6}\left(\frac{u\pi i}{2\omega}\right)^2 + \cdots\right\},$$

and consequently

$$\frac{1}{(z - z^{-1})^2} = -\frac{\omega^2}{\pi^2 u^2}\left\{1 + \frac{1}{3 \cdot 4}\frac{u^2\pi^2}{\omega^2} + \cdots\right\}.$$

We note that

$$-\frac{\pi^2}{\omega^2}\left(\frac{1}{z - z^{-1}}\right)^2 = \frac{1}{u^2} + \frac{1}{3 \cdot 4}\frac{\pi^2}{\omega^2} + ((u^2)).$$

* See Schwarz, *Formeln und Lehrsätze*, etc., p. 10.

If we write this value in the above expression for $\wp u$, we have

$$\frac{1}{u^2} + * + ((u^2)) = -\frac{\eta}{\omega} + \frac{1}{u^2} + \frac{1}{12}\frac{\pi^2}{\omega^2} + ((u^2))$$
$$- \frac{\pi^2}{\omega^2}\sum_{n=1}^{n=\infty}\frac{2h^{2n}}{(1-h^{2n})^2} + ((u^2)).$$

It follows that *

$$-\frac{\eta}{\omega}' + \frac{1}{12}\frac{\pi^2}{\omega^2} - \frac{\pi^2}{\omega^2}\sum_{n=1}^{n=\infty}\frac{2h^{2n}}{(1-h^{2n})^2} = 0,$$

or

$$2\eta\omega = \pi^2\left\{\frac{1}{6} - \sum_{n=1}^{n=\infty}\frac{4h^{2n}}{(1-h^{2n})^2}\right\}.$$

The above expression for $\wp u$ is not unique, since the period 2ω may be chosen in an indefinitely large number of ways.

ART. 287. Since the series derived in the last Article is uniformly convergent, we may integrate term by term. If in this integration we make a suitable choice of the constants, we again have a convergent series.

Multiplying the series by $-du$, it follows through integration that

$$\zeta u = \frac{\sigma'}{\sigma}(u) = \frac{1}{u} + * + ((u^2))$$
$$= \eta\frac{u}{\omega} + \frac{\pi i}{2\omega}\left[\frac{z+z^{-1}}{z-z^{-1}} + \sum_{n=1}^{n=\infty}\left\{\frac{2h^{2n}z^{-2}}{1-h^{2n}z^{-2}} - \frac{2h^{2n}}{1-h^{2n}}\right\}\right.$$
$$\left. - \sum_{n=1}^{n=\infty}\left\{\frac{2h^{2n}z^2}{1-h^{2n}z^2} - \frac{2h^{2n}}{1-h^{2n}}\right\}\right],$$

where the choice of constants has been such that the constant terms occurring in the expressions under the summation signs, when expanded in ascending powers of u, are zero, this being already the case on the left-hand side of the equation.

The above formula simplified may be written †

$$\zeta u = \frac{\sigma'}{\sigma}(u) = \frac{\eta}{\omega}u + \frac{\pi i}{2\omega}\left\{\frac{z+z^{-1}}{z-z^{-1}} + \sum_{n=1}^{n=\infty}\frac{2h^{2n}z^{-2}}{1-h^{2n}z^{-2}} - \sum_{n=1}^{n=\infty}\frac{2h^{2n}z^2}{1-h^{2n}z^2}\right\}.$$

If with Eisenstein (*loc. cit.*, p. 215) we note that

$$\frac{2h^{2n}z^{-2}}{1-h^{2n}z^{-2}} = \frac{2h^{2n}z^{-2}}{1-h^{2n}z^{-2}} + 1 - 1 = \frac{zh^{-n} + z^{-1}h^n}{zh^{-n} - z^{-1}h^n} - 1,$$

* Schwarz, *loc. cit.*, p. 8.
† Schwarz, *loc. cit.*, p. 10.

and further that

$$z = e^{\frac{\pi i u}{2\omega}}, \quad h = e^{\pi i \frac{\omega'}{\omega}}, \quad zh^{-n} = e^{\frac{\pi i}{2\omega}(u - 2n\omega')},$$

$$\frac{z + z^{-1}}{2} = \cos\frac{\pi}{2u}\omega, \quad \frac{z - z^{-1}}{2i} = \sin\frac{\pi}{2u}\omega,$$

it is seen that the above expression may be written *

$$\zeta u = \frac{\sigma'}{\sigma}(u) = \frac{\eta}{\omega}u + \frac{\pi}{2\omega}\left[\cot\frac{u\pi}{2\omega} + \sum_{n=1}^{n=\infty}\left\{\cot\frac{\pi}{2\omega}(u - 2n\omega') - i\right\}\right.$$
$$\left. + \sum_{n=1}^{n=\infty}\left\{\cot\frac{\pi}{2\omega}(u + 2n\omega') + i\right\}\right].$$

It is evident that the constant term of the series is zero; for if u is changed into $-u$, the right-hand side of the series takes its opposite value and is consequently an *odd* function of u.

If u is increased by 2ω, the quantity z becomes $-z$, for

$$e^{\frac{(u+2\omega)\pi i}{2\omega}} = e^{\pi i} e^{\frac{u\pi i}{2\omega}} = -z.$$

It follows at once that

$$\frac{\sigma'}{\sigma}(u + 2\omega) = 2\eta + \frac{\sigma'}{\sigma}(u),$$

or

$$\zeta(u + 2\omega) = \zeta u + 2\eta.$$

Writing $u = -\omega$ in this formula we have (cf. Art. 258)

$$\zeta\omega = \eta,$$

where η is finite since $\frac{\sigma'}{\sigma}(\omega)$ is finite.

We saw that

$$\wp(u + 2\omega') = \wp u.$$

Multiply both sides of this formula by $-du$ and integrate. It follows that

$$\frac{\sigma'}{\sigma}(u + 2\omega') = \frac{\sigma'}{\sigma}(u) + 2\eta',$$

or

$$\zeta(u + 2\omega') = \zeta u + 2\eta',$$

where η' is the constant of integration.

Again writing $u = -\omega'$, we have

$$\zeta\omega' = \eta'.$$

By interchanging ω and ω' in the preceding Article, it may be shown that

$$2\eta'\omega' = \pi^2\left\{\frac{1}{6} - \sum_{n=1}^{n=\infty}\frac{4h_0^{2n}}{(1 - h_0^{2n})^2}\right\},$$

where $h_0 = e^{\pi i \frac{\omega}{\omega'}}$.

* Schwarz, *loc. cit.*, p. 10; see also Halphen, *Fonct. Ellipt.*, t. I, p. 425; Tannery et Molk, *Fonct. Ellipt.*, t. II, p. 237.

From the formula

$$\wp u = -\frac{d^2}{du^2}\log \sigma u = -\frac{d}{du}\frac{\sigma'}{\sigma}(u)$$

it follows that (cf. Art. 258)

$$\zeta u = \frac{\sigma'}{\sigma}(u) = -\int^u \wp u\, du = \int^s \frac{s ds}{\sqrt{S}},$$

where $\wp u = s$ and $du = -\dfrac{ds}{\sqrt{S}}$.

The constant of integration on the right-hand side is so chosen that for sufficiently large values of s the series on the right-hand side is (cf. Art. 279)

$$\zeta u = \int^s \frac{s ds}{\sqrt{S}} = \sqrt{s}\left[1 + * - \frac{g_2}{24}\frac{1}{s^2} - \frac{g_3}{40}\frac{1}{s^3} + \cdots\right].$$

ART. 288. If u is increased by $2\omega'$, then $z = e^{\frac{u\pi i}{2\omega}}$ becomes $z \cdot h$. We consequently have

$$\zeta(u + 2\omega') = \zeta u + 2\eta' = \frac{\eta}{\omega}u + \frac{\pi i}{2\omega}\left\{\frac{hz + h^{-1}z^{-1}}{hz - h^{-1}z^{-1}} + \sum_{n=1}^{n=\infty}\frac{2h^{2n-2}z^{-2}}{1 - h^{2n-2}z^{-2}}\right.$$
$$\left. - \sum_{n=1}^{n=\infty}\frac{2h^{2n+2}z^2}{1 - h^{2n+2}z^2}\right\} + \frac{2\eta\omega'}{\omega}.$$

Comparing this formula with the one given above for ζu, we note that here under the first summation the new initial term is

$$\frac{2z^{-2}}{1 - z^{-2}}, \text{ which may be written } = \frac{z + z^{-1}}{z - z^{-1}} - 1,$$

and consequently the first summation is transformed into

$$\frac{z + z^{-1}}{z - z^{-1}} - 1 + \sum_{n=1}^{n=\infty}\frac{2h^{2n}z^{-2}}{1 - h^{2n}z^{-2}},$$

while the second summation

$$-\sum_{n=1}^{n=\infty}\frac{2h^{2n+2}z^2}{1 - h^{2n+2}z^2} \text{ becomes } \frac{2h^2z^2}{1 - h^2z^2} - \sum_{n=1}^{n=\infty}\frac{2h^{2n}z^2}{1 - h^{2n}z^2}.$$

We further note that

$$\frac{hz + h^{-1}z^{-1}}{hz - h^{-1}z^{-1}} = \frac{h^2z^2 + 1}{h^2z^2 - 1} = -\frac{h^2z^2 + 1}{1 - h^2z^2}.$$

It follows at once that

$$\zeta(u + 2\omega) = \zeta u + 2\eta' = \zeta u + \frac{2\eta\omega'}{\omega} + \frac{\pi i}{2}\left\{-\frac{h^2z^2 + 1}{1 - h^2z^2} - 1 + \frac{2h^2z^2}{1 - h^2z^2}\right\},$$

so that

$$2\eta' = \frac{2\eta\omega'}{\omega} + \frac{\pi i}{2}\left\{\frac{h^2z^2-1}{1-h^2z^2} - 1\right\},$$

or finally (cf. Art. 259)

$$\eta\omega' - \eta'\omega = \tfrac{1}{2}\pi i.$$

We have assumed always (Art. 86) that $\mathsf{R}\left(\frac{\omega'}{\omega i}\right) > 0$.

ART. 289. Following a method given by Forsyth (*Theory of Functions*, p. 257) we offer another method of proving the formula last written.

Consider the period-parallelogram with vertices 0, 2ω, $2\omega'$, $2\omega'' = 2\omega + 2\omega'$.

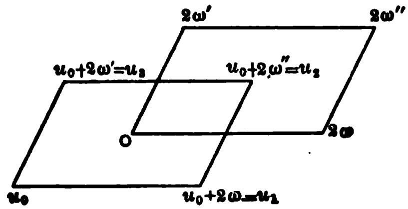


Fig. 71.

By sliding this parallelogram parallel with itself, it may be caused to take a position such that for all points on its boundary and within the interior (except the point $u = 0$) the function ζu has the character of an integral function, being of the form

$$\zeta u = \frac{1}{u} - \frac{1}{60}g_2u^3 - \frac{1}{140}g_3u^5 - \cdots.$$

It follows that

$$\int \zeta u\, du = 2\pi i,$$

where the integration has been taken over a small circle about $u = 0$.

Since this integral is the same as that taken over the parallelogram $u_0u_1u_2u_3$, we have

$$2\pi i = \int_{u_0}^{u_1}\zeta u\, du + \int_{u_1}^{u_2}\zeta u\, du + \int_{u_2}^{u_3}\zeta u\, du + \int_{u_3}^{u_0}\zeta u\, du;$$

or

$$2\pi i = \int_{u_0}^{u_1}[\zeta u - \zeta(u+2\omega')]du + \int_{u_0}^{u_3}[\zeta(u+2\omega) - \zeta u]du$$

$$= \int_{u_0}^{u_1} - 2\eta'\, du + \int_{u_0}^{u_3} 2\eta\, du = -4\eta'\omega + 4\eta\omega'.$$

ART. 290. If we multiply by du the expression

$$\frac{\sigma'}{\sigma}(u+2\omega) = \frac{\sigma'}{\sigma}(u) + 2\eta,$$

we have through integration

$$\log\sigma(u+2\omega) = \log\sigma u + 2\eta u + c,$$

or

$$\sigma(u+2\omega) = \sigma u e^{2\eta u}e^{c}.$$

If the value $-\omega$ is given to u, it is seen that

$$e^c = -e^{2\eta\omega}.$$

We consequently have

$$\sigma(u+2\omega) = -e^{2\eta(u+\omega)}\sigma(u).$$

If $-u$ is written for u in this formula, we have

$$\sigma(u-2\omega) = -e^{-2\eta(u-\omega)}\sigma u.$$

Combining these formulas into one formula, we may write

$$\text{(A)} \qquad \sigma(u \pm 2\omega) = -e^{\pm 2\eta(u\pm\omega)}\sigma(u).$$

In a similar manner it may be shown that

$$\text{(B)} \qquad \sigma(u \pm 2\omega') = -e^{\pm 2\eta'(u\pm\omega')}\sigma(u).$$

Further, if $2\tilde{\omega} = 2p\omega + 2q\omega'$, where p and q are positive or negative integers (including zero), it is seen that

$$\sigma(u+2\tilde{\omega}) = \sigma(u+2p\omega+2q\omega') = Ce^{2p\eta u}e^{2q\eta' u}\sigma u.$$

Writing

$$2p\eta + 2q\eta' = 2\tilde{\eta},$$

it follows that

$$\sigma(u+2\tilde{\omega}) = Ce^{2\tilde{\eta}u}\sigma u.$$

To determine the constant C, write $u = -\tilde{\omega} + v$, where v is a very small quantity. It follows that

$$\sigma(\tilde{\omega}+v) = -Ce^{-2\tilde{\eta}\tilde{\omega}+2\tilde{\eta}v}\sigma(\tilde{\omega}-v).$$

If we develop by Taylor's Theorem, it is seen that

$$\text{(C)} \qquad \sigma(\tilde{\omega}+v) = \sigma(\tilde{\omega}) + v\sigma'(\tilde{\omega}) + \cdots = -Ce^{-2\tilde{\eta}\tilde{\omega}+2\tilde{\eta}v}\sigma(\tilde{\omega}-v).$$

Two cases are possible:

(1) either $|\sigma(\tilde{\omega})| > 0$, or

(2) $|\sigma(\tilde{\omega})| = 0$.

In the first case we have by writing $v = 0$,

$$\sigma(\tilde{\omega}) = -Ce^{-2\tilde{\eta}\tilde{\omega}}\sigma(\tilde{\omega}).$$

It follows that

$$C = -e^{2\tilde{\eta}\tilde{\omega}},$$

and consequently

$$\sigma(u+2\tilde{\omega}) = -e^{2\tilde{\eta}(u+\tilde{\omega})}\sigma(u).$$

In the second case we have by developing both sides of (C)

$$\sigma'(\tilde{\omega}) + ((v)) = Ce^{-2\tilde{\eta}\tilde{\omega}+2\tilde{\eta}v}[\sigma'(\tilde{\omega}) + ((v))];$$

or by making $v = 0$,

$$C = +e^{2\tilde{\eta}\tilde{\omega}}.$$

It follows that

$$\sigma(u+2\tilde{\omega}) = \pm e^{2\tilde{\eta}(u+\tilde{\omega})}\sigma(u)$$

according as we have case (2) or case (1) respectively.

The quantity $\sigma(\tilde{\omega})$ vanishes when p and q are even integers. We may therefore write the general formula

$$\sigma(u + 2p\omega + 2q\omega') = (-1)^{pq+p+q}e^{2(p\eta+q\eta')(u+p\omega+q\omega')}\sigma u.$$

ART. 291. We derived in Art. 287 the formula

$$\frac{\sigma'}{\sigma}(u) = \zeta u = \frac{\eta}{\omega}u + \frac{\pi i}{2\omega}\left\{\frac{z+z^{-1}}{z-z^{-1}} + \sum_n \frac{2h^{2n}z^{-2}}{1-h^{2n}z^{-2}} - \sum_n \frac{2h^{2n}z^2}{1-h^{2n}z^2}\right\},$$

which is uniformly convergent within the period-parallelogram (vertices excluded). If this series is integrated term by term, it follows that

$$\log\sigma u = \frac{1}{2}\frac{\eta}{\omega}u^2 + C + \log\frac{z-z^{-1}}{2i} + \sum_n \log\frac{1-h^{2n}z^{-2}}{1-h^{2n}} + \sum_n \log\frac{1-h^{2n}z^2}{1-h^{2n}}.$$

When $u = 0$, we have $z = 1$, so that

$$[\log\sigma u]_{u=0} = C + \left[\log\sin\frac{\pi u}{2\omega}\right]_{u=0},$$

or

$$\log\left[\frac{u + ((u^5))}{\frac{\pi u}{2\omega} + ((u^3))}\right]_{u=0} = C.$$

It follows that *

$$C = \log\left(\frac{2\omega}{\pi}\right)$$

and

$$(1) \qquad \sigma u = \frac{2\omega}{\pi}e^{2\eta\omega v^2}\frac{z-z^{-1}}{2i}\prod_n\frac{1-h^{2n}z^{-2}}{1-h^{2n}}\prod_n\frac{1-h^{2n}z^2}{1-h^{2n}},$$

where $u = 2\omega v$.

Writing $\frac{\omega'}{\omega} = \tau$, it is seen that

$$\frac{1-h^{2n}z^{-2}}{1-h^{2n}} = \frac{\sin[(v-\tau n)\pi]}{\frac{h^{-n}-h^n}{2i}}z^{-1} = \frac{\sin[(\tau n - v)\pi]}{\sin n\tau\pi}z^{-1},$$

with a similar formula for $\frac{1-h^{2n}z^2}{1-h^{2n}}$.

It follows that †

$$(2) \qquad \sigma u = e^{2\eta\omega v^2}\frac{2\omega}{\pi}\sin v\pi\prod_n\frac{\sin[(n\tau-v)\pi]}{\sin n\tau\pi}e^{-v\pi i}\prod_n\frac{\sin[(n\tau+v)\pi]}{\sin n\tau\pi}e^{v\pi i},$$

or

$$\sigma u = e^{2\eta\omega v^2}\frac{2\omega}{\pi}\sin\pi v\prod_{n=1}^{n=\infty}\left(1 - \frac{\sin^2\pi v}{\sin^2\pi n\tau}\right).$$

* Compare this function with Eisenstein's χ-function, *loc. cit.*, p. 216.

† Schwarz, *loc. cit.*, p. 8. Formulas (2) and (3) are precisely the same as those derived by Jacobi for H(u) [Werke, I, pp. 141–142].

The formula (2) may be written

$$\sigma u = e^{2\eta\omega v^2}\frac{2\omega}{\pi}\sin v\pi \prod_n \frac{1 - 2h^{2n}\cos 2v\pi + h^{4n}}{(1-h^{2n})^2}. \tag{3}$$

Since 2ω may be chosen in an infinite number of ways, it is seen that σu may be expressed in an indefinite number of ways in the form of a simply infinite product. Through logarithmic differentiation of formula (3) it follows that

$$\zeta u = \frac{\pi}{2\omega}\cot v\pi + 2\eta v + \frac{2\pi}{\omega}\sum_{n=1}^{n=\infty}\frac{h^{2n}\sin 2v\pi}{1 - 2h^{2n}\cos 2v\pi + h^{4n}}.$$

Noting that

$$\frac{1}{1-u} = 1 + u + u^2 + \cdots + u^m + \cdots \qquad (|u| < 1),$$

it is evident, if $u = r(\cos\theta + i\sin\theta)$, that

$$\frac{2r\sin\theta}{1 - 2r\cos\theta + r^2} = \sum_{m=1}^{m=\infty} 2r^m \sin m\theta,$$

an identity which is true for complex as well as for real values of r.

If we put $r = h^{2n}$, we have

$$\sum_{n=1}^{n=\infty}\frac{h^{2n}\sin\theta}{1 - h^{2n}\cos\theta + h^{4n}} = \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} h^{2nm}\sin n\theta$$
$$= \sum_{n=1}^{n=\infty}\frac{h^{2n}}{1-h^{2n}}\sin n\theta;$$

and consequently

$$\zeta u = \frac{\pi}{2\omega}\cot v\pi + \frac{\eta u}{\omega} + \frac{2\pi}{\omega}\sum_{n=1}^{n=\infty}\frac{h^{2n}}{1-h^{2n}}\sin 2nv\pi.$$

If we differentiate with regard to u, we have

$$\wp u = \frac{\pi^2}{4\omega^2}\operatorname{cosec}^2 v\pi - \frac{\eta}{\omega} - \frac{2\pi^2}{\omega^2}\sum_{n=1}^{n=\infty}\frac{nh^{2n}}{1-h^{2n}}\cos 2nv\pi. \tag{A}$$

The right-hand side of this equation is

$$\wp u = \frac{1}{u^2} + \frac{g_2}{20}u^2 + \frac{g_3}{28}u^4 + \cdots,$$

while the expansion of $\operatorname{cosec}^2 t$ is

$$\operatorname{cosec}^2 t = \frac{1}{t^2} + \frac{1}{3} + \frac{4}{3\cdot 20}t^2 + \frac{8}{27\cdot 28}t^4 + \cdots.$$

By equating like powers of u on either side of (A), we have *

$$\left(\frac{\omega}{\pi}\right)^4 g_2 = \frac{1}{12} + 20\sum_{n=1}^{n=\infty}\frac{n^3h^{2n}}{1-h^{2n}}, \quad \left(\frac{\omega}{\pi}\right)^6 g_3 = \frac{1}{216} - \frac{7}{3}\sum_{n=1}^{n=\infty}\frac{n^5h^{2n}}{1-h^{2n}}.$$

* Harkness and Morley, *Theory of Functions*, p. 321; Halphen, *Fonct. Ellipt.*, t. I, Chap. 13

ART. 292. *Homogeneity.* — Write the functions σu, ζu, $\wp u$ in the forms

$$\sigma u = \sigma(u;\, \omega, \omega') = \sigma(u;\, g_2, g_3),$$
$$\zeta u = \zeta(u;\, \omega, \omega') = \zeta(u;\, g_2, g_3),$$
$$\wp u = \wp(u;\, \omega, \omega') = \wp(u;\, g_2, g_3).$$

It follows at once from the infinite product through which the function σu is defined (Art. 272) that

$$\sigma(\lambda u;\, \lambda\omega, \lambda\omega') = \lambda\sigma(u;\, \omega, \omega'),$$

where λ is any quantity real or imaginary.

We also have

$$\sigma'(\lambda u;\, \lambda\omega, \lambda\omega') = \sigma'(u;\, \omega, \omega'),$$

and consequently

$$\zeta(\lambda u;\, \lambda\omega, \lambda\omega') = \frac{1}{\lambda}\zeta(u;\, \omega, \omega'),$$

$$\wp(\lambda u;\, \lambda\omega, \lambda\omega') = \frac{1}{\lambda^2}\wp(u;\, \omega, \omega').$$

In the formulas

$$g_2 = 2^2 \cdot 3 \cdot 5 \sum{}' \frac{1}{w^4}, \quad g_3 = 2^2 \cdot 5 \cdot 7 \sum{}' \frac{1}{w^6},$$

when ω and ω' are replaced by $\lambda\omega$ and $\lambda\omega'$, w becomes λw, so that g_2 and g_3 are transformed into

$$\frac{g_2}{\lambda^4} \text{ and } \frac{g_3}{\lambda^6}.$$

It is also seen that

$$\sigma\left(\lambda u; \frac{g_2}{\lambda^4}, \frac{g_3}{\lambda^6}\right) = \lambda\sigma(u; g_2, g_3),$$

$$\wp\left(\lambda u; \frac{g_2}{\lambda^4}, \frac{g_3}{\lambda^6}\right) = \frac{1}{\lambda^2}\wp(u; g_2, g_3).$$

The above formulas are particularly useful when in Volume II we make a distinction between the real and imaginary values of the argument.

ART. 293. *Degeneracy.* — When $\omega' = \infty$, we saw in Art. 283 that

$$\wp u = \left(\frac{\pi}{2\,\omega}\right)^2 \left[\frac{1}{\sin^2 \dfrac{\pi u}{2\,\omega}} - \frac{1}{3}\right].$$

We further have

$$g_2 = 2^2 \cdot 3 \cdot 5 \sum{}' \frac{1}{2^4 m^4 \omega^4}, \quad g_3 = 2^2 \cdot 5 \cdot 7 \sum{}' \frac{1}{2^6 m^6 \omega^6}.$$

From Chapter I we have

$$\sum{}' \frac{1}{m^4} = \frac{\pi^4}{3^2 \cdot 5}, \quad \sum{}' \frac{1}{m^6} = \frac{2\,\pi^6}{3^3 \cdot 5 \cdot 7}.$$

It follows that

$$g_2=\frac{1}{3}\left(\frac{\pi^2}{2\,\omega^2}\right)^2,\quad g_3=\frac{1}{3^3}\left(\frac{\pi^2}{2\,\omega^2}\right)^3,$$

and consequently

$$\Delta = g_2{}^3 - 27\,g_3{}^2 = 0.$$

The discriminant being zero, the roots of the polynomial

$$4\,s^3 - g_2 s - g_3 = 0 = 4(s-e_1)(s-e_2)(s-e_3)$$

are equal. Further, since

$$e_1+e_2+e_3=0 \quad\text{and}\quad e_1>e_2>e_3,$$

the quantity e_1 must be positive and e_3 negative.

Two cases are possible: *either* e_2 coincides with e_3, *or* e_2 coincides with e_1.

In the *first* case: $e_2=e_3=-\frac{1}{2}e_1$; $g_2=3\,e_1{}^2$, $g_3=e_1{}^3$, $g_3>0$;

$$e_2=e_3=-\frac{3\,g_3}{2\,g_2};\quad \left(\frac{\pi}{2\,\omega}\right)^2=\frac{9\,g_3}{2\,g_1};\quad k^2=\frac{e_2-e_3}{e_1-e_3}=0,\quad sn\,u=\sin u,\quad K=\frac{\pi}{2}.$$

We also have

$$\zeta u=\frac{\sigma' u}{\sigma u}=\frac{\pi}{2\,\omega}\cot\frac{\pi u}{2\,\omega}+\frac{1}{3}\left(\frac{\pi}{2\,\omega}\right)^2 u,$$

$$2\,\eta\omega=\frac{\pi^2}{6},$$

$$\sigma u = e^{\frac{1}{6}\left(\frac{\pi u}{2\omega}\right)^2}\frac{2\,\omega}{\pi}\sin\frac{\pi u}{2\,\omega}.$$

In the *second* case: $e_2=e_1=-\frac{1}{2}e_3$; $g_3<0$,

$$g_2=3\,e_3{}^2,\quad g_3=e_3{}^3;\quad k=1,\quad sn\,u=\frac{e^u-e^{-u}}{e^u+e^{-u}},\quad K=\infty,\quad \omega=\infty.$$

$$\wp u=\frac{3\,g_3}{g_2}-\frac{9}{2}\frac{g_3}{g_2}\left(\frac{e^v+e^{-v}}{e^v-e^{-v}}\right)^2,\quad \text{where } v=iu\sqrt{\frac{9\,g_3}{2\,g_2}},$$

$$\zeta u=\frac{3\,g_3}{2\,g_2}u-\frac{\pi i}{2\,\omega'}\frac{e^{-\pi t}+e^{\pi t}}{e^{-\pi t}-e^{\pi t}},\quad \text{where}\quad t=\frac{iu}{2\,\omega'},$$

$$\eta'\omega'=\frac{\pi^2}{12},$$

$$\sigma u=\frac{2\,\omega'}{i\pi}\,\frac{e^{\frac{\pi i u}{2\omega'}}-e^{-\frac{\pi i u}{2\omega'}}}{2}\,e^{-\frac{1}{6}\left(\frac{\pi i u}{2\omega'}\right)^2}.$$

When the roots of the polynomial

$$4\,s^3-g_2 s-g_3=0$$

are equal, it may be shown directly that the values of $s=\wp u$ derived from the integral

(1)
$$u=\int_s^\infty\frac{ds}{\sqrt{4\,s^3-g_2 s-g_3}}$$

agree with the results above.

When both periods are infinite, then $g_2 = 0 = g_3$ and $e_1 = 0 = e_2 = e_3$. The integral (1) becomes

$$u = \int_s^\infty \frac{ds}{\sqrt{4\,s^3}}, \quad \text{or} \quad s = \frac{1}{u^2} = \wp u,$$

$$\zeta u = \frac{1}{u}, \quad \sigma u = u.$$

EXAMPLES

1. By making $\omega' = \infty$ in the formula

$$\sigma u = u \prod_w{}' \left\{ \left(1 - \frac{u}{w}\right) e^{\frac{u}{w} + \frac{1}{2}\frac{u^2}{w^2}} \right\}$$

derive the results of Arts. 283 *et seq.* (Halphen, *loc. cit.*, Chap. 13).

2. If
$$I = \int_0^\infty \frac{ds}{\sqrt{4\,s^3 - g_2 s}},$$

show that

$$2\sqrt{2}\,\sqrt[4]{-g_2}\,I = \mathrm{B}\left(\frac{1}{4}, \frac{1}{4}\right) = \frac{\Gamma\left(\frac{1}{4}\right)^2}{\Gamma\left(\frac{1}{2}\right)}.$$

3. If $F(t)$ is any rational function of $t = e^{\frac{\pi i u}{\omega}}$, such that $F(0) = 0 = F(\infty)$, show that

$$\phi(u) = F(t) + \sum_{n=1}^{n=\infty} F(th^{2n}) + \sum_{n=1}^{n=\infty} F(th^{-2n}) + C$$

is a one-valued doubly period function of u.

CHAPTER XVI

THE ADDITION-THEOREMS

ARTICLE 294. It is the purport of this treatise to consider as far as possible the ultimate meaning of the functions which have been introduced. The simplest functional elements have been found in the Jacobi Theta-functions which are made the foundation of the theory. It is therefore natural *first* to develop the addition-theorems from this standpoint.

We have seen in Art. 90 that there exists a linear homogeneous equation with coefficients that are independent of the variable among any $n+1$ intermediary functions $\Phi(u)$ of the nth order, which have the same periods. We may next make an application of this theorem for the case $n = 2$. If in Art. 87 we write

$$a = 2K, \quad b = 2iK', \quad n = 2,$$

it follows that

$$\text{(I)} \qquad \begin{cases} \Phi(u + 2K) = \Phi(u), \\ \Phi(u + 2iK') = e^{-\frac{2\pi i}{K}(u+iK')}\Phi(u). \end{cases}$$

Among any three functions of the second order which satisfy these functional equations there must exist a linear homogeneous equation with coefficients that are independent of the variable.*

Three such functions are

$$\Theta^2(u), \quad \mathrm{H}^2(u) \quad \text{and} \quad \Theta(u-v)\Theta(u+v),$$

where v is an arbitrary parameter.

It follows that

$$C\Theta(u+v)\Theta(u-v) + C_1\Theta^2(u) + C_2\mathrm{H}^2(u) = 0,$$

where the C's are quantities independent of u. The C's may, however, be functions of v.

None of these quantities can be zero; if, for example, $C = 0$, we would have

$$\frac{\mathrm{H}(u)}{\Theta(u)} = \text{Constant},$$

which is not true.

* See Hermite in Serret's *Calcul*, t. II, p. 797; and Koenigsberger, *Elliptische Functionen*, p. 368.

Writing

$$\frac{C_1}{C} = -f(v), \frac{C_2}{C} = -g(v),$$

we have

$$\Theta(u+v)\Theta(u-v) = f(v)\Theta^2(u) + g(v)\mathrm{H}^2(u).$$

If we consider $f(v)\Theta^2(u) + g(v)\mathrm{H}^2(u)$ as a function of v, say $\Psi(v)$, we have

$$\Psi(v) = \Theta(u+v)\Theta(u-v).$$

It follows that

$$\Psi(v+2K) = \Psi(v)$$

and

$$\Psi(v+2iK') = e^{-\frac{2\pi i}{K}(v+iK')}\Psi(v),$$

from which it is seen that $\Psi(v)$ satisfies the functional equations (I).

If we write $v + 2K$ in the equation

(II) $$\Theta(u+v)\Theta(u-v) = f(v)\Theta^2(u) + g(v)\mathrm{H}^2(u),$$

we have

$$\Theta(u+v)\Theta(u-v) = f(v+2K)\Theta^2(u) + g(v+2K)\mathrm{H}^2(u);$$

and consequently through subtraction it follows that

$$[f(v+2K) - f(v)]\Theta^2(u) + [g(v+2K) - g(v)]\mathrm{H}^2(u) = 0.$$

As this relation is true for all values of u, we must have

$$f(v+2K) = f(v),$$
$$g(v+2K) = g(v).$$

On the other hand, if in the equation (II) we write $v + 2iK'$ for v, we have in a similar manner

$$f(v+2iK') = e^{\frac{2\pi i}{K}(v+iK')}f(v),$$

$$g(v+2iK') = e^{\frac{2\pi i}{K}(v+iK')}g(v).$$

It follows that $f(v)$ and $g(v)$ satisfy the functional equations (I) that were satisfied by $\Theta^2(u)$ and $\mathrm{H}^2(u)$.

We thus have the following relations:

$$f(v) = \alpha\mathrm{H}^2(v) + \beta\Theta^2(v),$$
$$g(v) = \gamma\mathrm{H}^2(v) + \delta\Theta^2(v),$$

where $\alpha, \beta, \gamma, \delta$ are constants.

When these relations are written in the equation above, we have

(1) $$\Theta(u+v)\Theta(u-v) = [\alpha\mathrm{H}^2(v) + \beta\Theta^2(v)]\Theta^2(u) + [\gamma\mathrm{H}^2(v) + \delta\Theta^2(v)]\mathrm{H}^2(u).$$

To determine the constants α, β, γ, δ, write $v = 0$. We then have

$$\Theta^2(u) = \beta\Theta^2(0)\Theta^2(u) + \delta\Theta^2(0)\,\mathrm{H}^2(u),$$

or

$$\Theta^2(u)[1 - \beta\Theta^2(0)] = \delta\Theta^2(0)\,\mathrm{H}^2(u),$$

a relation which can exist only if

$$1 - \beta\Theta^2(0) = 0 \quad \text{and} \quad \delta\Theta^2(0) = 0.$$

We thus have

$$\beta = \frac{1}{\Theta^2(0)} \quad \text{and} \quad \delta = 0.$$

If next we write $u = 0$ in the above equation, we have $\alpha = 0$. To determine γ, we write the values of α, β, δ just found, in (1), then write $u = v + iK'$ and note that $\Theta(iK') = 0$. It follows that

$$\gamma = -\frac{1}{\Theta^2(0)}.$$

These values of α, β, γ, δ when written in the equation (1) give us the formula

$$\Theta^2(0)\,\Theta(u+v)\,\Theta(u-v) = \Theta^2(v)\,\Theta^2(u) - \mathrm{H}^2(v)\,\mathrm{H}^2(u),$$

which is fundamental in the Jacobi theory (see Jacobi, Werke, I, p. 227. formula 20).

ART. 295. We introduced in Art. 208 the following notation:

$$\begin{aligned}
\Theta(2Ku) &= \vartheta_0(u), & \vartheta_0(0) &= \vartheta_0,\\
\mathrm{H}(2Ku) &= \vartheta_1(u), & \vartheta_1(0) &= \vartheta_1,\\
\mathrm{H}_1(2Ku) &= \vartheta_2(u), & \vartheta_2(0) &= \vartheta_2,\\
\Theta_1(2Ku) &= \vartheta_3(u), & \vartheta_3(0) &= \vartheta_3.
\end{aligned}$$

We also saw in Art. 215 that

$$\frac{1}{\sqrt{k}} = \frac{\Theta_1(0)}{\mathrm{H}_1(0)} = \frac{\vartheta_3(0)}{\vartheta_2(0)} = \frac{\vartheta_3}{\vartheta_2},$$

$$\sqrt{k'} = \frac{\Theta(0)}{\Theta_1(0)} = \frac{\vartheta_0(0)}{\vartheta_3(0)} = \frac{\vartheta_0}{\vartheta_3};$$

and in Art. 217 that

$$sn\,2Ku = \frac{\vartheta_3}{\vartheta_2}\frac{\vartheta_1(u)}{\vartheta_0(u)},$$

$$cn\,2Ku = \frac{\vartheta_0}{\vartheta_2}\frac{\vartheta_2(u)}{\vartheta_0(u)},$$

$$dn\,2Ku = \frac{\vartheta_0}{\vartheta_3}\frac{\vartheta_3(u)}{\vartheta_0(u)}.$$

The addition formula above for the function Θ may be written

(1) $$\vartheta_0^2\vartheta_0(u+v)\,\vartheta_0(u-v) = \vartheta_0^2(u)\,\vartheta_0^2(v) - \vartheta_1^2(u)\,\vartheta_1^2(v),$$

if in the original formula we write $2Ku$ for u and $2Kv$ instead of v. In a similar manner we may derive

(2) $\vartheta_2\vartheta_3\vartheta_1(u+v)\vartheta_0(u-v) = \vartheta_1(u)\vartheta_0(u)\vartheta_2(v)\vartheta_3(v) + \vartheta_2(u)\vartheta_3(u)\vartheta_1(v)\vartheta_0(v)$,

(3) $\vartheta_0\vartheta_2\vartheta_2(u+v)\vartheta_0(u-v) = \vartheta_0(u)\vartheta_2(u)\vartheta_0(v)\vartheta_2(v) - \vartheta_1(u)\vartheta_3(u)\vartheta_1(v)\vartheta_3(v)$,

(4) $\vartheta_0\vartheta_3\vartheta_3(u+v)\vartheta_0(u-v) = \vartheta_0(u)\vartheta_3(u)\vartheta_0(v)\vartheta_3(v) - \vartheta_2(u)\vartheta_1(u)\vartheta_2(v)\vartheta_1(v)$.

All four of the above formulas were also derived in the table (C) of Art. 211.

ART. 296. If we divide equation (2) above by (1) we have

$$\frac{\vartheta_2\vartheta_3}{\vartheta_0^2}\frac{\vartheta_1(u+v)\vartheta_0(u-v)}{\vartheta_0(u+v)\vartheta_0(u-v)} = \frac{\vartheta_1(u)\vartheta_0(u)\vartheta_2(v)\vartheta_3(v) + \vartheta_2(u)\vartheta_3(u)\vartheta_1(v)\vartheta_0(v)}{\vartheta_0^2(u)\vartheta_0^2(v) - \vartheta_1^2(u)\vartheta_1^2(v)},$$

that is,

$$\frac{\vartheta_2\vartheta_3}{\vartheta_0^2}\frac{\vartheta_2}{\vartheta_3} sn\,[2K(u+v)] = \frac{\frac{\vartheta_2}{\vartheta_3} sn\,2Ku\frac{\vartheta_2}{\vartheta_0} cn\,2Kv\frac{\vartheta_3}{\vartheta_0} dn\,2Kv + \frac{\vartheta_2}{\vartheta_0} cn\,2Ku\frac{\vartheta_3}{\vartheta_0} dn\,2Ku\frac{\vartheta_2}{\vartheta_3} sn\,2Kv}{1 - \frac{\vartheta_2^2}{\vartheta_3^2} sn^2\,2Ku\frac{\vartheta_2^2}{\vartheta_3^2} sn^2\,2Kv},$$

or

$$sn\,[2K(u+v)] = \frac{sn\,2Ku\,cn\,2Kv\,dn\,2Kv + cn\,2Ku\,dn\,2Ku\,sn\,2Kv}{1 - k^2\,sn^2\,2Ku\,sn^2\,2Kv}:$$

If we divide the equation (3) by (1) we have

$$cn\,[2K(u+v)] = \frac{cn\,2Ku\,cn\,2Kv - sn\,2Ku\,sn\,2Kv\,dn\,2Ku\,dn\,2Kv}{1 - k^2\,sn^2 2Ku\,sn^2 2Kv};$$

and similarly when (4) is divided by (1) we have

$$dn\,[2K(u+v)] = \frac{dn\,2Ku\,dn\,2Kv - k^2 sn^2\,2Ku\,sn\,2Kv\,cn\,2Ku\,cn\,2Kv}{1 - k^2 sn^2\,2Ku\,sn^2\,2Kv}.$$

If we write u and v for $2Ku$ and $2Kv$, we have

$$sn(u+v) = \frac{sn\,u\,cn\,v\,dn\,v + cn\,u\,dn\,u\,sn\,v}{1 - k^2 sn^2 u\,sn^2 v}.$$

Further, since

$$\frac{d}{du} sn\,u = cn\,u\,dn\,u,$$

it follows that

$$sn(u+v) = \frac{sn\,u\,\dfrac{d\,sn\,v}{dv} + sn\,v\,\dfrac{d\,sn\,u}{du}}{1 - k^2 sn^2 u\,sn^2 v}.$$

We have thus shown *that $sn(u+v)$ is a rational function of $sn\,u$, $sn\,v$ and the first derivatives of these functions* (see Art. 158).

Remark. — If for brevity in the formula above we put $sn\,u = s$, $sn\,v = s'$; $cn\,u = c$, $cn\,v = c'$; $dn\,u = d$, $dn\,v = d'$, it becomes

$$sn(u+v) = \frac{sc'd' + s'cd}{1 - k^2 s^2 s'^2}.$$

We further have

$$cn^2(u+v) = 1 - sn^2(u+v) = \frac{(1-k^2s^2s'^2)^2 - (sc'd' + s'cd)^2}{(1-k^2s^2s'^2)^2}$$

$$= \frac{(cc' - ss'dd')^2}{(1-k^2s^2s'^2)^2},$$

so that

$$cn(u+v) = \pm \frac{cc' - ss'dd'}{1 - k^2s^2s'^2}.$$

Writing $v = 0$, and consequently $s' = 0$ and $c' = 1$ in this formula it follows that $cn\, u = \pm c$, so that the positive sign must be taken. We may derive the formula for $dn(u+v)$ in a similar manner.

ART. 297. *Addition-theorem for the elliptic integrals of the second kind.* — From the formula

$$\Theta^2(0)\,\Theta(u+v)\,\Theta(u-v) = \Theta^2(u)\,\Theta^2(v) - \mathrm{H}^2(u)\,\mathrm{H}^2(v)$$

we have at once

$$\frac{\Theta^2(0)\,\Theta(u+v)\,\Theta(u-v)}{\Theta^2(u)\,\Theta^2(v)} = 1 - k^2 sn^2u\, sn^2v.$$

This formula differentiated logarithmically with respect to u and v respectively becomes

$$\frac{\Theta'(u+v)}{\Theta(u+v)} + \frac{\Theta'(u-v)}{\Theta(u-v)} - 2\frac{\Theta'(u)}{\Theta(u)} = -\frac{2k^2 sn\,u\, cn\,u\, dn\,u\, sn^2v}{1 - k^2sn^2u\, sn^2v},$$

$$\frac{\Theta'(u+v)}{\Theta(u+v)} - \frac{\Theta'(u-v)}{\Theta(u-v)} - 2\frac{\Theta'(v)}{\Theta(v)} = -\frac{2k^2 sn\,v\, cn\,v\, dn\,v\, sn^2u}{1 - k^2sn^2u\, sn^2v}.$$

Through addition we have

$$\frac{\Theta'(u+v)}{\Theta(u+v)} - \frac{\Theta'(u)}{\Theta(u)} - \frac{\Theta'(v)}{\Theta(v)} = -k^2\, sn\,u\, sn\,v\, sn(u+v).$$

Since

$$\mathrm{Z}(u) = \frac{\Theta'(u)}{\Theta(u)},$$

it follows that

$$\mathrm{Z}(u+v) = \mathrm{Z}(u) + \mathrm{Z}(v) - k^2 sn\,u\, sn\,v\, sn(u+v).$$

Noting that

$$\mathrm{Z}(u) = E(u) - u\frac{E}{K},$$

we also have

$$E(u+v) = E(u) + E(v) - k^2\, sn\,u\, sn\,v\, sn(u+v).$$

ADDITION-THEOREMS FOR THE WEIERSTRASSIAN FUNCTIONS.

ART. 298. The addition-theorem for the $\wp$-function may be derived as follows: We note that the difference

$$\wp u - \wp v$$

is a one-valued doubly periodic function which becomes infinite of the second order at the origin and the congruent points. For all other points this difference is finite. The points $u = \pm v + 2\mu\omega + 2\mu'\omega' (\mu, \mu'$ integers) are the zeros of the function $\wp u - \wp v$.

Another function which has the same zeros is

$$\phi(u) = \frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2 u}.$$

Further, since

$$\sigma(u + 2\omega) = -e^{2\eta(u+\omega)}\sigma u, \quad \sigma(u + 2\omega') = -e^{2\eta'(u+\omega')}\sigma u,$$

it follows that

$$\phi(u + 2\omega) = \phi(u)$$

and

$$\phi(u + 2\omega') = \phi(u).$$

We note that the functions $\phi(u)$ and $\wp u - \wp v$ have the same periods.

The developments of these functions in the neighborhood of the origin are

$$\wp u - \wp v = \frac{1}{u^2} - \wp v + ((u^2)),$$

$$\phi(u) = \left\{ -\frac{\sigma^2 v}{u^2} + ((u^2)) \right\}.$$

It is further seen that the function

$$\phi_1(u) = -\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2 u\,\sigma^2 v}$$

is doubly periodic and becomes infinite in the same manner and at the same points as $\wp u - \wp v$.

Other developments are

$$\sigma(v + u) = \sigma v + u\sigma' v + \frac{u^2}{2}\sigma'' v + \cdots,$$

$$\sigma(-v + u) = -\sigma v + u\sigma' v - \frac{u^2}{2}\sigma'' v + \cdots,$$

$$\phi_1(u) = \frac{\sigma^2 v + [\sigma v\sigma'' v - (\sigma' v)^2]u^2 + ((u^4))}{\sigma^2(v)u^2 + ((u^6))},$$

or

$$\phi_1(u) = \frac{1}{u^2} + \frac{\sigma v\sigma'' v - (\sigma' v)^2}{\sigma^2 v} + ((u^2)).$$

Since

$$\frac{\sigma v \sigma'' v - (\sigma' v)^2}{\sigma^2 v} = - \wp v,$$

we may write

$$\phi_1(u) = \frac{1}{u^2} - \wp v + ((u^2)).$$

This value substituted in

$$\wp u - \wp v - \phi_1(u) = \text{Constant},$$

shows that the constant is zero.

We therefore have

$$\wp u - \wp v = - \frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2 u \sigma^2 v},$$

a formula of great elegance and importance.*

ART. 299. If the formula above be differentiated logarithmically respectively with regard to u and v, we have

$$\text{(A)} \qquad \frac{\sigma'}{\sigma}(u+v) + \frac{\sigma'}{\sigma}(u-v) - 2\frac{\sigma'}{\sigma}(u) = \frac{\wp' u}{\wp u - \wp v}$$

and

$$\text{(B)} \qquad \frac{\sigma'}{\sigma}(u+v) - \frac{\sigma'}{\sigma}(u-v) - 2\frac{\sigma'}{\sigma}(v) = - \frac{\wp' v}{\wp u - \wp v}.$$

Through the addition and subtraction of formulas (A) and (B) are derived the formulas †

$$\text{(C)} \qquad \frac{\sigma'}{\sigma}(u+v) = \frac{\sigma'}{\sigma}(u) + \frac{\sigma'}{\sigma}(v) + \frac{1}{2}\frac{\wp' u - \wp' v}{\wp u - \wp v},$$

and

$$\text{(D)} \qquad \frac{\sigma'}{\sigma}(u-v) = \frac{\sigma'}{\sigma}(u) - \frac{\sigma'}{\sigma}(v) + \frac{1}{2}\frac{\wp' u + \wp' v}{\wp u - \wp v}.$$

These formulas are the addition-theorems for the function $\frac{\sigma'}{\sigma}(u) = \zeta(u)$. Compare them with those given in Art. 297. The function ζu does not have an algebraic addition-theorem. ‡

If we differentiate again the formula (C) with respect to u and v, we have

$$\text{(E)} \qquad \wp(u \pm v) = \wp u - \frac{1}{2}\frac{\partial}{\partial u}\left[\frac{\wp' u \mp \wp' v}{\wp u - \wp v}\right]$$

$$= \wp u - \frac{1}{2}\frac{(\wp u - \wp v)\wp'' u - (\wp' u \mp \wp' v)\wp' u}{(\wp u - \wp v)^2}$$

and

$$\text{(F)} \qquad \wp(u \pm v) = \wp v \mp \frac{1}{2}\frac{\partial}{\partial v}\left[\frac{\wp' u \mp \wp' v}{\wp u - \wp v}\right].$$

* See Schwarz, *loc. cit.*, p. 13.

† Schwarz, *loc. cit.*, p. 13.

‡ Daniels, *Amer. Journ. Math.*, Vol. VI, p. 268.

It follows, since

$$(\wp' u)^2 = 4\wp^3 u - g_2\wp u - g_3 \quad\text{and}\quad \wp'' u = 6\wp^2 u - \tfrac{1}{2}g_2,$$

that the formula (E) becomes

$$\text{(E}') \quad \wp(u\pm v) = \wp u - \frac{(\wp u - \wp v)(6\wp^2 u - \frac{1}{2}g_2) - 4\wp^3 u + g_2\wp u + g_3 \pm \wp' u\,\wp' v}{2(\wp u - \wp v)^2},$$

while formula (F) may be written

$$\text{(F}') \quad \wp(u\pm v) = \wp v + \frac{(\wp u - \wp v)(6\wp^2 v - \frac{1}{2}g_2) + 4\wp^3 v - g_2\wp v - g_3 \mp \wp' u\,\wp' v}{2(\wp u - \wp v)^2}.$$

We have thus expressed $\wp(u\pm v)$ rationally through $\wp u$, $\wp v$, $\wp' u$, $\wp' v$ (see again Art. 158).

ART. 300. Through the addition of the formulas (E′) and (F′) we have

$$\text{(G)} \qquad \wp(u\pm v) = \frac{2(\wp u\,\wp v - \frac{1}{4}g_2)(\wp u + \wp v) - g_3 \mp \wp' u\wp' v}{2(\wp u - \wp v)^2}.$$

The function $\wp(u+v)$ is only infinite if u is equal or congruent to $-v$. Since $\wp u$ is finite at this point, it follows from the formula

$$\wp(u+v) = \wp u - \frac{1}{2}\frac{\partial}{\partial u}\left\{\frac{\wp' u - \wp' v}{\wp u - \wp v}\right\},$$

that the partial differential quotient which appears on the right-hand side must be infinite for the value $u = -v$.

To observe the nature of this infinity, write

$$u = -v + h.$$

It follows that

$$\frac{\wp' u - \wp' v}{\wp u - \wp v} = \frac{2\wp' v - h\wp'' v + \cdots}{h\wp' v - \frac{1}{2}h^2\wp'' v + \cdots} = \frac{2}{h} + * + ((h))$$

and that

$$\frac{\partial}{\partial u}\left\{\frac{\wp' u - \wp' v}{\wp u - \wp v}\right\} = -\frac{2}{h^2} + \cdots.$$

Noting these results we may obtain another formula for $\wp(u\pm v)$ as follows:

The function

$$\wp(u+v) - \frac{1}{4}\left(\frac{\wp' u - \wp' v}{\wp u - \wp v}\right)^2$$

is one-valued and doubly periodic. It is also finite at the point $u = -v$ and the congruent points. We further note that this function remains finite at the point $u = +v$. At the point $u = 0$ the function becomes infinite as $-\dfrac{1}{u^2}$. If then we add to the above expression the function $\wp u$, we have a doubly periodic function which remains finite everywhere

in the finite portion of the plane and is therefore (see Art. 83) a constant. It is easily shown that this constant is $-\wp v$.

We may consequently write

$$\wp(u \pm v) = \frac{1}{4}\left[\frac{\wp' u \mp \wp' v}{\wp u - \wp v}\right]^2 - \wp u - \wp v.$$

ART. 301. If in the formula just written we put $u + v$ for u and $-v$ for v, we have

$$\wp u + \wp(u+v) + \wp v = \frac{1}{4}\left[\frac{\wp'(u+v) + \wp' v}{\wp(u+v) - \wp v}\right]^2.$$

It follows that

$$\frac{\wp' u - \wp' v}{\wp u - \wp v} = \pm \frac{\wp'(u+v) + \wp' v}{\wp(u+v) - \wp v}.$$

If both sides of this equation are developed in powers of u, it is seen that the *negative* sign must be used.

In determinant form this formula may be written

$$\begin{vmatrix} 1, & \wp u, & \wp' u \\ 1, & \wp v, & \wp' v \\ 1, & \wp(u+v), & -\wp'(u+v) \end{vmatrix} = 0.$$

By differentiating with regard to v the formula

$$\wp(u \pm v) = \wp u - \frac{1}{2}\frac{\partial}{\partial u}\frac{\wp' u \mp \wp' v}{\wp u - \wp v},$$

we have

$$\wp'(u \pm v) = -\frac{1}{2}\frac{\partial^2}{\partial u\, \partial v}\frac{\wp' u \mp \wp' v}{\wp u - \wp v}$$

$$= \left\{\frac{(\wp' v)^2}{(\wp v - \wp u)^3} - \frac{\wp'' v}{2(\wp v - \wp u)^2}\right\}\wp' u \pm \left\{\frac{(\wp' u)^2}{(\wp u - \wp v)^3} - \frac{\wp'' u}{2(\wp u - \wp v)^2}\right\}\wp' v.$$

Remark.—If in the formula (F′) of Art. 299 we write ω in the place of v and observe that

$$4\wp^3\omega - g_2\wp\omega - g_3 = 4(\wp\omega - e_1)(\wp\omega - e_2)(\wp\omega - e_3) = 0,$$

it is seen that

$$\wp(u \pm \omega) - e_1 = \frac{6\wp^2\omega - \frac{1}{2}g_2}{2(\wp u - e_1)},$$

or

$$\wp(u \pm \omega) - e_1 = \frac{1}{2}\frac{6e_1^2 - \frac{1}{2}g_2}{\wp u - e_1}.$$

From the relation

$$\wp'' u = 6\wp^2 u - \tfrac{1}{2}g_2$$

it follows that

$$\wp''\omega = 6e_1^2 - \tfrac{1}{2}g_2,$$

and consequently that

$$\wp(u \pm \omega) - e_1 = \frac{1}{2}\frac{\wp''\omega}{\wp u - e_1}.$$

Further, since

$$(\wp' u)^2 = 4(\wp u - e_1)(\wp u - e_2)(\wp u - e_3),$$

and therefore also

$$\wp'' u = 2[(\wp u - e_1)(\wp u - e_2) + (\wp u - e_1)(\wp u - e_3) + (\wp u - e_2)(\wp u - e_3)],$$

it follows that

$$\wp''\omega = 2(e_1 - e_2)(e_1 - e_3).$$

We consequently have

$$\wp(u \pm \omega) - e_1 = \frac{(e_1 - e_2)(e_1 - e_3)}{\wp u - e_1},$$

and similarly

$$\wp(u \pm \omega'') - e_2 = \frac{(e_2 - e_1)(e_2 - e_3)}{\wp u - e_2},$$

$$\wp(u \pm \omega') - e_3 = \frac{(e_3 - e_1)(e_3 - e_2)}{\wp u - e_3}.$$

ART. 302. The reciprocal of formula (G), Art. 300, is

$$\frac{1}{\wp(u \pm v)} = \frac{2(\wp u - \wp v)^2}{2(\wp u \wp v - \frac{1}{4} g_2)(\wp u + \wp v) - g_3 \mp \wp' u \wp' v}$$

$$= \frac{2(\wp u - \wp v)^2 \left\{ (2 \wp u \wp v - \frac{1}{4} g_2)(\wp u + \wp v) - g_3 \pm \wp' u \wp' v \right\}}{[2(\wp u \wp v - \frac{1}{4} g_2)(\wp u + \wp v) - g_3]^2 - \wp'^2 u\, \wp'^2 v}.$$

Noting that

$$(\wp' u)^2 = 4 \wp^3 u - g_2 \wp u - g_3 \quad \text{and} \quad (\wp' v)^2 = 4 \wp^3 v - g_2 \wp v - g_3,$$

it is seen that

$$[2(\wp u \wp v - \tfrac{1}{4} g_2)(\wp u + \wp v) - g_3]^2 - [4 \wp^3 u - g_2 \wp u - g_3][4 \wp^3 v - g_2 \wp v - g_3]$$
$$= 4(\wp u - \wp v)^2 [\wp^2 u\, \wp^2 v + \tfrac{1}{2} g_2 \wp u \wp v + \tfrac{1}{16} g_2^2 + g_3(\wp u + \wp v)];$$

and consequently

$$\frac{1}{\wp(u \pm v)} = \frac{2(\wp u \wp v - \frac{1}{4} g_2)(\wp u + \wp v) - g_3 \pm \wp' u \wp' v}{2(\wp u \wp v + \frac{1}{4} g_2)^2 + 2 g_3(\wp u + \wp v)}.$$

If we write $u = v$, we have

$$\wp(2u) = \frac{(\wp^2 u + \frac{1}{4} g_2)^2 + 2 g_3 \wp u}{4 \wp^3 u - g_2 \wp u - g_3}.$$

It also follows that

$$\wp(2u) - \wp u = \frac{-3 \wp^4 u + \frac{3}{2} g_2 \wp^2 u + 3 g_3 \wp u + \frac{1}{16} g_2^2}{4 \wp^3 u - g_2 \wp u - g_3}$$
$$= -\frac{1}{4} \frac{d^2}{du^2} \log \wp' u.$$

From the formula just written we have

$$2 \wp(2u) du = 2\left(\wp u - \frac{1}{4} \frac{d^2}{du^2} \log \wp' u\right) du.$$

Integrating we have
$$\frac{\sigma'}{\sigma}(2u) = 2\frac{\sigma'}{\sigma}(u) + \frac{1}{2}\frac{d}{du}\log \wp' u + C$$
$$= 2\frac{\sigma'}{\sigma}(u) + \frac{1}{2}\frac{\wp'' u}{\wp' u} + C.$$

Developing both sides of this expression in ascending powers of u, it is seen that the constant $C = 0$.

We therefore have
$$\frac{\sigma'}{\sigma}(2u) = 2\frac{\sigma'}{\sigma}(u) + \frac{1}{2}\frac{\wp'' u}{\wp' u}.$$

This formula multiplied by $2\,du$ and integrated becomes
$$\log \sigma(2u) = 4 \log \sigma u + \log \wp' u + \log c,$$
so that
$$\sigma(2u) = c(\sigma u)^4 \wp' u.$$
It follows that
$$2u + * + ((u^5)) = c(u^4 + * \cdots)\left(-\frac{2}{u^3} + ((u))\right),$$
from which it is seen that $c = -1$ and consequently
$$\frac{\sigma(2u)}{(\sigma u)^4} = -\wp' u.$$

ART. 303. *Historical.*—It was known through the works of Fagnano, Landen, Jacob Bernoulli and others that the expressions for sin $(\alpha + \beta)$, sin $(\alpha - \beta,)$ etc., gave a means of adding or subtracting the arcs of circles, and that between the limits of two integrals that express lengths of arc of a lemniscate an algebraic relation exists, such that the arc of a lemniscate although a transcendent of higher order, may be doubled or halved just as the arc of a circle by means of geometric construction.

It was natural to inquire if the ellipse, hyperbola, etc., did not have similar properties. Investigating such properties Euler made the remarkable discovery of the addition-theorem of elliptic integrals (see *Nov. Comm. Petrop.* VI, pp. 58–84, 1761; and VII, p. 3; VIII, p. 83).

Euler shows that if
$$\int_0^x \frac{d\xi}{\sqrt{R(\xi)}} + \int_0^y \frac{d\xi}{\sqrt{R(\xi)}} = \int_0^a \frac{d\xi}{\sqrt{R(\xi)}},$$
where $R(\xi)$ is a rational integral function of the fourth degree in ξ, there exists among the upper limits x, y, a of the integrals an algebraic relation which is the addition-theorem of the arcs of an ellipse and is the algebraic solution (cf. again Euler, *Nov. Comm.* Vol. X, pp. 3–56) of the differential equation
$$\frac{d\xi}{\sqrt{R(\xi)}} + \frac{d\eta}{\sqrt{R(\eta)}} = 0.$$

Euler states that the above results were obtained *not* by any method, but *potius tentando, vel divinando*, and suggested that mathematicians seek a

direct proof. The numerous discoveries of Euler are systematized in his work *Institutiones Calculi Integralis*, Vol. I, Sectio Secunda, Caput VI.

The fourth volume (p. 446) contains an extension of the addition-theorem to the integrals of the second and third kinds. This work must therefore have proved of great value to Legendre in the development of his theory. In every case geometrical application of the formulas was made by Euler for the comparison of elliptic arcs.

The suggestion made by Euler that one should find a direct method of integrating the differential equation proposed by him, was carried out by Lagrange, who by direct methods integrated this equation and in a manner which elicited the great admiration of Euler (see *Miscell. Taurin.* IV, 1768; or Serret's *Œuvres de Lagrange*, t. II, p. 533).

The addition-theorem for elliptic integrals gave to the elliptic functions a meaning in higher analysis similar to that which the cyclometric and logarithmic functions had enjoyed for a long time.

ART. 304. We may consider next some of the general investigations which led Euler to the discovery of the addition-theorem and then give his solution and the one of Lagrange.

If we differentiate the equation

$$\text{(I)} \qquad Ax^2 + 2\,Bxy + Cy^2 + 2\,Dx + 2\,Ey + F = 0,$$

we have

$$\text{(II)} \qquad (Bx + Cy + E)\,dy + (Ax + By + D)dx = 0.$$

From (I) we have

$$x = -\frac{By + D}{A} \pm \frac{1}{A}\sqrt{(By + D)^2 - (Cy^2 + 2\,Ey + F)A},$$

$$y = -\frac{Bx + E}{C} \pm \frac{1}{C}\sqrt{(Bx + E)^2 - (Ax^2 + 2\,Dx + F)C}.$$

These values substituted in (II) give

$$\text{(III)} \qquad \frac{dx}{\sqrt{F(x)}} + \frac{dy}{\sqrt{G(y)}} = 0,$$

where

$$F(x) = (Bx + E)^2 - (Ax^2 + 2\,Dx + F)C,$$

$$G(y) = (By + D)^2 - (Cy^2 + 2\,Ey + F)A.$$

If $A = C$ and $D = E$, then $G(y)$ becomes $F(y)$. The differential equation (III) becomes thereby

$$\text{(III}'\text{)} \qquad \frac{dx}{\sqrt{F(x)}} + \frac{dy}{\sqrt{F(y)}} = 0,$$

and its algebraic integral is

$$\text{(I}'\text{)} \qquad A(x^2 + y^2) + 2\,Bxy + 2\,D(x + y) + F = 0.$$

Suppose next that $R(x) = ax^2 + 2bx + c$ is given and it is required to find the integral of

$$\frac{dx}{\sqrt{R(x)}} + \frac{dy}{\sqrt{R(y)}} = 0.$$

We must so determine the constants A, B, F, D that

$$ax^2 + 2bx + c = (Bx + D)^2 - A(Ax^2 + 2Dx + F).$$

By equating like powers of x, we have three relations existing among the four quantities A, B, F, D. We may therefore determine B, F, D in terms of A.

It follows that the differential equation

$$\frac{dx}{\sqrt{R(x)}} + \frac{dy}{\sqrt{R(y)}} = 0$$

is always integrable through an algebraic equation (I′) of the second degree which is symmetric in x and y and contains an arbitrary constant A. By the comparison of this algebraic equation with a transcendental equation which we shall determine later, we derive the associated addition-theorem.

If further we observe that $-aR(x) = (b^2 - ac)\left[1 - \left(\frac{ax+b}{\sqrt{b^2-ac}}\right)^2\right]$ and put $\frac{ax+b}{\sqrt{b^2-ac}} = z$, then

$$\int_{x_0, \sqrt{R(x_0)}}^{x, \sqrt{R(x)}} \frac{dx}{\sqrt{R(x)}} = u, \text{ say,}$$

becomes, if we take the *minus* sign with the root,

$$au = \int_{z_0, s_0}^{z, s} \frac{dz}{\sqrt{1 - z^2}}, \quad \text{where} \quad s^2 = 1 - z^2,$$

or

$$\frac{dz}{du} = a\sqrt{1 - z^2}.$$

If s is not a one-valued function of z, there must be a second branch of the function, which in the Riemann surface is represented on a second leaf, so that if z_1 represents the variable z in this leaf, we have

$$\frac{dz_1}{du} = -a\sqrt{1 - z_1{}^2}.$$

ART. 305. It is evident that we may write the differential equation

$$\frac{dx}{\sqrt{ax^2 + 2bx + c}} + \frac{dy}{\sqrt{ay^2 + 2by + c}} = 0$$

in the form

$$\frac{d\xi}{\sqrt{1 - \xi^2}} + \frac{d\eta}{\sqrt{1 - \eta^2}} = 0,$$

or

$$\sqrt{1 - \eta^2} + \sqrt{1 - \xi^2}\,\frac{d\eta}{d\xi} = 0.$$

If η is a function of ξ which satisfies this differential equation, then is

$$\int\left[\sqrt{1-\eta^2}+\sqrt{1-\xi^2}\,\frac{d\eta}{d\xi}\right]d\xi = C,$$

where C is the constant of integration. Integrating by parts we have at once

$$C = \xi\sqrt{1-\eta^2}+\eta\sqrt{1-\xi^2}+\int\xi\eta\left[\frac{d\eta}{d\xi}\,\frac{1}{\sqrt{1-\eta^2}}+\frac{1}{\sqrt{1-\xi^2}}\right]d\xi,$$

or

$$C = \xi\sqrt{1-\eta^2}+\eta\sqrt{1-\xi^2}.$$

This is the algebraic integral of the differential equation and corresponds to the integral (I′) of Art. 304, which latter equation was derived through experimenting by Euler. To determine the corresponding transcendental integral write

(1) $$u = \int_{0,1}^{\xi,\sigma}\frac{d\xi}{\sqrt{1-\xi^2}}, \quad \text{where } \sigma = \sqrt{1-\xi^2}, \quad \text{and}$$

(2) $$v = \int_{0,1}^{\eta,\tau}\frac{d\eta}{\sqrt{1-\eta^2}}, \quad \text{where } \tau = \sqrt{1-\eta^2}.$$

It follows that $\xi = \sin u$ and $\eta = \sin v$.

The differential equation

$$\frac{d\xi}{\sqrt{1-\xi^2}}+\frac{d\eta}{\sqrt{1-\eta^2}} = 0$$

becomes

$$du + dv = 0.$$

We therefore have

$$\int_{0,1}^{\xi,\sigma}\frac{d\xi}{\sqrt{1-\xi^2}}+\int_{0,1}^{\eta,\tau}\frac{d\eta}{\sqrt{1-\eta^2}} = c,$$

or

$$u + v = c,$$

which is the transcendental integral of the above differential equation.

We so determine the constant C in the algebraic integral

$$C = \xi\sqrt{1-\eta^2}+\eta\sqrt{1-\xi^2},$$

that for $\xi = 0$, $\sigma = +1$ the variable η takes the definite value η_0. It follows at once that

$$C = \eta_0.$$

When the values $\xi = 0$, $\sigma = +1$ are written in the upper limit of the integral (1), it is seen that

$$u = 0,$$

and since $u + v = c$, it follows that

$$c = \int_{0,1}^{\eta_0,\sqrt{1-\eta_0^2}}\frac{d\eta}{\sqrt{1-\eta^2}},$$

or

$$\eta_0 = \sin c = \sin(u+v).$$

On the other hand, since

$$\eta_0 = C = \xi\sqrt{1-\eta^2} + \eta\sqrt{1-\xi^2},$$

we have

$$\sin(u+v) = \sin u \cos v + \sin v \cos u,$$

which is the addition-theorem for the sine-function.

ART. 306. In a similar manner Euler derived the addition-theorem for $sn\, u$ as follows.

Suppose we have given the quadratic equation

$$\text{(I)} \qquad A\eta^2 + 2B\eta + C = F(\xi, \eta) = 0,$$

where

$$A = a_0\xi^2 + 2a_1\xi + a_2,$$
$$B = b_0\xi^2 + 2b_1\xi + b_2,$$
$$C = c_0\xi^2 + 2c_1\xi + c_2.$$

By arranging the terms according to powers of ξ, the same quadratic equation may be written

$$A'\xi^2 + 2B'\xi + C' = F(\xi, \eta) = 0,$$

where

$$A' = a_0\eta^2 + 2b_0\eta + c_0,$$
$$B' = a_1\eta^2 + 2b_1\eta + c_1,$$
$$C' = a_2\eta^2 + 2b_2\eta + c_2.$$

Differentiating (I) we have

$$\frac{\partial F}{\partial \xi}d\xi + \frac{\partial F}{\partial \eta}d\eta = 0,$$

or

$$(A'\xi + B')\,d\xi + (A\eta + B)d\eta = 0.$$

It follows * at once that

$$\frac{d\xi}{A\eta + B} + \frac{d\eta}{A'\xi + B'} = 0.$$

On the other hand we have

$$\eta = -\frac{B}{A} \pm \frac{1}{A}\sqrt{B^2 - AC},$$

or

$$A\eta + B = \sqrt{B^2 - AC},$$

where both signs may be associated with the root; and similarly we have

$$A'\xi + B' = \sqrt{B'^2 - A'C'}.$$

We thus derive † the equation

$$\text{(II)} \qquad \frac{d\xi}{\sqrt{B^2 - AC}} + \frac{d\eta}{\sqrt{B'^2 - A'C'}} = 0,$$

* See Euler, *loc. cit.*, or Enneper, *Elliptische Funktionen*, p. 186.

† See Euler, *Institutiones Calc. Int.*, Vol. I, Sectio Secunda, Caput VI; or Lagrange (1766–69), Œuvres (Serret, Paris, 1868), t. II, p. 533. Halphen (*Fonct. Ellipt.*, Vol. II, Chap. IX) calls such an equation an Euler-equation and remarks that by the discovery of the general integral of this equation "Euler sowed the first germ of the theory of elliptic functions" (in 1761).

or
$$\frac{d\xi}{\sqrt{(b_0\xi^2+2b_1\xi+b_2)^2-(a_0\xi^2+2a_1\xi+a_2)(c_0\xi^2+2c_1\xi+c_2)}}$$
$$+\frac{d\eta}{\sqrt{(a_1\eta^2+2b_1\eta+c_1)^2-(a_0\eta^2+2b_0\eta+c_0)(a_2\eta^2+2b_2\eta+c_2)}}=0.$$

If we put
$$a_1=b_0,\quad a_2=c_0,\quad b_2=c_1,$$
the expressions under the roots take the same form, while equation (I) becomes *

$$\text{(I}'\text{)}\qquad a_0\xi^2\eta^2+2b_0\xi\eta(\xi+\eta)+c_0(\xi^2+\eta^2)+4b_1\xi\eta+2c_1(\xi+\eta)+c_2=0.$$

If the differential equation which we wish to integrate is

$$\text{(III)}\qquad \frac{d\xi}{\sqrt{R(\xi)}}+\frac{d\eta}{\sqrt{R(\eta)}}=0,$$

where $R(t)=P_0t^4+P_1t^3+P_2t^2+P_3t+P_4$, we may make this equation identical with (II) by writing
$$B^2-AC=R(\xi),$$
or $(b_0\xi^2+2b_1\xi+b_2)^2-(a_0\xi^2+2a_1\xi+a_2)(c_0\xi^2+2c_1\xi+c_2)=R(\xi)$.

We therefore have the conditions
$$\begin{aligned}
P_0&=b_0^2-a_0c_0,\\
P_1&=4b_0b_1-2a_0c_1-2a_1c_0,\\
P_2&=2b_0b_2+4b_1^2-a_0c_2-4a_1c_1-a_2c_0,\\
P_3&=4b_1b_2-2a_1c_2-2a_2c_1,\\
P_4&=b_2^2-a_2c_2.
\end{aligned}$$

Thus in addition to the three conditions $a_1=b_0$, $a_2=c_0$, $b_2=c_1$ we have the above five conditions among the nine quantities $a_0, b_0, c_0, a_1, b_1, c_1, a_2, b_2, c_2$.

It is evident that when these conditions have been satisfied there remains an arbitrary constant in the equation (I′), which equation is the algebraic integral of (III).

ART. 307. In particular let the equation (III) have the form

$$\text{(III)}'\qquad \frac{d\xi}{\sqrt{(1-\xi^2)(1-k^2\xi^2)}}+\frac{d\eta}{\sqrt{(1-\eta^2)(1-k^2\eta^2)}}=0.$$

Noting from above that $a_1=b_0$, $a_2=c_0$, $b_2=c_1$, we have
$$\begin{aligned}
b_0^2-a_0c_0&=k^2,\\
(2b_1-c_0)b_0-a_0c_1&=0,\\
4b_1^2-a_0c_2-c_0^2-2b_0c_1&=-(1+k^2),\\
(2b_1-c_0)c_1-b_0c_2&=0,\\
c_1^2-c_0c_2&=1.
\end{aligned}$$

* See Cayley, *loc. cit.*, p. 341.

We observe that (III′) remains unchanged if ξ and η are replaced by $-\xi$ and $-\eta$. It follows that (I′) must remain unaltered by this transformation. We must therefore have

$$b_0 = 0, \quad c_1 = 0.$$

The relations just written are consequently

$$-c_0c_2 = 1, \quad 4b_1^2 - a_0c_2 - c_0^2 + 1 + k^2 = 0, \quad -a_0c_0 = k^2,$$

or

$$b_0 = 0, \quad c_1 = 0, \quad c_2 = -\frac{1}{c_0}, \quad a_0 = -\frac{k^2}{c_0},$$

$$4b_1^2 = \frac{(1-c_0^2)(k^2-c_0^2)}{c_0^2}.$$

Writing these values in equation (I′) we have

$$-\frac{1}{c_0} - \frac{k^2}{c_0}\xi^2\eta^2 + c_0(\xi^2+\eta^2) = -\frac{2}{c_0}\sqrt{(1-c_0^2)(k^2-c_0^2)}\,\xi\eta,$$

or

$$[1 + k^2\xi^2\eta^2 + c_0^2(\xi^2+\eta^2)]^2 = 4[k^2 - (1+k^2)c_0^2 + c_0^4]\xi^2\eta^2.$$

Arranged in powers of $\frac{1}{c_0}$, this equation is

$$\frac{(1-k^2\xi^2\eta^2)^2}{c_0^4} - \frac{2(1+k^2\xi^2\eta^2)(\xi^2+\eta^2) - 4(1+k^2)\xi^2\eta^2}{c_0^2} + (\xi^2-\eta^2)^2 = 0,$$

or

$$\frac{1}{c_0} = \frac{\xi\sqrt{1-\eta^2}\sqrt{1-k^2\eta^2} + \eta\sqrt{1-\xi^2}\sqrt{1-k^2\xi^2}}{1-k^2\xi^2\eta^2},$$

which is the algebraic integral of (III′). After deriving the transcendental integral Euler proceeded to the addition-theorem in practically the same manner as is given in the next Article.

ART. 308. Professor Darboux * proceeded to the above algebraic integral as follows: He assumed that

$$\frac{d\xi}{du} = \sqrt{Z(\xi)} \quad \text{or} \quad u = \int_{0,1}^{\xi,\sqrt{Z(\xi)}} \frac{d\xi}{\sqrt{Z(\xi)}}, \tag{i}$$

where
$$Z(\xi) = (1-\xi^2)(1-k^2\xi^2),$$

and required that ξ be determined as a function of u.

He further introduced an auxiliary variable v, such that

$$\frac{d\eta}{dv} = \sqrt{Z(\eta)} \quad \text{or} \quad v = \int_{0,1}^{\eta,\sqrt{Z(\eta)}} \frac{d\eta}{\sqrt{Z(\eta)}}. \tag{ii}$$

* Darboux, *Ann. de l'École Norm.*, IV, p. 85 (1867).

We therefore have from (III′)

$$du + dv = 0,$$

or

$$u + v = c, \quad v = -u + c,$$

where c is a constant.

It follows that

$$\frac{d\eta}{du} = -\sqrt{R(\eta)},$$

so that ξ and η are functions of u, both being integrals of the equation

$$\left(\frac{dx}{du}\right)^2 = \mathrm{Z}(x).$$

We next form

$$\frac{d^2\xi}{du^2} = \frac{1}{2}\frac{\mathrm{Z}'(\xi)}{\sqrt{\mathrm{Z}(\xi)}}\frac{d\xi}{du} = \frac{1}{2}\mathrm{Z}'(\xi)$$

$$= -(1 + k^2)\xi + 2k^2\xi^3$$

and

$$\frac{d^2\eta}{du^2} = -(1 + k^2)\eta + 2k^2\eta^3.$$

We have immediately

$$\eta\frac{d^2\xi}{du^2} - \xi\frac{d^2\eta}{du^2} = 2k^2\xi\eta(\xi^2 - \eta^2),$$

$$\eta^2\left(\frac{d\xi}{du}\right)^2 - \xi^2\left(\frac{d\eta}{du}\right)^2 = \eta^2 - \xi^2 + k^2\xi^2\eta^2(\xi^2 - \eta^2).$$

Through division it follows that

$$\frac{\eta\dfrac{d^2\xi}{du^2} - \xi\dfrac{d^2\eta}{du^2}}{\eta^2\left(\dfrac{d\xi}{du}\right)^2 - \xi^2\left(\dfrac{d\eta}{du}\right)^2} = \frac{2k^2\xi\eta}{-1 + k^2\xi^2\eta^2},$$

or

$$\frac{\eta\dfrac{d^2\xi}{du^2} - \xi\dfrac{d^2\eta}{du^2}}{\eta\dfrac{d\xi}{du} - \xi\dfrac{d\eta}{du}} = \frac{2k^2\xi\eta\left(\eta\dfrac{d\xi}{du} + \xi\dfrac{d\eta}{du}\right)}{k^2\xi^2\eta^2 - 1}.$$

This expression, when integrated, becomes

$$\frac{\eta\dfrac{d\xi}{du} - \xi\dfrac{d\eta}{du}}{1 - k^2\xi^2\eta^2} = C,$$

where C is a *constant.*

Further, since

$$\frac{d\xi}{du} = \sqrt{(1 - \xi^2)(1 - k^2\xi^2)}, \quad \frac{d\eta}{du} = \sqrt{(1 - \eta^2)(1 - k^2\eta^2)},$$

we have at once

$$\text{(M)} \qquad \frac{\eta\sqrt{(1 - \xi^2)(1 - k^2\xi^2)} + \xi\sqrt{(1 - \eta^2)(1 - k^2\eta^2)}}{1 - k^2\xi^2\eta^2} = C,$$

which is the algebraic integral of (III′).

The addition-theorem may be derived as follows: If in the relation

$$u + v = c$$

we write for u and v their values from (i) and (ii), we have

$$\text{(N)} \qquad \int_{0,1}^{\xi,\sqrt{Z(\xi)}} \frac{d\xi}{\sqrt{Z(\xi)}} + \int_{0,1}^{\eta,\sqrt{Z(\eta)}} \frac{d\eta}{\sqrt{Z(\eta)}} = c.$$

This is also an integral of (III′) but in transcendental form.

Suppose next that η becomes η_0 for the values $\xi = 0$, $\sqrt{Z(\xi)} = 1$. It follows from (M) that

$$\eta_0 = C,$$

and from (N) that

$$\text{(P)} \qquad \int_{0,1}^{\eta_0,\sqrt{Z(\eta_0)}} \frac{d\eta}{\sqrt{Z(\eta)}} = c.$$

If we write

$$\xi = sn\, u, \qquad \eta = sn\, v,$$
$$\sqrt{1 - \xi^2} = cn\, u, \qquad \sqrt{1 - \eta^2} = cn\, v,$$
$$\sqrt{1 - k^2\xi^2} = dn\, u, \qquad \sqrt{1 - k^2\eta^2} = dn\, v,$$

then from (P) we have

$$\eta_0 = sn\, c.$$

But since $c = u + v$ and also $\eta_0 = C$, the equation (M) may be written

$$\frac{snv\, cnu\, dnu + snu\, cnv\, dnv}{1 - k^2 sn^2u\, sn^2v} = sn(u + v).$$

Write

$$D = 1 - k^2 sn^2u\, sn^2v$$

and note, since $1 = sn^2u + cn^2u$, that

$$D = cn^2u + sn^2u\, dn^2v = D_1, \text{ say},$$

and

$$D = cn^2v + sn^2v\, dn^2u = D_2;$$

and also that

$$D^2 = D_1 D_2.$$

It follows that

$$cn^2(u + v) = 1 - sn^2(u + v)$$
$$= \frac{D^2 - (snu\, cnv\, dnv + snv\, cnu\, dnu)^2}{D^2},$$

or (cf. Art. 296)

$$cn(u + v) = \frac{cnu\, cnv - snu\, snv\, dnu\, dnv}{1 - k^2 sn^2u\, sn^2v}.$$

Similarly, if we note that

$$D = dn^2u + k^2sn^2u\, cn^2v = D_3$$

and that

$$D = dn^2v + k^2sn^2v\, cn^2u = D_4,$$

we may derive from

$$dn^2(u+v) = 1 - k^2sn^2(u+v)$$

the formula

$$dn(u+v) = \frac{dn\,u\, dn\,v - k^2sn\,u\, cn\,u\, sn\,v\, cn\,v}{1 - k^2sn^2u\, sn^2v}.$$

ART. 309. A direct process for finding the algebraic integral was given by Lagrange as follows:

For brevity write $X = a + bx + cx^2 + dx^3 + ex^4,$
$Y = a + by + cy^2 + dy^3 + ey^4.$

The differential equation to be integrated is of the form

(I) $$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} = 0.$$

Considering x and y as functions of u, we have as in Art. 308

$$\frac{dx}{du} = \sqrt{X} \quad \text{and} \quad \frac{dy}{du} = -\sqrt{Y}.$$

It follows * that

$$2\frac{d^2x}{du^2} = b + 2\,cx + 3\,dx^2 + 4\,ex^3,$$

$$2\frac{d^2y}{du^2} = b + 2\,cy + 3\,dy^2 + 4\,ey^3.$$

If next we introduce two new variables defined by

$$p = x + y \quad \text{and} \quad q = x - y,$$

we have

$$\frac{d^2p}{du^2} = \frac{d^2x}{du^2} + \frac{d^2y}{du^2} = b + cp + \tfrac{3}{4}\,d(p^2 + q^2) + \tfrac{1}{2}\,e(p^3 + 3\,pq^2),$$

$$\frac{dp}{du}\cdot\frac{dq}{du} = X - Y = bq + cpq + \tfrac{1}{4}\,qd(3\,p^2 + q^2) + \tfrac{1}{2}\,epq(p^2 + q^2).$$

It is seen at once that

$$q\frac{d^2p}{du^2} - \frac{dp}{du}\frac{dq}{du} = \tfrac{1}{2}\,q^3d + epq^3,$$

or

$$\frac{2}{q^2}\frac{d^2p}{du^2}\frac{dp}{du} - \frac{2}{q^3}\left(\frac{dp}{du}\right)^2\frac{dq}{du} = (d + 2\,ep)\frac{dp}{du}.$$

* See Cayley, *loc. cit.*, p. 337.

The integral of this expression is

$$\frac{1}{q^2}\left(\frac{dp}{du}\right) = C + pd + ep^2,$$

where C is the constant of integration.

Writing for q, $\frac{dp}{du}$, p their values, we see that the general integral of (I) is

$$\text{(II)} \qquad \left(\frac{\sqrt{X}-\sqrt{Y}}{x-y}\right)^2 = C + d(x+y) + e(x+y)^2.$$

Cayley (*Elliptic Functions*, p. 338) gives several interesting forms of this algebraic integral and of the addition-theorem.

ART. 310. The formula (II) above suggests at once a form for the integral of the corresponding differential equation in the Weierstrassian theory.

Write $$(a, b, c, d, e) = (-g_3, -g_2, 0, 4, 0)$$

and consider the integral

$$\text{(I')} \qquad \frac{ds}{\sqrt{4s^3 - g_2 s - g_3}} + \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}} = 0.$$

The algebraic integral is seen at once to be

$$\text{(II')} \qquad \left[\frac{\sqrt{4s^3 - g_2 s - g_3} - \sqrt{4t^3 - g_2 t - g_3}}{s-t}\right]^2 - 4(s+t) = C.$$

Writing

$$du = \frac{ds}{\sqrt{4s^3 - g_2 s - g_3}}, \quad dv = \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}},$$

the transcendental integral is

$$\text{(T)} \qquad u + v = c,$$

where $$s = \wp u, \quad t = \wp v = \wp(c-u) = \wp(u-c).$$

When these values are substituted in the algebraic integral, it becomes

$$\text{(A)} \qquad \left[\frac{\wp' u - \wp'(c-u)}{\wp u - \wp(c-u)}\right]^2 - 4\wp u - 4\wp(c-u) = C.$$

From (A) it follows (Art. 300) that $C = 4\wp(c)$, and from (T) we have

$$\wp(u+v) = \frac{1}{4}\left[\frac{\wp' u - \wp' v}{\wp u - \wp v}\right]^2 - \wp u - \wp v,$$

or

$$\wp(u+v) = \frac{2(\wp u \wp v - \frac{1}{4} g_2)(\wp u + \wp v) - g_3 - \wp' u \wp' v}{2(\wp u - \wp v)^2}.$$

ART. 311. Equate to zero the determinant *

$$\begin{vmatrix} 1, \wp u, \wp' u \\ 1, \wp v, \wp' v \\ 1, \wp w, \wp' w \end{vmatrix} = \wp' w(\wp v - \wp u) + \wp' v(\wp u - \wp w) + \wp' u(\wp w - \wp v) = 0.$$

Squaring we have

$$(\wp' w)^2(\wp u - \wp v)^2 - \left\{\wp' v(\wp u - \wp w) - \wp' u(\wp v - \wp w)\right\}^2 = 0,$$

or

$$(\wp u - \wp v)^2 [4\wp^3 w - g_2 \wp w - g_3] - [\wp' v \wp u - \wp' u \wp v - \wp w(\wp' v - \wp' u)]^2 = 0,$$

an equation which is satisfied for $w = u, v$ and also (Art. 301) for $-w = u + v$; that is for $\wp w = \wp u, \wp v, \wp(u + v)$.

The equation

$$(\wp u - \wp v)^2 \left\{s - \wp u\right\} \left\{s - \wp v\right\} \left\{s - \wp(u + v)\right\} = 0$$

has the same zeros, viz., $s = pu, pv, p(u + v)$; and since the coefficients of $(\wp w)^3$ and s^3 are the same in both equations, the two equations, since they can differ only by a multiplicative constant (Art. 83), must have all their coefficients the same.

The coefficients of $(\wp w)^2$ and s^2 give immediately

$$-(\wp' u - \wp' v)^2 = 4(\wp u - \wp v)^2 \left\{-\wp u - \wp v - \wp(u+v)\right\},$$

or

$$\wp(u + v) = \frac{1}{4}\left\{\frac{\wp' u - \wp' v}{\wp u - \wp v}\right\}^2 - \wp u - \wp v.$$

ART. 312. In Art. 193 we derived the formulas

$$\begin{array}{lll}
\bar{u}(+1) = -3K & \text{or} & sn(-3K) = 1, \\
\bar{u}\left(+\frac{1}{k}\right) = -3K - iK' & \text{or} & sn(-3K - iK') = \frac{1}{k}, \\
\bar{u}(\infty, \infty) = -iK' & \text{or} & sn(-iK') = \infty, \\
\bar{u}(0, 1) = 0 & \text{or} & sn(0) = 0, \\
\bar{u}(-1) = -K & \text{or} & sn(-K) = -1, \\
\bar{u}\left(-\frac{1}{k}\right) = -K - iK' & \text{or} & sn(-K - iK') = -\frac{1}{k}, \\
\bar{u}(\infty, -\infty) = -2K - iK' & \text{or} & sn(-2K - iK') = \infty, \\
\bar{u}(0, -1) = -2K & \text{or} & sn(-2K) = 0.
\end{array}$$

By means of these formulas and the addition-theorems we may verify the formulas IX–XV of Chapter XI.

* See Daniels, *Am. Journ. of Math.*, Vol. VI, p. 269.

ART. 313. *Duplication.*—In the addition-theorems above if we write $v = u$, we deduce the following formulas:

$$sn\,2u = \frac{2\,sn\,u\,cn\,u\,dn\,u}{D}, \qquad \text{where } D = 1 - k^2 sn^4 u,$$

$$cn\,2u = \frac{cn^2u - sn^2u\,dn^2u}{D},$$

$$dn\,2u = \frac{dn^2u - k^2 sn^2u\,cn^2u}{D}.$$

Writing $sn\,u = s$, $cn\,u = c$, $dn\,u = d$, we have

$$1 - cn\,2u = \frac{2\,s^2(1 - k^2s^2)}{D} = \frac{2\,s^2d^2}{D},$$

$$1 + cn\,2u = \frac{2\,c^2}{D}, \quad 1 - dn\,2u = \frac{2\,k^2s^2c^2}{D},$$

$$1 + dn\,2u = \frac{2\,d^2}{D}.$$

ART. 314. *Dimidiation.*—From the above formulas we deduce at once

$$sn^2u = \frac{1 - cn\,2u}{1 + dn 2u} \quad \text{or} \quad sn^2\frac{1}{2}u = \frac{1 - cn\,u}{1 + dn\,u},$$

$$cn^2u = \frac{dn\,2u + cn\,2u}{1 + dn\,2u}, \quad dn^2u = \frac{k'^2 + dn\,2u + k^2 cn\,2u}{1 + dn\,2u}.$$

Changing u to $\frac{1}{2}u$ we have formulas* for the determination of $sn(\frac{1}{2}K)$, $sn(\frac{1}{2}iK')$, $sn\,(\frac{3}{2}K)$, etc.; for example

$$sn\,\frac{K}{2} = \sqrt{\frac{1 - cn\,K}{1 + dn\,K}} = \sqrt{\frac{1}{1 + k'}},$$

$$cn\,\frac{iK'}{2} = \sqrt{\frac{dn\,iK' + cn\,iK'}{1 + dn\,iK'}} = \sqrt{\frac{-ikI - iI}{1 - ikI}} = \sqrt{\frac{1 + k}{k}}.$$

[Table of Formulas, No. XVII.]

In a similar manner we have

$$sn\left(\frac{1}{2}K + \frac{1}{2}iK'\right) = \sqrt{\frac{1 + k'}{k}}\left(\frac{1 + k + ik'}{1 + k + k'}\right) = \frac{\sqrt{k + ik'}}{\sqrt{k}},$$

where we have written

$$k = \sqrt{1 + k'}\sqrt{1 - k'}, \qquad 1 = \sqrt{k + ik'}\sqrt{k - ik'},$$

and noted that

$$\sqrt{k - ik'} + \sqrt{k + ik'} = \sqrt{1 - k'} + \sqrt{1 + k'}.$$

* See Table of Formulas, No. XVII.

ART. 315. To determine the value of the $\wp$-function for the quarter-periods, we note that

$$\wp u = e_3 + \frac{e_1 - e_3}{sn^2(\sqrt{e_1 - e_3} \cdot u)}; \ \wp' u = -2\frac{(e_1 - e_3)^{\frac{3}{2}} cn(\sqrt{e_1 - e_3} \cdot u) dn(\sqrt{e_1 - e_3} \cdot u)}{sn^3(\sqrt{e_1 - e_3} \cdot u)},$$

$$k = \sqrt{\frac{e_2 - e_3}{e_1 - e_3}}, \qquad k' = \sqrt{\frac{e_1 - e_2}{e_1 - e_3}}.$$

We have for example

$$\wp\left(\frac{\omega}{2}\right) = e_3 + \frac{e_1 - e_3}{sn^2\left(\frac{K}{2}\right)} = e_3 + (e_1 - e_3)(1 + k')$$

$$= e_1 + \sqrt{(e_1 - e_3)(e_1 - e_2)};$$

$$\wp'\left(\frac{\omega + \omega'}{2}\right) = -2(e_1 - e_3)^{\frac{3}{2}} \frac{cn\left(\frac{K + iK'}{2}\right) dn\left(\frac{K + iK'}{2}\right)}{sn^3\left(\frac{K + iK'}{2}\right)}$$

$$= -2i(e_1 - e_3)^{\frac{3}{2}} kk'(k - ik')$$

or

$$\wp'\left(\frac{\omega + \omega'}{2}\right) = -2(e_1 - e_2)\sqrt{e_2 - e_3} - 2i(e_2 - e_3)\sqrt{e_1 - e_2},$$

a formula which is incorrectly derived and given by Halphen, *Fonct. Ellipt.*, t. I, p. 54.

ART. 316. We also find that

$$\wp(u + \omega) = e_3 + \frac{e_1 - e_3}{sn^2(v + K)} \qquad [v = u\sqrt{e_1 - e_3}]$$

$$= e_3 + \frac{(e_1 - e_3) dn^2 v}{cn^2 v} = e_3 + \frac{(e_1 - e_3)(\wp u - e_2)}{\wp u - e_1}$$

$$= e_3 + \frac{(e_1 - e_3)(e_1 - e_2)}{\wp u - e_1} + e_1 - e_3.$$

It follows, if we write $S(t) = 4(t - e_1)(t - e_2)(t - e_3)$, that

$$\wp(u + \omega) = e_1 + \frac{1}{4}\frac{S'(e_1)}{\wp u - e_1},$$

and similarly

$$\wp(u + \omega'') = e_2 + \frac{1}{4}\frac{S'(e_2)}{\wp u - e_2},$$

$$\wp(u + \omega') = e_3 + \frac{1}{4}\frac{S'(e_3)}{\wp u - e_3}.$$

If further we let $P_\lambda(u) = \wp u - e_\lambda$ $(\lambda = 1, 2, 3)$, we may derive at once the formulas *

$$P_1(u+\omega) = \frac{1}{4}\frac{S'(e_1)}{P_1(u)},$$

$$P_2(u+\omega) = (e_1 - e_2)\frac{P_3(u)}{P_1(u)},$$

$$P_3(u+\omega) = (e_1 - e_3)\frac{P_2(u)}{P_1(u)},$$

$$P_1(u+\omega'') = (e_2 - e_1)\frac{P_3(u)}{P_2(u)},$$

$$P_2(u+\omega'') = \frac{1}{4}\frac{S'(e_2)}{P_2(u)},$$

$$P_3(u+\omega'') = (e_2 - e_3)\frac{P_1(u)}{P_2(u)},$$

$$P_1(u+\omega') = (e_3 - e_1)\frac{P_2(u)}{P_3(u)},$$

$$P_2(u+\omega') = (e_3 - e_2)\frac{P_1(u)}{P_3(u)},$$

$$P_3(u+\omega') = \frac{1}{4}\frac{S'(e_3)}{P_3(u)}.$$

EXAMPLES

1. Show that $$sn(u+v) = \frac{cn^2u - cn^2v}{sn\,v\,cn\,u\,dn\,u - sn\,u\,cn\,v\,dn\,v}.$$

2. Show that $$\frac{1}{cn(u+v)} + \frac{1}{cn(u-v)} = \frac{2k'^2 cn\,u\,cn\,v}{dn^2u\,dn^2v - k'^2}.$$

3. Prove that $$\frac{cn(u+v)}{sn(u+v)} + \frac{cn(u-v)}{sn(u-v)} = \frac{2\,sn\,u\,cn\,u\,dn\,v}{sn^2u - sn^2v}.$$

4. Prove that $$sn(u+v) - sn(u-v) = \frac{1}{k}\frac{\partial}{\partial u}\log\frac{1+k\,sn\,u\,sn\,v}{1-k\,sn\,u\,sn\,v}.$$

5. Prove that $$\tan \operatorname{am}\frac{u+v}{2} = \frac{sn\,u\,dn\,v + sn\,v\,dn\,u}{cn\,u + cn\,v}.$$

6. Verify the formulas given in the Table of Formulas, No. LXIII.

7. Derive the addition-theorem for the $\wp$-function from that of the sn-function.

8. Show that
$$sn^2v - sn^2u = \frac{\Theta^2(0)}{k}\frac{\mathrm{H}(v-u)\,\mathrm{H}(v+u)}{\Theta^2(u)\,\Theta^2(v)}.$$

* See also Art. 327.

9. If $\operatorname{am} a = \alpha$, $\operatorname{am} b = \beta$, $\operatorname{am}(a+b) = \sigma$, show that

$$(1)\quad \sin\alpha\sin\beta\,\Delta\sigma + \cos\sigma = \cos\alpha\cos\beta,$$

$$(2)\quad \cos\beta\cos\sigma + \Delta\alpha\sin\beta\sin\sigma = \cos\alpha,$$

$$(3)\quad \Delta\sigma + k^2\sin\alpha\sin\beta\cos\sigma = \Delta\alpha\,\Delta\beta.$$

(Lagrange.)

10. Show that the algebraic integral of

$$\frac{dx}{\sqrt{X}} = \pm\frac{dy}{\sqrt{Y}},$$

where

$$X = a_0x^4 + 4\,a_1x^3 + 6\,a_2x^2 + 4\,a_3x + a_4,$$
$$Y = a_0y^4 + 4\,a_1y^3 + 6\,a_2y^2 + 4\,a_3y + a_4,$$

may be expressed in the form of the symmetric determinant

$$\begin{vmatrix} 0, & 1, & -\dfrac{x+y}{2}, & xy \\ 1, & a_0, & a_1, & a_2 - 2c \\ -\dfrac{x+y}{2}, & a_1, & a_2 + c, & a_3 \\ xy, & a_2 - 2c, & a_3, & a_4 \end{vmatrix} = 0,$$

where c is an arbitrary constant (Richelot, *Crelle*, Bd. 44, p. 277; Stieltjes, *Bull. des Sciences Math.*, t. XII, pp. 222–227).

CHAPTER XVII

THE SIGMA-FUNCTIONS

ARTICLE 317. In Chapter XIV we derived the function σu from a certain theta-function and we then proceeded to the other sigma-functions. In Chapter XV the function σu was defined through an infinite product which followed from the definition of the $\wp$-function and the characteristic properties of the sigma-function were thus established.

We shall now prescribe these characteristic properties of the sigma-functions and derive therefrom directly the functions themselves.*

In Art. 298 it was shown that

$$\wp u - \wp v = -\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2 u\,\sigma^2 v}.$$

We write $v = \tilde{\omega}$, where $2\,\tilde{\omega} = 2\,p\omega + 2\,q\omega'$. The quantities p and q are integers, and here one of them at least is taken odd, so that $\tilde{\omega}$ is different from a period.

Since $\tilde{\omega}$ is a half period, we may write

$$\wp\tilde{\omega} = e_\lambda \qquad (\lambda = 1, 2, 3).$$

The formula above becomes

$$\wp u - e_\lambda = -\frac{\sigma(u+\tilde{\omega})\,\sigma(u-\tilde{\omega})}{\sigma^2 u\,\sigma^2\tilde{\omega}}.$$

In Art. 290 we derived the formula

$$\sigma(u + 2\,\tilde{\omega}) = \mp\, e^{2\tilde{\eta}(u+\tilde{\omega})}\,\sigma u,$$

where $2\,\tilde{\eta} = 2\,p\eta + 2\,q\eta'$, and the negative or positive sign was to be taken according as $\sigma\tilde{\omega}$ was different from or equal to zero.

In the present case we must therefore take the negative sign; and if $u - \tilde{\omega}$ is written for u, it follows that

$$\sigma(u + \tilde{\omega}) = -\,e^{2\tilde{\eta}u}\,\sigma(u - \tilde{\omega}).$$

We consequently have

$$\wp u - e_\lambda = e^{2\tilde{\eta}u}\frac{\sigma^2(u-\tilde{\omega})}{\sigma^2 u\,\sigma^2\tilde{\omega}} = \left(\frac{e^{\tilde{\eta}u}\,\sigma(u-\tilde{\omega})}{\sigma u\,\sigma\tilde{\omega}}\right)^2.$$

* Hermite (p. 753 of Serret's *Calculus*, 2d volume, 1900) writes: "Nothing is more important nor more worthy of interest than a careful study of a process by which, starting with notions previously acquired, one comes to the knowledge of a new function which becomes the origin of a new order of analytic notions."

If $\sigma_\lambda u$ is defined through the equation

$$\sigma_\lambda u = e^{\tilde{\eta} u} \frac{\sigma(\tilde{\omega} - u)}{\sigma \tilde{\omega}},$$

we have

$$\wp u - e_\lambda = \left(\frac{\sigma_\lambda u}{\sigma u}\right)^2.$$

The quantities η and η' are defined as in Art. 259. As there are only three incongruent half-periods, we have the three new functions

$$\sigma_\lambda u \qquad (\lambda = 1, 2, 3).$$

When $u = 0$ it is seen that $\sigma_\lambda u = 1$. We defined in Art. 272 the function σu through the relation

$$\wp u = -\frac{d^2 \log \sigma u}{d^2 u} = -\frac{d}{du}\frac{\sigma' u}{\sigma u} = \frac{\sigma u \sigma'' u - (\sigma' u)^2}{(\sigma u)^2}.$$

If then we require that the sigma-functions be one-valued, analytic functions which have the character of integral transcendental functions, it is seen that $\wp u$, $\wp u - e_\lambda$ may be expressed through the quotient of such functions (Art. 262).

Art. 318. By means of Laurent's Theorem we may express at once the function σu through a Fourier Series as follows:

If $f(t)$ is a one-valued, finite and continuous function within and on the boundaries of a ring inclosed between two circles, it may be developed in a series * consisting of an infinite number of positive and negative terms in the form

$$f(t) = \sum_{k=-\infty}^{k=+\infty} c_k (t-a)^k \quad (c_k \text{ constant}).$$

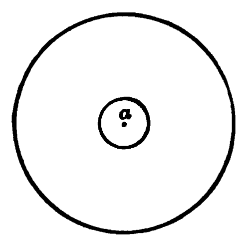


Fig. 72.

We shall next assume that the interior circle is arbitrarily small, so that the above series is convergent for the entire larger circle with the exception of the point a.

Let $F(u)$ be a function defined for the whole or a part of the u-plane and suppose that this function is one-valued, finite, continuous and simply periodic having the period p, say.

We then have

$$F(u + p) = F(u).$$

If we write (cf. Art. 67) $t = e^{\frac{2\pi i}{p} u}$, or $u = \frac{p}{2\pi i}\log t$, we have

$$F(u) = f(t).$$

* Osgood, *loc. cit.*, p. 295.

The function $f(t)$ is one-valued, for if a definite value $\bar{t}$ is given to t, then

$$u = \frac{p}{2\pi i}\log \bar{t} + kp \quad (k \text{ an integer}).$$

But for all such values the function $F(u)$ retains the same value, since p is its period. It follows that $F(u)$ is one-valued.

Further, if t describes a circle, so that $t = re^{i\phi}$, then is

$$u = \frac{p}{2\pi i}\log r + \frac{p\phi}{2\pi} + kp,$$

or
$$u = b + m\phi \quad (b \text{ and } m \text{ constants});$$

and consequently u describes a straight line [Art. 60].

From the relation $u\dfrac{2\pi i}{p} = \log t$ it is seen that for $t = 0$ and also for $t = \infty$ we have $u = \infty$; and since $u = \infty$ is an essential singularity of $F(u)$, it follows that $t = 0$ and $t = \infty$ are singularities of $f(t)$.

Since zero is a singular point of $f(t)$, we have from above the expansion

$$f(t) = \sum_{k=-\infty}^{k=+\infty} c_k t^k$$

and therefore
$$F(u) = \sum_{k=-\infty}^{k=+\infty} c_k e^{k\frac{2\pi i}{p}u}.$$

Art. 319. We write
$$\phi(u) = \sigma u e^{A+Bu+Cu^2},$$

and we shall so determine the constants A, B, C that $\phi(u)$ has the period 2ω. This function $\phi(u)$ is one-valued, finite and continuous for the finite portion of the u-plane.

From the formula
$$\phi(u + 2\omega) = \phi(u)$$
we have, since
$$\sigma(u + 2\omega) = -e^{2\eta(u+\omega)}\sigma u,$$
the formula
$$-e^{2\eta(u+\omega)+A+B(u+2\omega)+C(u+2\omega)^2}\sigma u = e^{A+Bu+Cu^2}\sigma u,$$
or
$$e^{2\eta(u+\omega)+2B\omega+4Cu\omega+4C\omega^2} = -1.$$
It follows that

$$2\eta(u + \omega) + 2B\omega + 4Cu\omega + 4C\omega^2 = (2k + 1)\pi i,$$

where k is an integer; and consequently

$$2\eta + 4C\omega = 0, \quad \text{or} \quad C = -\frac{1}{2}\frac{\eta}{\omega};$$

and
$$2\eta\omega + 2B\omega + 4C\omega^2 = (2k + 1)\pi i;$$
or, if $k = 0$,
$$B = \frac{\pi i}{2\omega}.$$

The remaining constant A being arbitrary, may be taken equal to zero.

We then have

$$\phi(u) = \sigma u\, e^{-\frac{\eta}{2\omega}u^2 + \frac{\pi i}{2\omega}u}.$$

We further write $u = 2\,\omega v$ and put

$$\phi(u) = \psi(v).$$

Since $\phi(u + 2\,\omega) = \phi(u)$, it follows that

$$\phi[2\,\omega(v + 1)] = \phi(2\,\omega v),$$

or

$$\psi(v + 1) = \psi(v),$$

and consequently from the last Article

$$\psi(v) = \sum_{k=-\infty}^{k=+\infty} C_k e^{k2\pi i v} \qquad (p = 1),$$

a series which is uniformly convergent within the finite portion of the v-plane.

To determine the coefficients C_k, we note that

$$\sigma(u + 2\,\omega') = -\, e^{2\eta'(u+\omega')}\sigma u,$$

and consequently

$$\phi(u + 2\,\omega') = -\, e^{2\eta'(u+\omega') - \frac{\eta}{2\omega}(u+2\omega')^2 + \frac{\pi i}{2\omega}(u+2\omega')}\sigma u;$$

or

$$\phi(u + 2\,\omega') = -\, \phi(u)\, e^{2\eta'(u+\omega') - 2\eta u\frac{\omega'}{\omega} - \frac{2\omega'^2}{\omega}\eta + \pi i\frac{\omega'}{\omega}}.$$

Since $\eta\omega' - \omega\eta' = \dfrac{\pi i}{2}$, it follows that

$$\phi(u + 2\,\omega') = -\, e^{-\frac{\pi i u}{\omega}}\phi(u).$$

Writing $\dfrac{\omega'}{\omega} = \tau$, we have

$$\phi(2\,\omega v + 2\,\omega') = -\, e^{-\pi i 2v}\psi(v),$$

or

$$\psi(v + \tau) = -\, e^{-i\pi 2v}\psi(v).$$

Since

$$\psi(v) = \sum_{k=-\infty}^{k=+\infty} C_k e^{2k\pi i v},$$

we therefore have

$$\sum_{k=-\infty}^{k=+\infty} C_k e^{2k\pi i(v+\tau)} = e^{-\pi i(2v+1)}\sum_{k=-\infty}^{k=+\infty} C_k e^{2k\pi i v},$$

or

$$-\sum_{k=-\infty}^{k=+\infty} C_k e^{2k\pi i v} = \sum_{k=-\infty}^{k=+\infty} C_k e^{(2k+2)\pi i v} e^{2k\pi i\tau}.$$

If the coefficients of $e^{2\lambda\pi i v}$ on either side of this equation are equated, we have

$$-C_\lambda = C_{\lambda-1}\,e^{2(\lambda-1)\pi i\tau},$$

which is a transcendental equation of differences.

In the formula

$$C_\lambda = C_{\lambda-1}\,e^{2(\lambda-1)\pi i\tau+\pi i},$$

change λ to $\lambda+1$ and write $\log C_\lambda = B_\lambda$. We then have

$$B_{\lambda+1} = B_\lambda + 2\lambda\pi i\tau + \pi i.$$

Suppose that $\chi(\lambda) = A_0 + A_1\lambda + A_2\lambda^2$ and consequently that

$$\chi(\lambda+1) - \chi(\lambda) = A_1 + 2A_2\lambda + A_2 = 2\lambda\pi i\tau + \pi i.$$

It follows that

$$A_2 = \pi i\tau \quad \text{and} \quad A_1 = \pi i(1-\tau).$$

As A_0 remains arbitrary, we choose it equal to zero. These values substituted in $\chi(\lambda)$ give

$$\chi(\lambda) = \pi i(1-\tau)\lambda + \pi i\tau\lambda^2.$$

Let us further write $B_\lambda - \chi(\lambda) = E_\lambda$. We then have

$$E_{\lambda+1} = B_{\lambda+1} - \chi(\lambda+1) = B_\lambda + 2\lambda\pi i\tau + \pi i - [\chi(\lambda) + 2\lambda\pi i\tau + \pi i]$$
$$= B_\lambda - \chi(\lambda) = E_\lambda.$$

We note that

$$E_{\lambda+1} = E_\lambda = \cdots = E_0.$$

Further, since

$$B_\lambda = \chi(\lambda) + E_0,$$

or

$$\log C_\lambda = \chi(\lambda) + E_0,$$

we have

$$C_\lambda = e^{E_0}e^{\chi(\lambda)}.$$

Writing $e^{E_0} = C$, it follows that $C_\lambda = Ce^{\pi i(1-\tau)\lambda+\pi i\tau\lambda^2}$, and consequently

$$\psi(v) = C\sum_{k=-\infty}^{k=+\infty} e^{\pi i(1-\tau)k+\pi i\tau k^2}\,e^{2k\pi i v}.$$

Further, since

$$\psi(v) = e^{-\frac{\eta}{2\omega}u^2+\frac{\pi i}{2\omega}u}\,\sigma u,$$

it is seen that

$$\sigma u = \sigma(2\omega v) = e^{2\eta\omega v^2-\pi i v}\,C\sum_{k=-\infty}^{k=+\infty} e^{\pi i(1-\tau)k+\pi i\tau k^2}\,e^{2k\pi i v}$$
$$= e^{2\eta\omega v^2}C\sum_{k=-\infty}^{k=+\infty} e^{\pi i(1-\tau)k+\pi i\tau k^2}\,e^{(2k-1)\pi i v}$$
$$= e^{2\eta\omega v^2}C\sum_{k=-\infty}^{k=+\infty} e^{\pi i k}\,e^{\pi i\tau\left(\frac{2k-1}{2}\right)^2}\,e^{(2k-1)\pi i v}\,e^{-\frac{\pi i\tau}{4}}$$
$$= e^{2\eta\omega v^2}e^{-\frac{\pi i\tau}{4}}\,C\sum_{k=-\infty}^{k=\infty}(-1)^k e^{\pi i\tau\left(\frac{2k-1}{2}\right)^2}\,e^{(2k-1)\pi i v}.$$

Letting $-e^{-\frac{\pi i\tau}{4}}C = c$ and substituting $k+1$ for k, we have

$$\sigma(2\omega v) = e^{2\eta\omega v^2}c\sum_{k=-\infty}^{k=+\infty}(-1)^k e^{\pi i\tau\left(\frac{2k+1}{2}\right)^2}e^{(2k+1)\pi i v}.$$

We note that σu is an odd function and we shall assume that the constant c is such that the coefficient in the first term in the expansion of σu is unity, that is,

$$\sigma u = 1 \cdot u + \cdots.$$

The sigma-function is thus completely determined.

ART. 320. If we write $\tilde{\omega} = \omega, \omega'', \omega'$, we have directly from Art. 317 the formulas

$$\sigma_1 u = e^{\eta u}\frac{\sigma(\omega - u)}{\sigma\omega} = e^{-\eta u}\frac{\sigma(u+\omega)}{\sigma\omega},$$

$$\sigma_2 u = e^{\eta'' u}\frac{\sigma(\omega'' - u)}{\sigma\omega''} = e^{-\eta'' u}\frac{\sigma(u+\omega'')}{\sigma\omega''},$$

$$\sigma_3 u = e^{\eta' u}\frac{\sigma(\omega' - u)}{\sigma\omega'} = e^{-\eta' u}\frac{\sigma(u+\omega')}{\sigma\omega'},$$

where $\omega'' = \omega + \omega'$ and $\eta'' = \eta + \eta'$.

The argument $2\omega(v + \frac{1}{2})$ corresponds to the argument $u + \omega$. We may consequently write

$$\sigma(u+\omega) = e^{2\eta\omega(v+\frac{1}{2})^2}c\sum_{k=-\infty}^{k=+\infty}(-1)^k e^{\frac{\pi i\tau}{4}(2k+1)^2}e^{(2k+1)\pi i(v+\frac{1}{2})},$$

so that

$$\sigma_1 u = e^{-2\eta\omega v}e^{2\eta\omega v^2+2\eta\omega v+\frac{\eta\omega}{2}}\frac{c}{\sigma\omega}\sum_{k=-\infty}^{k=+\infty}(-1)^k e^{\frac{\pi i\tau}{4}(2k+1)^2}e^{(2k+1)\pi i v}(e^{\pi i})^k e^{\frac{\pi i}{2}},$$

or

$$\sigma_1 u = \frac{c}{\sigma\omega}e^{2\eta\omega v^2+\frac{\eta\omega}{2}+\frac{\pi i}{2}}\sum_{k=-\infty}^{k=+\infty}(-1)^{2k}e^{\frac{1}{4}(2k+1)^2\pi i\tau}e^{(2k+1)\pi i v}.$$

If we write

$$\frac{c}{\sigma\omega}e^{\frac{\eta\omega}{2}+\frac{\pi i}{2}} = \beta_1,$$

then is

$$\sigma_1 u = \beta_1 e^{2\eta\omega v^2}\sum_{k=-\infty}^{k=+\infty}e^{\frac{1}{4}(2k+1)^2\pi i\tau}e^{(2k+1)\pi i v},$$

and similarly

$$\sigma_2 u = \beta_2 e^{2\eta\omega v^2}\sum_{k=-\infty}^{k=+\infty}e^{k^2\tau\pi i}e^{2k\pi i v},$$

$$\sigma_3 u = \beta_3 e^{2\eta\omega v^2}\sum_{k=-\infty}^{k=+\infty}(-1)^k e^{k^2\tau\pi i}e^{2k\pi i v}.$$

If with Weierstrass we write

$$e^{\tau\pi i} = h \quad \text{and} \quad e^{\pi i v} = z,$$

we have

$$\sigma u = ce^{2\eta\omega v^2} \sum_{k=-\infty}^{k=+\infty} (-1)^k h^{\frac{(2k+1)^2}{4}} z^{2k+1},$$

$$\sigma_1 u = \beta_1 e^{2\eta\omega v^2} \sum_{k=-\infty}^{k=+\infty} h^{\frac{(2k+1)^2}{4}} z^{2k+1},$$

$$\sigma_2 u = \beta_2 e^{2\eta\omega v^2} \sum_{k=-\infty}^{k=+\infty} h^{k^2} z^{2k},$$

$$\sigma_3 u = \beta_3 e^{2\eta\omega v^2} \sum_{k=-\infty}^{k=+\infty} (-1)^k h^{k^2} z^{2k}.$$

Using the notation of Jacobi: $h = e^{\pi i \frac{\omega'}{\omega}} = e^{-\pi\frac{K'}{K}} = q$, and writing with him

$$\begin{aligned}
\vartheta_1(v) &= \frac{1}{i}\sum_{k=-\infty}^{k=+\infty} (-1)^k q^{\frac{(2k+1)^2}{4}} z^{2k+1} \\
&= 2q^{\frac{1}{4}} \sin \pi v - 2q^{\frac{9}{4}} \sin 3\pi v + 2q^{\frac{25}{4}} \sin 5\pi v - \cdots, \\
\vartheta_2(v) &= \sum_{k=-\infty}^{k=+\infty} q^{\frac{(2k+1)^2}{4}} z^{2k+1} \\
&= 2q^{\frac{1}{4}} \cos \pi v + 2q^{\frac{9}{4}} \cos 3\pi v + 2q^{\frac{25}{4}} \cos 5\pi v + \cdots, \\
\vartheta_3(v) &= \sum_{k=-\infty}^{k=+\infty} q^{k^2} z^{2k} \\
&= 1 + 2q\cos 2\pi v + 2q^4 \cos 4\pi v + 2q^9 \cos 6\pi v + \cdots, \\
\vartheta_0(v) &= \sum_{k=-\infty}^{k=+\infty} (-1)^k q^{k^2} z^{2k} \\
&= 1 - 2q\cos 2\pi v + 2q^4 \cos 4\pi v - 2q^9 \cos 6\pi v + \cdots,
\end{aligned}$$

we have

$$\begin{aligned}
\sigma u &= \beta e^{2\eta\omega v^2} \vartheta_1(v) \qquad [\beta = ci], \\
\sigma_1 u &= \beta_1 e^{2\eta\omega v^2} \vartheta_2(v), \\
\sigma_2(u) &= \beta_2 e^{2\eta\omega v^2} \vartheta_3(v), \\
\sigma_3 u &= \beta_3 e^{2\eta\omega v^2} \vartheta_0(v).
\end{aligned}$$

ART. 321. By differentiating both sides of the formula above for σu and then writing $u = 0$, it is seen that

$$\frac{d\sigma u}{du} = [\beta\, 4\,\eta\omega v \vartheta_1(v) + \beta\vartheta_1'(v)] e^{2\eta\omega v^2} \frac{dv}{du},$$

or

$$\beta = \frac{2\omega}{\vartheta_1'(0)}.$$

Further since $\sigma_1(0) = 1 = \sigma_2(0) = \sigma_3(0)$, it follows that

$$\beta_1 = \frac{1}{\vartheta_2(0)}, \quad \beta_2 = \frac{1}{\vartheta_3(0)}, \quad \beta_3 = \frac{1}{\vartheta_0(0)}.$$

In Art. 340 it is shown that

$$\vartheta_1'(0) = 2\pi h^{\frac{1}{4}} \prod_{n=1}^{n=\infty} (1 - h^{2n})^3.$$

When this value is substituted in the formula above for σu, we have

$$\sigma u = e^{2\eta\omega v^2} \frac{2\omega}{\pi} \sin v\pi \prod_{n=1}^{n=\infty} \frac{1 - 2h^{2n}\cos 2\pi v + h^{4n}}{(1 - h^{2n})^2}.$$

In a similar manner if we write for β_1, β_2, β_3, their values we have

$$\sigma_1(u) = e^{2\eta\omega v^2} \cos v\pi \prod_{n=1}^{n=\infty} \frac{1 + 2h^{2n}\cos 2v\pi + h^{4n}}{(1 + h^{2n})^2},$$

$$\sigma_2 u = e^{2\eta\omega v^2} \prod_{n=1}^{n=\infty} \frac{1 + 2h^{2n-1}\cos 2v\pi + h^{4n-2}}{(1 + h^{2n-1})^2},$$

$$\sigma_3 u = e^{2\eta\omega v^2} \prod_{n=1}^{n=\infty} \frac{1 - 2h^{2n-1}\cos 2v\pi + h^{4n-2}}{(1 - h^{2n-1})^2}.$$

Take the logarithmic derivatives of $\sigma_1 u$, $\sigma_2 u$, $\sigma_3 u$ and equate the coefficients of u on either side of the resulting expressions. We then have

$$2\eta\omega = -2e_1\omega^2 + \pi^2 \left\{ \frac{1}{2} + \sum_{n=1}^{n=\infty} \frac{4h^{2n}}{(1 + h^{2n})^2} \right\},$$

$$2\eta\omega = -2e_2\omega^2 + \pi^2 \sum_{n=1}^{n=\infty} \frac{4h^{2n-1}}{(1 + h^{2n-1})^2},$$

$$2\eta\omega = -2e_3\omega^2 - \pi^2 \sum_{n=1}^{n=\infty} \frac{4h^{2n-1}}{(1 - h^{2n-1})^2}.$$

Since $e_1 + e_2 + e_3 = 0$, it follows from Art. 286 that

$$\frac{\pi^2}{3} \left\{ \frac{1}{2} + \sum_{n=1}^{n=\infty} \frac{4h^{2n}}{(1 + h^{2n})^2} + \sum_{n=1}^{n=\infty} \frac{4h^{2n-1}}{(1 + h^{2n-1})^2} - \sum_{n=1}^{n=\infty} \frac{4h^{2n-1}}{(1 - h^{2n-1})^2} \right\}$$
$$= 2\eta\omega = \frac{\pi^2}{3} \left\{ \frac{1}{2} - 3 \sum_{n=1}^{n=\infty} \frac{4h^{2n}}{(1 - h^{2n})^2} \right\}.$$

We note that σu is an *odd* function, while $\sigma_1 u$, $\sigma_2 u$ and $\sigma_3 u$ are even functions.

The zeros of these four functions are given in the Table of Formulas, No. XXXI.

ART. 322. If the formulas

$$\wp u - e_1 = \left(\frac{\sigma_1 u}{\sigma u}\right)^2, \quad \wp u - e_2 = \left(\frac{\sigma_2 u}{\sigma u}\right)^2, \quad \wp u - e_3 = \left(\frac{\sigma_3 u}{\sigma u}\right)^2$$

are multiplied together, we have in virtue of the equation

$$(\wp' u)^2 = 4(\wp u - e_1)(\wp u - e_2)(\wp u - e_3),$$

the formula

$$\pm \wp' u = 2\sqrt{\left(\frac{\sigma_1 u\, \sigma_2 u\, \sigma_3 u}{\sigma^3 u}\right)^2}.$$

To determine the sign to be used before the root, write $u = 0$ and it is seen that the negative sign must be employed. We thus have

(1) $$\wp' u = -2\frac{\sigma_1 u\, \sigma_2 u\, \sigma_3 u}{\sigma^3 u}.$$

In Art. 302 it was seen that

(2) $$\frac{\sigma(2u)}{\sigma^4 u} = -\wp' u.$$

It follows from (1) that

$$\sigma(2u) = 2\,\sigma u\, \sigma_1 u\, \sigma_2 u\, \sigma_3 u.$$

ART. 323. We may next note how the sigma-functions behave when the argument u is increased by a period.

Since

$$\sigma_1 u = e^{\eta u}\frac{\sigma(\omega - u)}{\sigma\omega} = e^{-\eta u}\frac{\sigma(\omega + u)}{\sigma\omega},$$

it follows that

$$\sigma_1(u+2\omega) = e^{\eta(u+2\omega)}\frac{\sigma(\omega - u - 2\omega)}{\sigma\omega}$$

$$= -e^{\eta(u+2\omega)}\frac{\sigma(\omega + u)}{\sigma\omega} = -e^{2\eta(u+\omega)}e^{-\eta u}\frac{\sigma(\omega + u)}{\sigma\omega},$$

or

$$\sigma_1(u + 2\omega) = -e^{2\eta(u+\omega)}\,\sigma_1(u),$$

and similarly *

$$\sigma_2(u + 2\omega) = e^{2\eta(u+\omega)}\,\sigma_2(u),$$

$$\sigma_3(u + 2\omega) = e^{2\eta(u+\omega)}\,\sigma_3(u),$$

$$\sigma(u + 2\omega) = -e^{2\eta(u+\omega)}\,\sigma u.$$

Formulas for $\sigma_1(u + 2\omega'')$, etc., are found in the Table of Formulas, No. XXVI.

* See Schwarz, *loc. cit.*, p. 22.

ART. 324. Let λ, μ, ν represent in any order the integers 1, 2, 3; then by Art. 262 we have

$$\left(\frac{\sigma_\lambda u}{\sigma u}\right)^2 = \wp u - e_\lambda, \qquad \left(\frac{\sigma_\mu u}{\sigma u}\right)^2 = \wp u - e_\mu,$$

$$\left(\frac{\sigma_\nu u}{\sigma u}\right)^2 = \wp u - e_\nu,$$

no two of the quantities λ, μ, ν being supposed equal.

By eliminating $\wp u$ from the second and the third of these formulas we have

$$\left(\frac{\sigma_\mu u}{\sigma u}\right)^2 - \left(\frac{\sigma_\nu u}{\sigma u}\right)^2 = -(e_\mu - e_\nu),$$

or

$$(\sigma_\mu u)^2 - (\sigma_\nu u)^2 + (e_\mu - e_\nu)\,\sigma^2 u = 0.$$

It is also seen that

$$(e_2 - e_3)\sigma_1^2 u + (e_3 - e_1)\sigma_2^2 u + (e_1 - e_2)\sigma_3^2 u = 0.$$

DIFFERENTIAL EQUATIONS WHICH ARE SATISFIED BY SIGMA-QUOTIENTS.

ART. 325. If the formula

$$\left(\frac{\sigma_\lambda u}{\sigma u}\right)^2 = \wp u - e_\lambda$$

is differentiated, and for $\wp' u$ its value in terms of the sigma-functions is substituted, it follows that

$$2\frac{\sigma_\lambda u}{\sigma u}\, d\,\frac{\sigma_\lambda u}{\sigma u} = -\,2\,\frac{\sigma_\lambda u}{\sigma u}\,\frac{\sigma_\mu u}{\sigma u}\,\frac{\sigma_\nu u}{\sigma u}\,du,$$

or

$$\frac{d}{du}\,\frac{\sigma_\lambda u}{\sigma u} = -\,\frac{\sigma_\mu u}{\sigma u}\,\frac{\sigma_\nu u}{\sigma u}.$$

If we differentiate the equation

$$\left(\frac{\sigma_\mu u}{\sigma_\nu u}\right)^2 = \frac{\wp u - e_\mu}{\wp u - e_\nu},$$

it follows that

$$2\,\frac{\sigma_\mu u}{\sigma_\nu u}\,\frac{d}{du}\,\frac{\sigma_\mu u}{\sigma_\nu u} = \frac{(\wp u - e_\nu) - (\wp u - e_\mu)}{(\wp u - e_\nu)^2}\,\wp' u$$

$$= \frac{e_\mu - e_\nu}{\left(\dfrac{\sigma_\nu u}{\sigma u}\right)^4}\left(-\,2\,\frac{\sigma_\lambda u\,\sigma_\mu u\,\sigma_\nu u}{(\sigma u)^3}\right),$$

or

$$\frac{d}{du}\,\frac{\sigma_\mu u}{\sigma_\nu u} = -(e_\mu - e_\nu)\,\frac{\sigma_\lambda u\,\sigma u}{\sigma_\nu u\,\sigma_\nu u}.$$

From

$$\left(\frac{\sigma u}{\sigma_\lambda u}\right)^2 = \frac{1}{\wp u - e_\lambda},$$

it follows that

$$2\frac{\sigma u}{\sigma_\lambda u}\frac{d}{du}\left(\frac{\sigma u}{\sigma_\lambda u}\right) = -\frac{\wp' u}{(\wp u - e_\lambda)^2} = \frac{2\dfrac{\sigma_\lambda u}{\sigma u}\dfrac{\sigma_\mu u}{\sigma u}\dfrac{\sigma_\nu u}{\sigma u}}{\left(\dfrac{\sigma_\lambda u}{\sigma u}\right)^4},$$

or

$$\frac{d}{du}\frac{\sigma u}{\sigma_\lambda u} = \frac{\sigma_\mu u}{\sigma_\lambda u}\frac{\sigma_\nu u}{\sigma_\lambda u}.$$

Since the equation

$$\sigma_\mu^2 u - \sigma_\nu^2 u + (e_\mu - e_\nu)\sigma^2 u = 0$$

may be written

$$\frac{\sigma_\mu^2 u}{\sigma_\lambda^2 u} = 1 - (e_\mu - e_\lambda)\frac{\sigma^2 u}{\sigma_\lambda^2 u},$$

and further since

$$\left(\frac{d}{du}\frac{\sigma u}{\sigma_\lambda u}\right)^2 = \left(\frac{\sigma_\mu u}{\sigma_\lambda u}\right)^2\left(\frac{\sigma_\nu u}{\sigma_\lambda u}\right)^2,$$

we have

$$\left(\frac{d}{du}\frac{\sigma u}{\sigma_\lambda u}\right)^2 = \left[1 - (e_\mu - e_\lambda)\left(\frac{\sigma u}{\sigma_\lambda u}\right)^2\right]\left[1 - (e_\nu - e_\lambda)\left(\frac{\sigma u}{\sigma_\lambda u}\right)^2\right].$$

In the same way it may be shown that

$$\left(\frac{d}{du}\frac{\sigma_\mu u}{\sigma_\nu u}\right)^2 = \left[1 - \frac{\sigma_\mu^2 u}{\sigma_\nu^2 u}\right]\left[e_\mu - e_\lambda + (e_\lambda - e_\nu)\frac{\sigma_\mu^2 u}{\sigma_\nu^2 u}\right],$$

and

$$\left(\frac{d}{du}\frac{\sigma_\lambda u}{\sigma u}\right)^2 = \left[\frac{\sigma_\lambda^2 u}{\sigma^2 u} + e_\lambda - e_\mu\right]\left[\frac{\sigma_\lambda^2 u}{\sigma^2 u} + e_\lambda - e_\nu\right].$$

It follows that

$$\frac{\sigma u}{\sigma_\lambda u},\quad \frac{1}{\sqrt{e_\mu - e_\lambda}}\frac{\sigma_\mu u}{\sigma_\nu u},\quad \frac{1}{\sqrt{e_\nu - e_\lambda}}\frac{\sigma_\nu u}{\sigma_\mu u},\quad \frac{1}{\sqrt{(e_\mu - e_\lambda)(e_\nu - e_\lambda)}}\frac{\sigma_\lambda u}{\sigma u}$$

are all particular solutions of the differential equation *

$$\text{(A)}\qquad \left(\frac{d\xi}{du}\right)^2 = [1 - (e_\mu - e_\lambda)\xi^2][1 - (e_\nu - e_\lambda)\xi^2].$$

ART. 326. If we write $\nu = 2$, $\mu = 1$, $\lambda = 3$ and

$$\xi = \frac{\sigma u}{\sigma_3 u},$$

the differential equation (A) becomes

$$\left(\frac{d\xi}{du}\right)^2 = [1 - (e_1 - e_3)\xi^2][1 - (e_2 - e_3)\xi^2].$$

* See Schwarz, *loc. cit.*, Art. 25; or Daniels, *Am. Journ. Math.*, Vol. VI, p. 180 and Vol. VII, p. 89.

Further write

$$(e_1 - e_3)\xi^2 = x^2, \quad \text{or} \quad \sqrt{e_1 - e_3}\,\xi = x,$$

and

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3}.$$

We then have

$$\left(\frac{dx}{d\sqrt{e_1 - e_3}\cdot u}\right)^2 = (1 - x^2)(1 - k^2x^2).$$

If

$$u = \int_0^x \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}},$$

then is $x = \sin am\, u$ or $x = sn(u, k)$.

We therefore have

(1) $$\xi = \frac{\sigma u}{\sigma_3 u} = \frac{1}{\sqrt{e_1 - e_3}}\, sn(\sqrt{e_1 - e_3}\cdot u, k).$$

Further, since

$$\sqrt{1 - x^2} = \sqrt{1 - (e_1 - e_3)\frac{\sigma^2 u}{\sigma_3{}^2 u}} = \sqrt{\frac{\sigma_3{}^2 u - (e_1 - e_3)\sigma^2 u}{\sigma_3{}^2 u}},$$

and as

$$\sigma_3{}^2 u - \sigma_1{}^2 u + (e_3 - e_1)\sigma^2 u = 0,$$

we have

(2) $$cn(\sqrt{e_1 - e_3}\cdot u, k) = \frac{\sigma_1 u}{\sigma_3 u};$$

and similarly

(3) $$dn(\sqrt{e_1 - e_3}\cdot u, k) = \frac{\sigma_2 u}{\sigma_3 u}.$$

ART. 327. If we write *

$$\xi_{\lambda 0}(u) = \frac{\sigma_\lambda u}{\sigma u} = \sqrt{\wp u - e_\lambda},$$

$$\xi_{0\lambda}(u) = \frac{\sigma u}{\sigma_\lambda u} = \frac{1}{\sqrt{\wp u - e_\lambda}},$$

$$\xi_{\mu\nu}(u) = \frac{\sigma_\mu u}{\sigma_\nu u} = \frac{\sqrt{\wp u - e_\mu}}{\sqrt{\wp u - e_\nu}},$$

we have at once

$$\xi_{\lambda 0}(u)\xi_{0\lambda}(u) = 1$$

and

$$\xi_{\mu\nu}(u) = \frac{\xi_{\mu 0}(u)}{\xi_{\nu 0}(u)} = \xi_{\mu 0}(u)\xi_{0\nu}(u) = \xi_{\mu\lambda}(u)\xi_{\lambda\nu}(u).$$

* See Enneper, *Elliptische Funktionen*, p. 160; or Tannery et Molk, *Fonct. Ellipt.*, t. II, Chap. IV.

It is also evident that

$$\wp u = e_\lambda + \xi_{\lambda 0}^2(u) = e_\mu + \xi_{\mu 0}^2(u) = e_\nu + \xi_{\nu 0}^2(u),$$

$$3\wp u = \xi_{\lambda 0}^2(u) + \xi_{\mu 0}^2(u) + \xi_{\nu 0}^2(u),$$

$$\left.\begin{aligned}\xi_{\lambda 0}(u + 2\omega_\lambda) &= \xi_{\lambda 0}(u),\\ \xi_{\lambda 0}(u + 2\omega_\mu) &= -\xi_{\lambda 0}(u)\end{aligned}\right\}\ \begin{array}{l}\text{where we write without regard to order}\\ \qquad\omega_\lambda, \omega_\mu, \omega_\nu \text{ for } \omega, \omega'', \omega'.\end{array}$$

$$\xi_{\mu\nu}(u + 2\omega_\lambda) = \xi_{\mu\nu}(u),$$

$$\xi_{\mu\nu}(u + 2\omega_\mu) = -\xi_{\mu\nu}(u);$$

$$\xi_{\lambda 0}'(u) = -\xi_{\mu 0}(u)\,\xi_{\nu 0}(u), \quad \xi_{0\lambda}'(u) = \xi_{\mu\lambda}(u)\,\xi_{\nu\lambda}(u),$$

$$\xi_{\mu\nu}'(u) = -(e_\mu - e_\nu)\,\xi_{0\nu}(u)\,\xi_{\lambda\nu}(u).$$

ART. 328. Through the equations *

$$(1)\qquad \sqrt{\wp u - e_1} = \frac{\sigma_1 u}{\sigma u}, \quad \sqrt{\wp u - e_2} = \frac{\sigma_2 u}{\sigma u}, \quad \sqrt{\wp u - e_3} = \frac{\sigma_3 u}{\sigma u},$$

the values of the three quantities $\sqrt{\wp u - e_\lambda}$ are defined as one-valued functions of u.

If we give to u the values ω, ω'', ω', it is seen that

$$(2)\qquad \begin{cases}\sqrt{e_1 - e_2} = \dfrac{\sigma_2\omega}{\sigma\omega} = \dfrac{e^{\eta''\omega}\sigma\omega'}{\sigma\omega\,\sigma\omega''}, & \sqrt{e_1 - e_3} = \dfrac{\sigma_3\omega}{\sigma\omega} = \dfrac{e^{-\eta'\omega}\sigma\omega''}{\sigma\omega\,\sigma\omega'}.\\[2ex] \sqrt{e_2 - e_1} = \dfrac{\sigma_1\omega''}{\sigma\omega''} = -\dfrac{e^{\eta\omega''}\sigma\omega'}{\sigma\omega\,\sigma\omega''}, & \sqrt{e_2 - e_3} = \dfrac{\sigma_3\omega''}{\sigma\omega''} = -\dfrac{e^{\eta'\omega''}\sigma\omega}{\sigma\omega'\,\sigma\omega''},\\[2ex] \sqrt{e_3 - e_1} = \dfrac{\sigma_1\omega'}{\sigma\omega'} = \dfrac{e^{-\eta\omega'}\sigma\omega''}{\sigma\omega\,\sigma\omega'}, & \sqrt{e_3 - e_2} = \dfrac{\sigma_2\omega'}{\sigma\omega'} = \dfrac{e^{\eta''\omega'}\sigma\omega}{\sigma\omega'\,\sigma\omega''}.\end{cases}$$

Through these formulas the six quantities on the left-hand side are uniquely determined.

We note that

$$\frac{\sqrt{e_3 - e_2}}{\sqrt{e_2 - e_3}} = -e^{\eta''\omega' - \eta'\omega''}.$$

On the other hand

$$\eta''\omega' - \eta'\omega'' = \eta\omega' - \eta'\omega = +\tfrac{1}{2}\pi i,$$

if

$$\mathsf{R}\left(\frac{\omega'}{i\omega}\right) > 0 \quad \text{(see Art. 288).}$$

Hence among the six quantities above there exist the relations

$$\sqrt{e_3 - e_2} = -i\sqrt{e_2 - e_3}, \quad \sqrt{e_3 - e_1} = -i\sqrt{e_1 - e_3}, \quad \sqrt{e_2 - e_1} = -i\sqrt{e_1 - e_2},$$

or

$$i\sqrt{e_3 - e_2} = \sqrt{e_2 - e_3}, \quad i\sqrt{e_3 - e_1} = \sqrt{e_1 - e_3}, \quad i\sqrt{e_2 - e_1} = \sqrt{e_1 - e_2}.$$

We have thus reduced the six roots without any ambiguity to the three roots $\sqrt{e_1 - e_2}$, $\sqrt{e_1 - e_3}$, $\sqrt{e_2 - e_3}$, which three roots are real and positive if the discriminant of $4s^3 - g_2 s - g_3 = 0$ is positive.

* Schwarz, *loc. cit.*, Art. 21.

Remark. For the sake of a greater symmetry some recent writers on this theory have written ω_1, ω_2, ω_3 for the quantities which at the outset with Weierstrass we denoted by $\omega, \omega'', \omega'$. When such formulas that result are compared with those given by Weierstrass, much confusion, in particular with regard to sign, arises; for example with these writers

$$\sqrt{e_3-e_2}=i\sqrt{e_2-e_3},\ \sqrt{e_3-e_1}=-i\sqrt{e_1-e_3},\ \sqrt{e_2-e_1}=i\sqrt{e_1-e_2}.$$

The explanation they give to $-\omega_2$ is not entirely satisfactory, especially if these quantities are defined on the Riemann Surface with reference to K and iK'.

ART. 329. From the equations (2) above it follows that

$$\text{(A)}\qquad \sigma\omega=\frac{e^{\frac{1}{2}\eta\omega}}{\sqrt[4]{e_1-e_3}\,\sqrt[4]{e_1-e_2}},\quad \sigma\omega''=\frac{\sqrt{i}\,e^{\frac{1}{2}\eta''\omega''}}{\sqrt[4]{e_2-e_3}\,\sqrt[4]{e_1-e_2}},$$

$$\sigma\omega'=\frac{i\,e^{\frac{1}{2}\eta'\omega'}}{\sqrt[4]{e_2-e_3}\,\sqrt[4]{e_1-e_3}}\quad (\text{where } i=e^{\frac{1}{2}\pi i}).$$

We note here (see also Art. 345) that the quantities

$$\sqrt[4]{e_2-e_3},\quad \sqrt[4]{e_1-e_3},\quad \sqrt[4]{e_1-e_2}$$

can take only such values whose squares

$$\sqrt[2]{e_2-e_3},\quad \sqrt[2]{e_1-e_3},\quad \sqrt[2]{e_1-e_2}$$

are uniquely determined through the equations (2) of Art. 328. Hence each of the fourth roots may take *two* and not *four* values; but as soon as the value of any one of these quantities is known, the values of the two others are uniquely determined through the formulas (A).

If in the formula

$$\sigma_1 u=\frac{e^{-\eta u}\sigma(\omega+u)}{\sigma\omega},$$

we put $u=-\frac{1}{2}\omega$, we have

$$\sigma_1\left(\frac{\omega}{2}\right)=\frac{e^{\frac{\eta\omega}{2}}\sigma\left(\frac{\omega}{2}\right)}{\sigma(\omega)}.$$

It follows that we may write formulas (A) in the form

$$\sqrt[4]{e_1-e_3}\,\sqrt[4]{e_1-e_2}=\frac{\sigma_1\left(\frac{\omega}{2}\right)}{\sigma\left(\frac{\omega}{2}\right)},\quad \sqrt[4]{e_2-e_3}\,\sqrt[4]{e_1-e_2}=\sqrt{i}\,\frac{\sigma_2\left(\frac{\omega''}{2}\right)}{\sigma\left(\frac{\omega''}{2}\right)},$$

$$\sqrt[4]{e_2-e_3}\,\sqrt[4]{e_1-e_3}=i\,\frac{\sigma_3\left(\frac{\omega'}{2}\right)}{\sigma\left(\frac{\omega'}{2}\right)},$$

which expressions may be used to determine the products of any two of the three fourth roots.

ART. 330. We may next derive a table of the four sigma-functions when the argument is increased or diminished by a quarter-period. It is assumed that the definite values derived above are given to the square and fourth roots that appear.

Take, for example, the formula

$$\sigma_1 u = \frac{e^{-\eta u}\sigma(\omega + u)}{\sigma\omega} = \frac{e^{\eta u}\sigma(\omega - u)}{\sigma\omega}.$$

We have at once

$$\sigma(u \pm \omega) = \pm\, e^{\pm \eta u}\sigma\omega\sigma_1 u = \pm \frac{1}{\sqrt[4]{e_1 - e_2}\cdot\sqrt[4]{e_1 - e_3}} e^{\pm\eta\left(u \pm \frac{\omega}{2}\right)}\sigma_1 u.$$

Further, since

$$(\sigma\omega)^2 = \frac{e^{\eta\omega}}{\sqrt[2]{e_2 - e_3}\sqrt[2]{e_1 - e_2}},$$

it follows that

$$\sigma_1(u \pm \omega) = \mp\sqrt{e_1 - e_3}\sqrt{e_1 - e_2}\, e^{\pm\eta u}\sigma\omega\sigma u = \mp\sqrt[4]{e_1 - e_2}\sqrt[4]{e_1 - e_3}\, e^{\pm(u \pm \frac{1}{2}\omega)\eta}\sigma u.$$

The formulas given in the Table of Formulas No. XXXIV should be verified.

ART. 331. It is seen that

$$\frac{\sigma(u + 2\omega)}{\sigma_3(u + 2\omega)} = -\frac{\sigma u}{\sigma_3 u}$$

and consequently

$$\frac{\sigma(u + 4\omega)}{\sigma_3(u + 4\omega)} = \frac{\sigma u}{\sigma_3 u}.$$

Also, since $\dfrac{\sigma(u + 2\omega')}{\sigma_3(u + 2\omega')} = \dfrac{\sigma u}{\sigma_3 u}$, it follows that 4ω and $2\omega'$ are periods of $\dfrac{\sigma u}{\sigma_3 u}$. A closer investigation shows that 4ω and $2\omega'$ are a *primitive* pair of periods of this function; for in the period-parallelogram with the sides 4ω and $2\omega'$ the function $\sigma_3 u$ becomes zero only on the points ω and $2\omega + \omega'$, being zero of the first order. Hence $\dfrac{\sigma u}{\sigma_3 u}$ becomes infinite of the first order on these points. Since only two infinities lie within the period-parallelogram with the sides 4ω and $2\omega'$, and since the smallest number of infinities within a primitive period-parallelogram is *two*, it follows that 4ω, $2\omega'$ form a pair of primitive periods of $\dfrac{\sigma u}{\sigma_3 u}$.

ART. 332. It follows at once from the formulas above that

$$\frac{\sigma(u + \omega)}{\sigma_3(u + \omega)} = \frac{1}{\sqrt{e_1 - e_3}}\frac{\sigma_1 u}{\sigma_2 u}.$$

This may also be seen from the formula of Art. 326

$$\frac{\sigma u}{\sigma_3 u} = \frac{1}{\sqrt{e_1 - e_3}} sn\left(u\sqrt{e_1 - e_3}, k\right).$$

Since $K = \sqrt{e_1 - e_3} \cdot \omega$ we have

$$\frac{\sigma(u+\omega)}{\sigma_3(u+\omega)} = \frac{1}{\sqrt{e_1-e_3}}\, sn(u\sqrt{e_1-e_3} + K, k)$$
$$= \frac{1}{\sqrt{e_1-e_3}}\, \frac{cn(u\sqrt{e_1-e_3}, k)}{dn(u\sqrt{e_1-e_3}, k)} = \frac{1}{\sqrt{e_1-e_3}}\, \frac{\sigma_1 u}{\sigma_2 u}.$$

For $u = 0$, it follows that

$$\frac{\sigma\omega}{\sigma_3\omega} = \frac{1}{\sqrt{e_1-e_3}},$$

and further that all values of u which satisfy the equation

$$\frac{\sigma u}{\sigma_3 u} = \frac{1}{\sqrt{e_1-e_3}}$$

are contained in the form

$$\omega + 4p\omega + 2q\omega',$$

where p and q are integers positive or negative, including zero.

We might define K more generally through the equation

$$K = \sqrt{e_1-e_3}(\omega + 4p\omega + 2q\omega'),$$

where it is assumed that $4p + 1$ and $2q$ have no common divisor. The quantity $\sqrt{e_1 - e_3}$ is to have the *same* value as given in formulas (2) of Art. 328 or the *opposite* value according as q is even or odd.

ART. 333. It also follow from the equation

$$\frac{\sigma u}{\sigma_3 u} = \frac{1}{\sqrt{e_1-e_3}} \sin \operatorname{am} (\sqrt{e_1-e_3} \cdot u, k)$$

that
$$\frac{1}{\sqrt{e_1-e_3}} = \frac{1}{\sqrt{e_1-e_3}} \sin \operatorname{am} (K, k),$$

or
$$sn(K, k) = 1 \quad \text{(see Art. 218)}.$$

The *coamplitude* is defined by Jacobi (*Werke* I, p. 81) through the formula (see Art. 221)

$$\operatorname{coam} (\sqrt{e_1-e_3} \cdot u, k) = \operatorname{am} (K - \sqrt{e_1-e_3} \cdot u, k),$$

or

$$\operatorname{coam} (\sqrt{e_1-e_3} \cdot u, k) = \operatorname{am} [\sqrt{e_1-e_3}(\omega + 4p\omega + 2q\omega' - u), k].$$

Since $4p\omega$ is a period for all the sigma-functions, it may be dropped from the argument u.

We then have

$$K = \sqrt{e_1-e_3}(\omega + 2q\omega'),$$

and

$$\operatorname{coam} (\sqrt{e_1-e_3} \cdot u, k) = \operatorname{am} [\sqrt{e_1-e_3}(\omega + 2q\omega' - u), k].$$

We may note that

$$\cos\operatorname{am}\left[\sqrt{e_1-e_3}\,(\omega+2q\omega'-u),k\right]=\frac{\sigma_1(-u+\omega+2q\omega',k)}{\sigma_3(-u+\omega+2q\omega',k)}$$

$$=\frac{\sigma_1(u-\omega-2q\omega',k)}{\sigma_3(u-\omega-2q\omega',k)}.$$

Since

$$\frac{\sigma_1(u-\omega)}{\sigma_3(u-\omega)}=\sqrt{e_1-e_2}\,\frac{\sigma u}{\sigma_2 u},$$

we have

$$\cos\operatorname{coam}\left[\sqrt{e_1-e_3}\cdot u,k\right]=\sqrt{e_1-e_2}\,\frac{\sigma(u-2q\omega',k)}{\sigma_2(u-2q\omega',k)},$$

and since

$$\frac{\sigma(u+2\omega')}{\sigma_2(u+2\omega')}=-\frac{\sigma u}{\sigma_2 u},$$

we have finally *

$$\cos\operatorname{coam}\left[\sqrt{e_1-e_3}\cdot u,k\right]=(-1)^q\sqrt{e_1-e_2}\,\frac{\sigma u}{\sigma_2 u}.$$

Making $q=0$, we have the set of formulas given in the table, No. LIV.

ART. 334. In Art. 79 we wrote (Cf. Schwarz, *loc. cit.*, Art. 33)

$$\tilde\omega=p\omega+q\omega',\quad \tilde\omega'=p'\omega+q'\omega',\quad \tilde\omega''=\tilde\omega+\tilde\omega',$$

where p, q, p', q' were any integers such that $pq'-qp'=1$; and it was seen that 2ω, $2\omega'$ and $2\tilde\omega$, $2\tilde\omega'$ formed equivalent pairs of primitive periods.

We shall further write

$$\tilde\eta=p\eta+q\eta',\quad \tilde\eta'=p'\eta+q'\eta',\quad \tilde\eta''=\tilde\eta+\tilde\eta'.$$

If in the place of the quantities

$$\omega,\ \omega',\ \omega''=\omega+\omega';\qquad \eta,\ \eta',\ \eta''=\eta+\eta';$$

we substitute

$$\tilde\omega,\ \tilde\omega',\ \tilde\omega''=\tilde\omega+\tilde\omega';\qquad \tilde\eta,\ \tilde\eta',\ \tilde\eta''=\tilde\eta+\tilde\eta',$$

it follows at once (Arts. 276, 271) that the invariants g_2, g_3 and the functions $\wp u$, σu remain unaltered.

Also owing to the equation

$$(\wp'u)^2=4[\wp u-\wp\omega][\wp u-\wp\omega''][\wp u-\wp\omega']=4[\wp u-e_1][\wp u-e_2][\wp u-e_3]$$

the collectivity of the three quantities e_1, e_2, e_3 remains unchanged and consequently also the collectivity of the three functions

$$\left(\frac{\sigma_i u}{\sigma u}\right)^2=\wp u-e_i\qquad (i=1,2,3),$$

although the indices 1, 2, 3 may be permuted.

* See Schwarz, *loc. cit.*, p. 30; or Daniels, *Am. Journ. Math.*, Vol. VII, p. 89.

We therefore have a set of more general formulas if in the preceding developments we write

$$\left.\begin{matrix} \tilde{\omega}, & \tilde{\omega}'', & \tilde{\omega}' \\ \tilde{\eta}, & \tilde{\eta}'', & \tilde{\eta}' \\ e_\lambda, & e_\mu, & e_\nu \\ \sigma_\lambda, & \sigma_\mu, & \sigma_\nu \\ v = \dfrac{u}{2\tilde{\omega}}, & & \tau = \dfrac{\tilde{\omega}'}{\tilde{\omega}} \end{matrix}\right\} \text{ in the place of } \left\{\begin{matrix} \omega, & \omega'', & \omega' \\ \eta, & \eta'', & \eta' \\ e_1, & e_2, & e_3 \\ \sigma_1, & \sigma_2, & \sigma_3 \\ v = \dfrac{u}{2\omega}, & & \tau = \dfrac{\omega'}{\omega} \end{matrix}\right.$$

where λ, μ, ν may take in any order the values 1, 2, 3. The corresponding changes must, of course, be made in z and h.

The following table contains the values of the indices λ, μ, ν for each of the six different cases which may arise (see also Halphen, *loc. cit.*, t. I., p. 262):

	Residue, mod. 2						
	p	q	p'	q'	λ	μ	ν
I	1	0	0	1	1	2	3
II	1	0	1	1	1	3	2
III	1	1	0	1	2	1	3
IV	1	1	1	0	2	3	1
V	0	1	1	1	3	1	2
VI	0	1	1	0	3	2	1

Addition-Theorems for the Sigma-Functions.

Art. 335. In a similar manner as was done in the case of the theta-functions (Arts. 210) we may derive theorems for the addition of the sigma-functions. These functions like the theta-functions do not have algebraic addition-theorems.

If in the identical relation

$$(\wp u - \wp u_1)(\wp u_2 - \wp u_3) + (\wp u - \wp u_2)(\wp u_3 - \wp u_1) + (\wp u - \wp u_3)(\wp u_1 - \wp u_2) \equiv 0$$

we make repeated application of the formula

$$\wp u - \wp v = -\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2 u\,\sigma^2 v},$$

we have

$$
\begin{aligned}
(1)\qquad &\sigma(u+u_1)\,\sigma(u-u_1)\,\sigma(u_2+u_3)\,\sigma(u_2-u_3)\\
&+\sigma(u+u_2)\,\sigma(u-u_2)\,\sigma(u_3+u_1)\,\sigma(u_3-u_1)\\
&+\sigma(u+u_3)\,\sigma(u-u_3)\,\sigma(u_1+u_2)\,\sigma(u_1-u_2)=0,
\end{aligned}
$$

an equation which is true for all values of the arbitrary quantities u, u_1, u_2, u_3.

Through the equations

$$
(2)\qquad
\begin{array}{llll}
u+u_1=a, & u-u_1=b, & u_2+u_3=c, & u_2-u_3=d,\\
u+u_2=a', & u-u_2=b', & u_3+u_1=c', & u_3-u_1=d',\\
u+u_3=a'', & u-u_3=b'', & u_1+u_2=c'', & u_1-u_2=d'',
\end{array}
$$

we may define three systems of four quantities each

$$a, b, c, d;\quad a', b', c', d';\quad a'', b'', c'', d'',$$

among which the following relations exist (cf. also Art. 210):

$$
(3)\qquad
\begin{array}{lll}
a'=\tfrac12(a+b+c+d) & a''=\tfrac12(a'+b'+c'+d') & a=\tfrac12(a''+b''+c''+d'')\\
b'=\tfrac12(a+b-c-d) & b''=\tfrac12(a'+b'-c'-d') & b=\tfrac12(a''+b''-c''-d'')\\
c'=\tfrac12(a-b+c-d) & c''=\tfrac12(a'-b'+c'-d') & c=\tfrac12(a''-b''+c''-d'')\\
d'=\tfrac12(-a+b+c-d) & d''=\tfrac12(-a'+b'+c'-d') & d=\tfrac12(-a''+b''+c''-d'')\\
\hline
a''=\tfrac12(a+b+c-d) & a=\tfrac12(a'+b'+c'-d') & a'=\tfrac12(a''+b''+c''-d'')\\
b''=\tfrac12(a+b-c+d) & b=\tfrac12(a'+b'-c'+d') & b'=\tfrac12(a''+b''-c''+d'')\\
c''=\tfrac12(a-b+c+d) & c=\tfrac12(a'-b'+c'+d') & c'=\tfrac12(a''-b''+c''+d'')\\
d''=\tfrac12(a-b-c-d) & d=\tfrac12(a'-b'-c'-d') & d'=\tfrac12(a''-b''-c''-d'')
\end{array}
$$

$$(3')\qquad a^2+b^2+c^2+d^2=a'^2+b'^2+c'^2+d'^2=a''^2+b''^2+c''^2+d''^2.$$

If in equation (1) instead of the quantities

$$u+u_1,\quad u-u_1,\quad u_2+u_3,\quad u_2-u_3$$

we write respectively

[1] a, b, c, d; [2] $a+\tilde\omega,\ b+\tilde\omega,\ c,\ d$;

[3] $a+\tilde\omega,\ b+\tilde\omega'',\ c-\tilde\omega',\ d$; [4] $a+\tilde\omega'',\ b+\tilde\omega'',\ c+\tilde\omega',\ d-\tilde\omega'$,

[5] $a+\tilde\omega+2\tilde\omega',\ b+\tilde\omega,\ c+\tilde\omega,\ d-\tilde\omega$; [6] $a+\tilde\omega,\ b+\tilde\omega,\ c+\tilde\omega,\ d-\tilde\omega$,

we have the following relations given by Schwarz, *loc. cit.*, § 38:

[A.]

$$
\begin{array}{lllllll}
[1]\ \sigma a\,\sigma b\,\sigma c\,\sigma d & + & \sigma a'\,\sigma b'\,\sigma c'\,\sigma d' & + & \sigma a''\,\sigma b''\,\sigma c''\,\sigma d'' & = 0, \\
[2]\ \sigma_\lambda a\,\sigma_\lambda b\,\sigma c\,\sigma d & + & \sigma_\lambda a'\,\sigma_\lambda b'\,\sigma c'\,\sigma d' & + & \sigma_\lambda a''\,\sigma_\lambda b''\,\sigma c''\,\sigma d'' & = 0, \\
[3]\ \sigma_\lambda a\,\sigma_\mu b\,\sigma_\nu c\,\sigma d & + & \sigma_\lambda a'\,\sigma_\mu b'\,\sigma_\nu c'\,\sigma d' & + & \sigma_\lambda a''\,\sigma_\mu b''\,\sigma_\nu c''\,\sigma d'' & = 0, \\
[4]\ \sigma_\mu a\,\sigma_\mu b\,\sigma_\nu c\,\sigma_\nu d & - & \sigma_\mu a'\,\sigma_\mu b'\,\sigma_\nu c'\,\sigma_\nu d' & + & (e_\mu - e_\nu)\sigma_\lambda a''\,\sigma_\lambda b''\,\sigma c''\,\sigma d'' & = 0,
\end{array}
$$

$$[5]\ (e_\mu - e_\nu)\sigma_\lambda a\,\sigma_\lambda b\,\sigma_\lambda c\,\sigma_\lambda d + (e_\nu - e_\lambda)\sigma_\mu a'\,\sigma_\mu b'\,\sigma_\mu c'\,\sigma_\mu d' + (e_\lambda - e_\mu)\sigma_\nu a''\,\sigma_\nu b''\,\sigma_\nu c''\,\sigma_\nu d'' = 0,$$

$$[6]\ \sigma_\lambda a\,\sigma_\lambda b\,\sigma_\lambda c\,\sigma_\lambda d - \sigma_\lambda a'\,\sigma_\lambda b'\,\sigma_\lambda c'\,\sigma_\lambda d' + (e_\lambda - e_\nu)(e_\lambda - e_\mu)\sigma a''\,\sigma b''\,\sigma c''\,\sigma d'' = 0.$$

From [A.] formula [2] follow without difficulty:

[B.]

$$(1)\qquad \sigma_\lambda w\,\sigma(u+v+w)\sigma(u-v) = \sigma(u+w)\sigma u\,\sigma_\lambda(v+w)\sigma_\lambda v - \sigma_\lambda(u+w)\sigma_\lambda u\,\sigma(v+w)\sigma v,$$

$$(2)\qquad \sigma w\,\sigma(u+v+w)\sigma_\lambda(u-v) = \sigma(u+w)\sigma_\lambda u\,\sigma(v+w)\sigma_\lambda v - \sigma_\lambda(u+w)\sigma u\,\sigma_\lambda(v+w)\sigma v.$$

Professor Schwarz, *loc. cit.*, p. 50, gives eighteen other such formulas. Write in [A.], [2] the values

$$
\begin{array}{llll}
a = 0, & b = 0, & c = u + v, & d = u - v, \\
a' = u, & b' = -u, & c' = v, & d' = v, \\
a'' = v, & b'' = -v, & c'' = u, & d'' = -u,
\end{array}
$$

and we have

[C.]

$$\sigma(u+v)\sigma(u-v) = \sigma^2 u\,\sigma_\lambda^2 v - \sigma_\lambda^2 u\,\sigma^2 v.$$

The other eight formulas given in the Table of Formulas LXII should be verified.

We note that these formulas are the analogues of the formulas (D) of Art. 211. Scheibner (*Crelle*, Bd. 102, p. 258) has derived the Weierstrassian formulas from those of Jacobi. A method by which the formulas of both Jacobi and Weierstrass may be derived is given by Kronecker (*Crelle*, Bd. 102, p. 260); see also Briot et Bouquet, *Traité des fonctions elliptiques*, pp. 485 *et seq.*

Expansion of the Sigma-functions in Powers of the Argument.

Art. 336. In Art. 281 we saw that

$$(1)\qquad \sigma u = u - \frac{g_2 u^5}{2^4\cdot 3\cdot 5} - \frac{g_3 u^7}{2^3\cdot 3\cdot 5\cdot 7} - \cdots,$$

and in Art. 279 we saw that the coefficients of u were rational functions of g_2 and g_3.

We may determine these coefficients as follows:*

If the equation

$$\left(\frac{d\wp u}{du}\right)^2 = 4\wp^3 u - g_2\wp u - g_3$$

be differentiated respectively with respect to u, g_2, g_3, we have

$$(a)\qquad \begin{cases} 2\dfrac{\partial^2\wp u}{\partial u^2} = 12(\wp u)^2 - g_2, \\ 2\dfrac{\partial\wp u}{\partial u}\dfrac{\partial^2\wp u}{\partial u\partial g_2} = [12(\wp u)^2 - g_2]\dfrac{\partial\wp u}{\partial g_2} - \wp u, \\ 2\dfrac{\partial\wp u}{\partial u}\dfrac{\partial^2\wp u}{\partial u\partial g_3} = [12(\wp u)^2 - g_2]\dfrac{\partial\wp u}{\partial g_3} - 1. \end{cases}$$

We also have

$$\frac{\partial}{\partial u}\left[\frac{\dfrac{\partial\wp u}{\partial g_2}}{\dfrac{\partial\wp u}{\partial u}}\right] = -\frac{1}{2}\frac{\wp u}{\left(\dfrac{\partial\wp u}{\partial u}\right)^2}, \qquad \frac{\partial}{\partial u}\left[\frac{\dfrac{\partial\wp u}{\partial g_3}}{\dfrac{\partial\wp u}{\partial u}}\right] = -\frac{1}{2}\frac{1}{\left(\dfrac{\partial\wp u}{\partial u}\right)^2}.$$

We further note that

$$(2)\qquad g_2^2\frac{\partial}{\partial u}\left[\frac{\dfrac{\partial\wp u}{\partial g_3}}{\dfrac{\partial\wp u}{\partial u}}\right] + 18g_3\frac{\partial}{\partial u}\left[\frac{\dfrac{\partial\wp u}{\partial g_2}}{\dfrac{\partial\wp u}{\partial u}}\right] = -3\wp u + \frac{\partial}{\partial u}\left(\frac{6\wp^2 u - g_2}{\dfrac{\partial\wp u}{\partial u}}\right).$$

If the equation (2) is integrated with respect to u, it becomes

$$g_2^2\frac{\partial\wp u}{\partial g_3} + 18g_3\frac{\partial\wp u}{\partial g_2} = 3\frac{\partial}{\partial u}\log\sigma u\frac{\partial\wp u}{\partial u} + 6\wp^2 u - g_2,$$

or

$$(3)\ g_2^2\frac{\partial^3\log\sigma u}{\partial u^2\partial g_3} + 18g_3\frac{\partial^3\log\sigma u}{\partial u^2\partial g_2} = 3\frac{\partial\log\sigma u}{\partial u}\frac{\partial^3\log\sigma u}{\partial u^3} - 6\left(\frac{\partial^2\log\sigma u}{\partial u^2}\right)^2 + g_2.$$

Noting that

$$\frac{d}{du}\log\sigma u = \frac{1}{u} - \frac{4g_2u^3}{2^4\cdot 3\cdot 5} - ((u^7)),$$

it is seen that the constant of integration that would appear in (3) is zero.

Since $\dfrac{\partial^2}{\partial u^2}\log\sigma u = -\wp u$, we have from (a)

$$6\left[\frac{\partial^2}{\partial u^2}\log\sigma u\right]^2 = -\frac{\partial^4\log\sigma u}{\partial u^4} + \frac{1}{2}g_2,$$

* See Weierstrass, *Zur Theorie der elliptischen Functionen*, *Berl. Monatsb.*, 1882, pp. 443–451; *Werke*, Bd. II, p. 245, and also Forsyth, *Quarterly Journ.*, Vol. XXII, pp. 1 *et seq.*; Hermite, *Crelle*, Bd. 85, p. 248; Meyer, *Crelle*, Bd. 56, p. 321; Enneper, *Ellipt. Funct.*, p. 166.

and observing the identity

$$\frac{1}{2}\frac{\partial^2}{\partial u^2}\left(\frac{\partial \log \sigma u}{\partial u}\right)^2 \equiv \frac{\partial \log \sigma u}{\partial u}\frac{\partial^3 \log \sigma u}{\partial u^3} + \left(\frac{\partial^2 \log \sigma u}{\partial u^2}\right)^2,$$

it is seen that the equation (3) may be written

$$g_2^2 \frac{\partial^3 \log \sigma u}{\partial u^2 \partial g_3} + 18\, g_3 \frac{\partial^3 \log \sigma u}{\partial u^2 \partial g_2} = \frac{3}{2}\frac{\partial^2}{\partial u^2}\left(\frac{\partial \log \sigma u}{\partial u}\right)^2 + \frac{3}{2}\frac{\partial^4 \log \sigma u}{\partial u^4} + \frac{1}{4} g_2.$$

This equation when integrated twice with regard to u becomes

$$g_2^2 \frac{\partial \log \sigma u}{\partial g_3} + 18\, g_3 \frac{\partial \log \sigma u}{\partial g_2} = \frac{3}{2}\left(\frac{\partial \log \sigma u}{\partial u}\right)^2 + \frac{3}{2}\frac{\partial^2 \log \sigma u}{\partial u^2} + \frac{1}{8} g_2 u^2,$$

or

$$g_2^2 \frac{\partial \log \sigma u}{\partial g_3} + 18\, g_3 \frac{\partial \log \sigma u}{\partial g_2} = \frac{3}{2}\frac{1}{\sigma u}\frac{\partial^2 \sigma u}{\partial u^2} + \frac{1}{8} g_2 u^2,$$

the constant of integration being zero.

It follows finally, since

$$\frac{\partial \log \sigma u}{\partial x} = \frac{1}{\sigma u}\frac{\partial \sigma u}{\partial x},$$

that

(4)
$$\frac{\partial^2 \sigma(u, g_2, g_3)}{\partial u^2} - 12\, g_3 \frac{\partial \sigma(u, g_2, g_3)}{\partial g_2} - \frac{2}{3} g_2^2 \frac{\partial \sigma(u, g_2, g_3)}{\partial g_3} + \frac{1}{12} g_2 u^2 \sigma(u, g_2, g_3) = 0.$$

Using this as a recursion formula Professor Schwarz (*loc. cit.*, p. 7) has calculated the terms of σu, up to the 35th power of u.

If with Halphen * we write

$$\sigma u = u + b_2 \frac{u^5}{5!} + b_3 \frac{u^7}{7!} + \cdots + b_n \frac{u^{2n+1}}{(2n+1)!} + \cdots,$$

we have

$$b_n = 12\, g_3 \frac{\partial b_{n-1}}{\partial g_2} + \frac{2}{3} g_2^2 \frac{\partial b_{n-1}}{\partial g_3} - \frac{(2n-1)(n-1)}{6} g_2 b_{n-2}.$$

To simplify the computation write

$$g_2 = 2h_2, \quad g_3 = \frac{2}{3} h_3,$$

and consequently

(5)
$$b_n = 4\left[h_3 \frac{\partial b_{n-1}}{\partial h_2} + h_2^2 \frac{\partial b_{n-1}}{\partial h_3}\right] - \frac{(2n-1)(n-1)}{3} h_2 b_{n-2}.$$

* Halphen, *loc. cit.*, t. I, p. 300.

It follows from (1) that $b_2 = -h_2$, $b_3 = -4h_3$; and from (5) we have

$$\begin{aligned}
b_4 &= -9h_2^2,\\
b_5 &= -24h_2h_3,\\
b_6 &= -3\cdot 2^5 h_3^2 + 3\cdot 23 h_2^3,\\
b_7 &= 2^2\cdot 3^2\cdot 19 h_2^2h_3,\\
b_8 &= 3h_2(2^7\cdot 23 h_3^2 + 107 h_2^3),\\
&\cdots\cdots\cdots
\end{aligned}$$

Expansions for *sn u*, *cn u* and *dn u* were given in Art. 226. These functions may be expressed as quotients of theta-functions. We have not, however, expressed the theta-functions in powers of u. As we have already given the expansions of $\wp u$, ζu, etc., in powers of u, it seems somewhat superfluous to expand $\sigma_\lambda u$ in powers of u.

From the formula

$$\sqrt{\wp u - e_\lambda} = \frac{\sigma_\lambda u}{\sigma u},$$

it follows that

$$\sigma_\lambda u = \sigma u\sqrt{\wp u - e_\lambda} = \sigma u\left[\frac{1}{u^2} - e_\lambda + \frac{g_2}{20}u^2 + \cdots\right]^{\frac{1}{2}},$$

or

$$\sigma_\lambda u = 1 - \tfrac{1}{2}e_\lambda u^2 - \tfrac{1}{48}(6e_\lambda^2 - g_2)u^4 - \cdots.$$

Methods including recursions formulas for the further expansion of these functions are found under the references given above. In particular attention is called to the formulas that result from the partial differentiations with regard to the invariants (given by Halphen, *loc. cit.*, t. I, Chapter IX; Frobenius and Stickelberger, *Crelle*, Bd. 92, p. 311).

EXAMPLES.

1. Show that

$$\sigma(2u) = \frac{2\sigma(u-\omega)\sigma(u+\omega'')\sigma(u-\omega')\sigma u}{\sigma\omega\,\sigma\omega''\sigma\omega'}.$$

2. Show that

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma^2u\sigma^2v} = \frac{d^2}{du^2}\log\sigma u - \frac{d^2}{dv^2}\log\sigma v.$$

3. Prove that (if $\omega_\lambda, \omega_\mu, \omega_\nu = \omega, \omega'', \omega'$ without respect to order)

(1) $$\xi_{\lambda 0}(u+\omega_\lambda) = -\sqrt{e_\lambda - e_\mu}\sqrt{e_\lambda - e_\nu}\,\xi_{0\lambda}(u),$$

(2) $$\xi_{\lambda 0}(u+\omega_\mu) = \sqrt{e_\mu - e_\lambda}\,\xi_{\nu\mu}(u),$$

(3) $$\xi_{\mu\nu}(u+\omega_\lambda) = \frac{\sqrt{e_\lambda - e_\mu}}{\sqrt{e_\lambda - e_\nu}}\,\xi_{\nu\mu}(u),$$

(4) $$\xi_{\mu\nu}(u+\omega_\mu) = -\sqrt{e_\mu - e_\lambda}\,\xi_{0\lambda}(u).$$

4. Verify the formulas

$$\xi_{0\nu}(u+v)=\frac{\xi_{0\nu}(u)\xi_{\lambda\nu}(v)\xi_{\mu\nu}(v)+\xi_{0\nu}(v)\xi_{\lambda\nu}(u)\xi_{\mu\nu}(u)}{1-(e_\nu-e_\lambda)(e_\nu-e_\mu)\xi^2{}_{0\nu}(u)\xi^2{}_{0\nu}(v)};$$

$$\xi_{\nu 0}(u+v)=\frac{\xi_{0\nu}(u)\xi_{\lambda\nu}(v)\xi_{\mu\nu}(v)-\xi_{0\nu}(v)\xi_{\lambda\nu}(u)\xi_{\mu\nu}(u)}{\xi^2{}_{0\nu}(u)-\xi^2{}_{0\nu}(v)},$$

$$\xi_{\mu\nu}(u+v)=\frac{\xi_{\mu\nu}(u)\xi_{\mu\nu}(v)-(e_\mu-e_\nu)\xi_{0\nu}(u)\xi_{\lambda\nu}(u)\xi_{0\nu}(v)\xi_{\lambda\nu}(v)}{1-(e_\nu-e_\lambda)(e_\nu-e_\mu)\xi^2{}_{0\nu}(u)\xi^2{}_{0\nu}(v)}.$$

5. Show that

$$e^{-\frac{\eta u^2}{2\omega}}\sigma_1 u=\cos\frac{\pi u}{2\omega}\prod_{m=1}^{m=\infty}\left(1-\frac{\sin^2\frac{\pi u}{2\omega}}{\cos^2\frac{m\pi\omega'}{\omega}}\right),$$

$$e^{-\eta\frac{u^2}{2\omega}}\sigma_2 u=\prod_{m=1}^{m=\infty}\left(1-\frac{\sin^2\frac{\pi u}{2\omega}}{\cos^2\left(m-\frac{1}{2}\right)\frac{\omega'\pi}{\omega}}\right),$$

$$e^{-\frac{\eta u^2}{2\omega}}\sigma_3 u=\prod_{m=1}^{m=8}\left(1-\frac{\sin^2\frac{\pi u}{2\omega}}{\sin^2\left(m-\frac{1}{2}\right)\frac{\omega'\pi}{\omega}}\right).$$

6. Show that

$$\frac{1}{2}\frac{\wp'u}{\wp u-e_l}=\frac{\sigma_\lambda' u}{\sigma_\lambda u}-\frac{\sigma' u}{\sigma u}=\frac{d}{du}\log\frac{\sigma_\lambda u}{\sigma u},$$

$$\frac{1}{2}\frac{(e_\mu-e_\nu)\wp'u}{(\wp u-e_\mu)(\wp u-e_\nu)}=\frac{\sigma_\mu' u}{\sigma_\mu u}-\frac{\sigma_\nu' u}{\sigma_\nu u}=\frac{d}{du}\log\frac{\sigma_\mu u}{\sigma_\nu u};$$

$$-\frac{(e_\lambda-e_\mu)(e_\lambda-e_\nu)}{\wp u-e_\lambda}-e_\lambda=\frac{d}{du}\frac{\sigma_\lambda' u}{\sigma_\lambda u}=\frac{d^2}{du^2}\log\sigma_\lambda u.$$

7. Show that

$$\frac{\partial F(\phi.k)}{\partial k}=-\frac{F(\phi,k)}{k}+\frac{E(\phi,k)}{kk'^2}-\frac{k}{k'^2}\frac{\sin\phi\cos\phi}{\Delta(\phi,k)},$$

where $F(\phi, k)$ and $E(\phi, k)$ are Legendre's integrals of the first and second kinds; and that

$$\frac{\partial E(\phi.k)}{\partial k}=-\frac{F(\phi,k)}{k}+\frac{E(\phi,k)}{k},$$

$$k(1-k^2)\frac{\partial^2 F(\phi,k)}{\partial k^2}-(3k^2-1)\frac{\partial F(\phi,k)}{\partial k}-kF(\phi,k)+k\frac{\partial}{\partial k}\frac{k\sin\phi\cos\phi}{\Delta(\phi,k)}=0,$$

$$k\frac{\partial^2 E(\phi,k)}{\partial k^2}+\frac{\partial E(\phi,k)}{\partial k}+\frac{k}{k'^2}E(\phi,k)-\frac{k}{k'^2}\frac{\sin\phi\cos\phi}{\Delta(\phi,k)}=0.$$

Write $\phi=\frac{\pi}{2}$ in these equations and note the results.

CHAPTER XVIII

THE THETA- AND SIGMA-FUNCTIONS WHEN SPECIAL VALUES ARE GIVEN TO THE ARGUMENT

ARTICLE 337. The theta-functions were expressed in Art. 209 through the following formulas :

$$\vartheta_0(u) = \prod_{m=1}^{m=\infty}(1 - q^{2m})\prod_{m=1}^{m=\infty}(1 - 2q^{2m-1}\cos 2\pi u + q^{4m-2}),$$

$$\vartheta_1(u) = 2q^{\frac{1}{4}}\sin \pi u\prod_{m=1}^{m=\infty}(1 - q^{2m})\prod_{m=1}^{m=\infty}(1 - 2q^{2m}\cos 2\pi u + q^{4m}),$$

$$\vartheta_2(u) = 2q^{\frac{1}{4}}\cos \pi u\prod_{m=1}^{m=\infty}(1 - q^{2m})\prod_{m=1}^{m=\infty}(1 + 2q^{2m}\cos 2\pi u + q^{4m}),$$

$$\vartheta_3(u) = \prod_{m=1}^{m=\infty}(1 - q^{2m})\prod_{m=1}^{m=\infty}(1 + 2q^{2m-1}\cos 2\pi u + q^{4m-2}).$$

For brevity we put

$$Q_0 = \prod_{m=1}^{m=\infty}(1 - q^{2m}), \quad Q_1 = \prod_{m=1}^{m=\infty}(1 + q^{2m}),$$

$$Q_2 = \prod_{m=1}^{m=\infty}(1 + q^{2m-1}), \quad Q_3 = \prod_{m=1}^{m=\infty}(1 - q^{2m-1}).$$

Since these quotients are absolutely convergent (Art. 17), we may write

$$Q_0Q_3 = \prod_{m=1}^{m=\infty}(1 - q^{m}), \quad Q_1Q_2 = \prod_{m=1}^{m=\infty}(1 + q^{m}),$$

and consequently

$$Q_0Q_1Q_2Q_3 = \prod_{m=1}^{m=\infty}(1 - q^{2m}) = Q_0.$$

It follows * that

$$Q_1Q_2Q_3 = 1.$$

* See the 16th Chapter of Euler, *Introdutio in analysin infinit.*

Making the argument equal to zero in the theta-functions, it is seen that

$$\vartheta_0(0) = \vartheta_0 = Q_0 Q_3^2,$$
$$\vartheta_1(0) = \vartheta_1 = 0,$$
$$\vartheta_2(0) = \vartheta_2 = 2\, q^{\frac{1}{4}} Q_0 Q_1^2,$$
$$\vartheta_3(0) = \vartheta_3 = Q_0 Q_2^2.$$

ART. 338. From the following formulas (see Art. 208),

$$\vartheta_0(u) = 1 + 2\sum_{m=1}^{m=\infty} (-1)^m q^{m^2} \cos 2\, m\pi u = \sum_{m=-\infty}^{m=+\infty} (-1)^m e^{m^2 \tau\pi i} e^{2m\pi u i},$$

$$\vartheta_1(u) = 2\sum_{m=0}^{m=\infty} (-1)^m q^{\frac{(2m+1)^2}{4}} \sin(2m+1)\pi u = \frac{1}{i}\sum_{m=-\infty}^{m=+\infty} (-1)^m e^{\frac{(2m+1)^2}{4}\tau\pi i} e^{(2m+1)\pi u i},$$

$$\vartheta_2(u) = 2\sum_{m=0}^{m=\infty} q^{\frac{(2m+1)^2}{4}} \cos(2\, m + 1)\pi u = \sum_{m=-\infty}^{m=+\infty} e^{\frac{(2m+1)^2}{4}} \quad e^{(2m+1)\pi u i},$$

$$\vartheta_3(u) = 1 + 2\sum_{m=1}^{m=\infty} q^{m^2} \cos 2\, m\pi u = \sum_{m=-\infty}^{m=+\infty} e^{m^2\tau\pi i} e^{2m\pi u i},$$

we have

$$\vartheta_0 = 1 + 2\sum_{m=1}^{m=\infty} (-1)^m q^m = 1 - 2\, q + 2\, q^4 - 2\, q^9 + \cdots,$$

$$[\vartheta_0''(u)]_{u=0} = \vartheta_0'' = -2\,\pi^2 \sum_{m=1}^{m=\infty} (-1)^m (2\, m)^2 q^{m^2} = 8\,\pi^2 (q - 4\, q^4 + 9\, q^9 - \cdots),$$

$$\vartheta_1' = 2\pi \sum_{m=0}^{m=\infty} (-1)^m (2\, m + 1) q^{\frac{(2m+1)^2}{4}} = 2\,\pi q^{\frac{1}{4}} (1 - 3\, q^2 + 5\, q^6 - 7\, q^{12} + \cdots),$$

$$\vartheta_1''' = -2\pi^3 \sum_{m=0}^{m=\infty} (-1)^m (2m+1)^3 q^{\frac{(2m+1)^2}{4}} = -2\pi^3 q^{\frac{1}{4}} (1 - 3^3 q^2 + 5^3 q^6 - 7^3 q^{12} + \cdots),$$

$$\vartheta_2 = 2\sum_{m=0}^{m=\infty} q^{\frac{(2m+1)^2}{4}} = 2\, q^{\frac{1}{4}} (1 + q^2 + q^6 + q^{12} + \cdots),$$

$$\vartheta_2'' = -2\,\pi^2 \sum_{m=0}^{m=\infty} (2\, m + 1)^2 q^{\frac{(2m+1)^2}{4}} = -2\,\pi^2 q^{\frac{1}{4}} (1 + 3^2 q^2 + 5^2 q^6 + 7^2 q^{12} + \cdots),$$

$$\vartheta_3 = 1 + 2\sum_{m=1}^{m=\infty} q^{m^2} = 1 + 2\, q + 2\, q^4 + 2\, q^9 + \cdots,$$

$$\vartheta_3'' = -2\,\pi^2 \sum_{m=1}^{m=\infty} (2\, m)^2 q^{m^2} = -8\,\pi^2 (q + 4\, q^4 + 9\, q^9 + \cdots).$$

ART. 339. Since the functions ϑ_0, ϑ_1', ϑ_2, ϑ_3 depend only upon one variable q, it is natural to expect that they are connected by three relations, which we would suppose are of a transcendental nature. Two of these relations *, however, as we shall show in the sequel, are algebraic, viz.,

$$\vartheta_2^4 + \vartheta_0^4 = \vartheta_3^4 \quad \text{and} \quad \vartheta_1' = \pi\vartheta_0\vartheta_2\vartheta_3.$$

The first of these follows at once from the equation (Cf. Art. 193)

$$k^2 + k'^2 = 1.$$

To derive the second we use the equation of Art. 295,

$$\vartheta_2\vartheta_3\vartheta_1(u+v)\vartheta_0(u-v) = \vartheta_1(u)\vartheta_0(u)\vartheta_2(v)\vartheta_3(v) + \vartheta_2(u)\vartheta_3(u)\vartheta_1(v)\vartheta_0(v).$$

Expanded in powers of u, it becomes

$$\begin{aligned}\vartheta_2\vartheta_3 \Big\{ &\vartheta_1(v)\vartheta_0(v) + [\vartheta_1'(v)\vartheta_0(v) - \vartheta_1(v)\vartheta_0'(v)]u \\ &+ \frac{1}{2!}[\vartheta_1''(v)\vartheta_0(v) - 2\vartheta_1'(v)\vartheta_0'(v) + \vartheta_1(v)\vartheta_0''(v)]u^2 + \cdots \Big\} \\ = \vartheta_2(v)\vartheta_3(v) &\Big\{ \vartheta_1'\vartheta_0 \cdot u + 0 \cdot u^2 + \cdots \Big\} \\ &+ \vartheta_0(v)\vartheta_1(v) \Big\{ \vartheta_2\vartheta_3 + 0 \cdot u + \frac{1}{2}[\vartheta_2''\vartheta_3 + \vartheta_2\vartheta_3'']u^2 + \cdots \Big\}.\end{aligned}$$

If the coefficients of u^2 on either side of this equation are equated, we have †

$$\begin{aligned}\vartheta_2\vartheta_3\{\vartheta_1''(v)\vartheta_0(v) - 2\vartheta_1'(v)\vartheta_0'(v) + \vartheta_1(v)\vartheta_0''(v)\} \\ = \vartheta_0(v)\,\vartheta_1(v)\{\vartheta_2''\vartheta_3 + \vartheta_2\vartheta_3''\},\end{aligned}$$

an expression which differentiated with regard to v becomes

$$\begin{aligned}\vartheta_2\vartheta_3\{\vartheta_1'''(v)\vartheta_0(v) + \vartheta_1''(v)\vartheta_0'(v) - 2\vartheta_1''(v)\vartheta_0'(v) - 2\vartheta_1'(v)\vartheta_0''(v) \\ + \vartheta_1'(v)\vartheta_0''(v) + \vartheta_1(v)\vartheta_0'''(v)\} \\ = \{\vartheta_0'(v)\vartheta_1(v) + \vartheta_0(v)\vartheta_1'(v)\}\{\vartheta_2''\vartheta_3 + \vartheta_2\vartheta_3''\}.\end{aligned}$$

If we put $v = 0$ in this equation, we have

$$\vartheta_2\vartheta_3\{\vartheta_1'''\vartheta_0 - \vartheta_1'\vartheta_0''\} = \vartheta_0\vartheta_1'\{\vartheta_2''\vartheta_3 + \vartheta_2\vartheta_3''\},$$

or

$$\frac{\vartheta_1'''}{\vartheta_1'} = \frac{\vartheta_0''}{\vartheta_0} + \frac{\vartheta_2''}{\vartheta_2} + \frac{\vartheta_3''}{\vartheta_3}.$$

* They are both due to Jacobi, Werke I, pp. 515–17.

† See Koenigsberger, *Ell. Funct.*, p. 380; or Burkhardt, *Ell. Funkt.*, p. 120.

ART. 340. It may next be proved that

$$\frac{\partial \vartheta_\alpha(u, \tau)}{\partial \tau} = \frac{1}{4\pi i}\frac{\partial^2 \vartheta_\alpha(u, \tau)}{\partial u^2} \qquad (\alpha = 0, 1, 2, 3).$$

Take, for example, the equation

$$\vartheta_0(u) = \sum_{m=-\infty}^{m=+\infty}(-1)^m e^{m^2\tau\pi i + 2m\pi i u} = \vartheta_0(u, \tau).$$

When differentiated with regard to τ, it becomes

$$(1) \qquad \frac{\partial \vartheta_0(u, \tau)}{\partial \tau} = \pi i \sum_{m=-\infty}^{m=+\infty} m^2(-1)^m e^{m^2\tau\pi i + 2m\pi i u}$$

$$= \frac{1}{4\pi i}\frac{\partial^2 \vartheta_0(u, \tau)}{\partial u^2}.$$

By Maclaurin's Theorem

$$\vartheta_0(u) = \vartheta_0 + \frac{u^2}{2}\vartheta_0'' + \cdots.$$

and consequently also

$$\frac{\partial \vartheta_0(u)}{\partial \tau} = \frac{\partial \vartheta_0}{\partial \tau} + \frac{u^2}{2}\frac{\partial \vartheta_0''}{\partial \tau} + \cdots.$$

If these values are substituted in (1), we have

$$\frac{\partial^2 \vartheta_0(u, \tau)}{\partial u^2} = 4\pi i\left\{\frac{\partial \vartheta_0}{\partial \tau} + \frac{u^2}{2}\frac{\partial \vartheta_0''}{\partial \tau} + \cdots\right\},$$

or writing $u = 0$,

$$\vartheta_0'' = 4\pi i \frac{\partial \vartheta_0}{\partial \tau}.$$

In a similar manner it may be shown that

$$\vartheta_\lambda'' = 4\pi i \frac{\partial \vartheta_\lambda}{\partial \tau} \qquad (\lambda = 0, 2, 3),$$

and also that

$$\vartheta_1''' = 4\pi i \frac{\partial \vartheta_1'}{\partial \tau}.$$

Writing these values in the last equation of the preceding Article and integrating we have

$$\vartheta_1' = C\vartheta_0\vartheta_2\vartheta_3.$$

If both sides of this equation are expanded in powers of q, it is seen that the constant $C = \pi$, and consequently that

$$\vartheta_1' = \pi\vartheta_0\vartheta_2\vartheta_3.$$

It is also seen from the results of the preceding Article that

$$\vartheta_1' = 2\pi q^{\frac{1}{4}} Q_0^3.$$

ART. 341. If the formula

$$sn\, u = \frac{\vartheta_3}{\vartheta_2} \frac{\vartheta_1\left(\frac{u}{2K}\right)}{\vartheta_0\left(\frac{u}{2K}\right)}$$

be differentiated with regard to u, we have

$$cn\, u\, dn\, u = \frac{\vartheta_3}{\vartheta_2}\left[\frac{\vartheta_1'\left(\frac{u}{2K}\right)}{\vartheta_0\left(\frac{u}{2K}\right)} - \frac{\vartheta_1\left(\frac{u}{2K}\right)}{\vartheta_0^2\left(\frac{u}{2K}\right)}\vartheta_0'\left(\frac{u}{2K}\right)\right]\frac{1}{2K}.$$

If in this expression we put $u = 0$, it follows that

$$1 = \frac{\vartheta_3}{\vartheta_2}\frac{\vartheta_1'}{\vartheta_0}\frac{1}{2K} \quad \text{or} \quad 1 = \frac{\vartheta_3}{\vartheta_2\vartheta_0}\pi\vartheta_0\vartheta_2\vartheta_3\frac{1}{2K}.$$

It is thus seen that

$$\vartheta_3 = \sqrt{\frac{2K}{\pi}}.$$

From the formula

$$\frac{\vartheta_2}{\vartheta_3} = \sqrt{k}, \quad \frac{\vartheta_0}{\vartheta_3} = \sqrt{k'},$$

if also follows that

$$\vartheta_2 = \sqrt{\frac{2Kk}{\pi}}, \quad \vartheta_0 = \sqrt{\frac{2Kk'}{\pi}},$$

$$\vartheta_1' = \frac{2K\sqrt{2Kkk'}}{\sqrt{\pi}}.$$

We note * (see also Art. 345) that

$$\sqrt[4]{k} = \sqrt{2}\,\sqrt[8]{q}\,\frac{Q_1}{Q_2}, \quad \sqrt[4]{k'} = \frac{Q_3}{Q_2};$$

or since

$$Q_1Q_2Q_3 = 1,$$

we have

$$\sqrt[4]{k} = \sqrt{2}\,\sqrt[8]{q}\,Q_1^2Q_3, \quad \sqrt[4]{k'} = Q_1Q_3^2,$$

$$\sqrt[12]{kk'} = \sqrt[6]{2}\,\sqrt[24]{q}\,Q_1Q_3;$$

and also

$$\frac{Q_0}{Q_1} = \sqrt{\frac{2K\sqrt{k}}{\pi}}, \quad q^{\frac{1}{4}}\frac{Q_0}{Q_2} = \sqrt{\frac{K}{\pi}\sqrt{kk'}}, \quad q^{\frac{1}{4}}\frac{Q_0}{Q_3} = \sqrt{\frac{K}{\pi}\sqrt{k}}.$$

It is seen that k and k' considered as functions of $q = e^{\tau\pi i}$ are *one-valued* functions of τ. From this point of view Kronecker found the origin of some of his most beautiful discoveries and Poincaré was also thus led to the discovery of the Fuchsian Functions.

* See Jacobi, *Werke* I, p. 146.

Hermite * wrote $\sqrt[4]{k} = \phi(\tau)$ and $\sqrt[4]{k'} = \psi(\tau)$, where from above $\phi(\tau)$ and $\psi(\tau)$ are *one-valued* functions of τ which may be expressed as quotients of two infinite products. These functions are of such importance that we may consider them more closely and at the same time introduce other interesting formulas for the elliptic functions.

ART. 342. From the equation

$$\vartheta_0(u, q) = \prod_{m=1}^{m=\infty}(1 - q^{2m})\prod_{m=1}^{m=\infty}(1 - 2q^{2m-1}\cos 2\pi u + q^{4m-2}),$$

it follows that

$$\vartheta_0(2u, q^2) = \prod_{m=1}^{m=\infty}(1 - q^{4m})\prod_{m=1}^{m=\infty}(1 - 2q^{2(2m-1)}\cos 4\pi u + q^{4(2m-1)}).$$

Since

$$1 - 2q^{2(2m-1)}\cos 4\pi u + q^{4(2m-1)}$$
$$= (1 - 2q^{2m-1}\cos 2\pi u + q^{2(2m-1)})(1 + 2q^{2m-1}\cos 2\pi u + q^{2(2m-1)}),$$

we have

$$\vartheta_0(2u, q^2) = \prod_{m=1}^{m=\infty}\frac{1 - q^{4m}}{(1 - q^{2m})^2}\vartheta_0(u, q)\,\vartheta_3(u, q),$$

or

(1) $$\vartheta_0(u, q)\vartheta_3(u, q) = \frac{Q_0}{Q_1}\vartheta_0(2u, q^2),$$

and similarly

(2) $$\vartheta_1(u, q)\vartheta_2(u, q) = \frac{Q_0}{Q_1}\vartheta_1(2u, q^2).$$

We also have from the product of $\vartheta_0(u, q)$ and $\vartheta_1(u, q)$ the formula

$$\vartheta_0(u, q)\vartheta_1(u, q) = 2q^{\frac{1}{4}}\sin \pi u\prod_{m=1}^{m=\infty}(1 - q^{2m})^2\prod_{m=1}^{m=\infty}(1 - 2q^m\cos 2\pi u + q^{2m});$$

and since

$$1 - 2q^m\cos 2\pi u + q^{2m} = 1 - 2(\sqrt{q})^{2m}\cos 2\pi u + (\sqrt{q})^{4m},$$

it follows that

$$\vartheta_0(u, q)\vartheta_1(u, q) = q^{\frac{1}{8}}\prod_{m=1}^{m=\infty}\frac{(1 - q^{2m})^2}{1 - q^m}\vartheta_1(u, \sqrt{q});$$

further noting that

$$\prod_{m=1}^{m=\infty}(1 - q^m) = \prod_{m=1}^{m=\infty}(1 - q^{2m})\prod_{m=1}^{m=\infty}(1 - q^{2m-1}),$$

we have

(3) $$\vartheta_0(u, q)\vartheta_1(u, q) = q^{\frac{1}{8}}\frac{Q_0}{Q_3}\vartheta_1(u, \sqrt{q}).$$

* Hermite, *Résolution de l'équation du cinquième degrè.* Œuvres, t. II, p. 7; and also *Sur la théorie des équations modulaires,* Œuvres, t. II, p. 38; see also Webber. *Elliptische Functionen,* pp. 147 and 327.

If for u we write $u + \frac{1}{2}$ in this equation, it becomes (see Art. 208)

$$(4) \qquad \vartheta_3(u, q)\vartheta_2(u, q) = q^{\frac{1}{4}} \frac{Q_0}{Q_3} \vartheta_2(u, \sqrt{q}).$$

If for q we write $qe^{\pi i} = -q = qe^{-\pi i}$, the quantity $q^{\frac{1}{4}} \frac{Q_0}{Q_3}$ becomes $q^{\frac{1}{4}} e^{-\frac{\pi i}{8}} \frac{Q_0}{Q_2}$,

and the equations (3) and (4) become

$$(5) \qquad \vartheta_3(u, q)\vartheta_1(u, q) = q^{\frac{1}{4}} e^{-\frac{\pi i}{8}} \frac{Q_0}{Q_2} \vartheta_1(u, e^{\frac{\pi i}{2}} \sqrt{q}),$$

$$(6) \qquad \vartheta_0(u, q)\vartheta_2(u, q) = q^{\frac{1}{4}} e^{-\frac{\pi i}{8}} \frac{Q_0}{Q_2} \vartheta_2(u, e^{\frac{\pi i}{2}} \sqrt{q}).$$

The six formulas above are given by Jacobi (*Seconde mémoire sur la rotation d'un corps.* Werke, II, p. 431).

In the formula

$$\vartheta_1(u) = 2 \sum_{m=0}^{m=\infty} (-1)^m q^{\frac{(2m+1)^2}{4}} \sin(2m + 1)\pi u,$$

the summation is taken over positive integers including zero. If we separate the even integers and the odd integers by writing $m = 2n$ and $m = -(2n + 1)$, we have

$$\vartheta_1(u) = 2 q^{\frac{1}{4}} \sum_{n=-\infty}^{n=+\infty} q^{4n^2+2n} \sin(4n + 1)\pi u,$$

and similarly

$$\vartheta_2(u) = 2 q^{\frac{1}{4}} \sum_{n=-\infty}^{n=+\infty} q^{4n^2+2n} \cos(4n + 1)\pi u.$$

Since

$$sn\, 2Ku = \frac{1}{\sqrt{k}} \frac{\vartheta_1(u)}{\vartheta_0(u)} = \frac{1}{\sqrt{k}} \frac{\vartheta_1(u)\vartheta_3(u)}{\vartheta_0(u)\vartheta_3(u)} = \frac{1}{\sqrt{k}} \frac{\vartheta_1(u)\vartheta_2(u)}{\vartheta_0(u)\vartheta_2(u)},$$

$$cn\, 2Ku = \sqrt{\frac{k'}{k}} \frac{\vartheta_2(u)}{\vartheta_0(u)} = \sqrt{\frac{k'}{k}} \frac{\vartheta_2(u)}{\vartheta_0(u)} \frac{\vartheta_3(u)}{\vartheta_3(u)} = \sqrt{\frac{k'}{k}} \frac{\vartheta_2(u)}{\vartheta_0(u)} \frac{\vartheta_3(u)}{\vartheta_3(u)},$$

$$dn\, 2Ku = \sqrt{k'} \frac{\vartheta_3(u)}{\vartheta_0(u)} = \sqrt{k'} \frac{\vartheta_3(u)\vartheta_2(u)}{\vartheta_0(u)\vartheta_2(u)} = \sqrt{k'} \frac{\vartheta_3(u)}{\vartheta_0(u)} \frac{\vartheta_1(u)}{\vartheta_1(u)},$$

it follows from the formulas above that

$$(7) \quad sn\, 2Ku = \frac{e^{-\frac{\pi i}{8}} \vartheta_1\left(u, e^{\frac{i\pi}{2}} \sqrt{q}\right)}{\sqrt{2\sqrt{k}}\, \vartheta_0(2u, q^2)} = \frac{\sqrt{2}\, q^{\frac{1}{4}}}{\sqrt[4]{k}} \frac{\sum (-1)^n q^{2n^2+n} \sin(4n+1)\pi u}{\sum (-1)^n q^{2n^2} \cos 4n\pi u},$$

where the summations on the right are over all integers from $n=-\infty$ to $n=+\infty$. The summations are taken over the same integers in the following formulas:

$$(8)\quad cn\,2Ku=\frac{1}{\sqrt{2}}\sqrt[4]{\frac{k'}{k}}\frac{\vartheta_2(u,\sqrt{q})}{\vartheta_0(2u,q^2)}=\sqrt{2}\sqrt[4]{\frac{k'}{k}}q^{\frac{1}{4}}\frac{\sum q^{2n^2+n}\cos(4n+1)\pi u}{\sum(-1)^{n^2}q^{2n^2}\cos 4n\pi u},$$

$$(9)\quad dn\,2Ku=\sqrt[4]{k'}e^{\frac{i\pi}{8}}\frac{\vartheta_2(u,\sqrt{q})}{\vartheta_2(u,e^{\pi i/2}\sqrt{q})}=\sqrt[4]{k'}\frac{\sum q^{2n^2+n}\cos(4n+1)\pi u}{\sum(-1)^n q^{2n^2+n}\cos(4n+1)\pi u};$$

$$(10)\quad sn\,2Ku=\sqrt{2}e^{\frac{i\pi}{8}}\frac{\vartheta_1(2u,q^2)}{\vartheta_2(u,e^{\pi i/2}\sqrt{q})}=\frac{\sqrt{2}}{\sqrt[4]{k^3}}q^{\frac{3}{4}}\frac{\sum q^{8n^2+4n}\sin(8n+2)\pi u}{\sum(-1)^n q^{2n^2+n}\cos(4n+1)\pi u},$$

$$(11)\quad cn\,2Ku=\sqrt{2}\sqrt{\frac{k'^3}{k^3}}\frac{\vartheta_1(2u,q^2)}{\vartheta_1(u,\sqrt{q})}=\sqrt{2}\sqrt[4]{\frac{k'^3}{k^3}}q^{\frac{3}{4}}\frac{\sum q^{8n^2+4n}\sin(8n+2)\pi u}{\sum q^{2n^2+n}\sin(4n+1)\pi u},$$

$$(12)\quad dn\,2Ku=\sqrt[4]{k'^3}e^{-\frac{i\pi}{8}}\frac{\vartheta_1(u,e^{\frac{i\pi}{2}}\sqrt{q})}{\vartheta_1(u,\sqrt{q})}=\sqrt{k'^3}\frac{\sum(-1)^n q^{2n^2+n}\sin(4n+1)\pi u}{\sum q^{2n^2+n}\sin(4n+1)\pi u}.$$

If we put $u=0$ in (8) and (9), we have

$$\sqrt[4]{k}=\sqrt{2}\,q^{\frac{1}{4}}\frac{\sum(-1)^n q^{2n^2+n}}{\sum(-1)^n q^{2n^2}},$$

$$\sqrt[4]{k'}=\frac{\sum(-1)^n q^{2n^2+n}}{\sum q^{2n^2+n}}.$$

Jacobi (Werke II, pp, 233–235) has given several different forms for these two quotients of infinite series.

If we write $u=0$ in (10) and (12) and determine the resulting indeterminate forms, we have *

$$\sqrt[4]{k^3}=2\sqrt{2}\,q^{\frac{3}{4}}\frac{\sum(4n+1)q^{8n^2+4n}}{\sum(-1)^n(4n+1)q^{2n^2+n}},$$

$$\sqrt[4]{k'^3}=\frac{\sum(4n+1)q^{2n^2+n}}{\sum(-1)^n(4n+1)q^{2n^2+n}}.$$

ART. 343. By equating the expressions for the theta-functions in the form of infinite products and in the form of infinite series we may derive interesting relations connecting the quantity q.

For example, in the case of $\vartheta_1(u)$ we have after division by $q^{\frac{1}{4}}$

$$(1)\quad \sin\pi u(1-q^2)(1-2q^2\cos 2\pi u+q^4)(1-q^4)(1-2q^4\cos 2\pi u+q^8)\cdots$$
$$=\sin\pi u-q^2\sin 3\pi u+q^6\sin 5\pi u-q^{12}\sin 7\pi u+q^{20}\sin 9\pi u-\cdots.$$

* See Hermite, Œuvres, t. II, p. 275.

If in this equation we put $u = \frac{1}{3}$ and divide by $\frac{1}{2}\sqrt{3}$, we have *

$$(1 - q^6)(1 - q^{12})(1 - q^{18}) \cdots = 1 - q^6 - q^{12} + q^{30} + q^{42} - \cdots;$$

or writing $q^6 = t$, it follows that

$$\prod_{m=1}^{m=\infty}(1 - t^m) = \sum_{m=-\infty}^{m=+\infty}(-1)^m t^{\frac{3m^2+m}{2}}. \tag{2}$$

Upon this formula depends the trisection of the elliptic functions.

If further we divide equation (1) by $\sin \pi u$ and then put $u = 0$, we have

$$[(1 - q^2)(1 - q^4)(1 - q^6) \cdots]^3 = 1 - 3q^2 + 5q^6 - 7q^{12} + 9q^{20} - \cdots.$$

Writing $q^2 = t$ in this equation, it follows that

$$\prod_{m=1}^{m=\infty}(1 - t^m)^3 = \sum_{m=0}^{m=\infty}(-1)^m(2m + 1)t^{\frac{m^2+m}{2}}. \tag{3}$$

If we compare the equations (2) and (3), it is seen (cf. Jacobi, Werke, I, p. 237) that

$$(1 - q - q^2 + q^5 + q^7 - q^{12} + \cdots)^3 = 1 - 3q + 5q^3 - 7q^6 + 9q^{10} - \cdots.$$

Further in equation (1) put $\sqrt{q}$ in the place of q.

We then have

$$\begin{aligned}(1 - q)(1 - q^2)(1 - q^3) \ldots \sin \pi u(1 - 2q\cos 2\pi u + q^2)&\\ (1 - 2q^2\cos 2\pi u + q^4)(1 - 2q^3\cos 2\pi u + q^6) \ldots&\\ = \sin \pi u - q\sin 3\pi u + q^3\sin 5\pi u - q^6\sin 7\pi u + \cdots&.\end{aligned}$$

Write in this equation $u = \frac{1}{2}$ and observe that

$$Q_3Q_0Q_1{}^2Q_2{}^2 = \frac{Q_0}{Q_3}; \text{ it follows that}$$

$$\frac{Q_0}{Q_3} = 1 + q + q^2 + q^6 + q^{10} + q^{15} + \cdots.$$

If we compare the two expressions for $\vartheta_0(u)$, we have

$$\begin{aligned}Q_0(1 - 2q\cos 2\pi u + q^2)(1 - 2q^3\cos 2\pi u + q^6) \ldots&\\ = 1 - 2q\cos 2\pi u + 2q^4\cos 4\pi u - 2q^9\cos 6\pi u + \cdots&.\end{aligned}$$

In this equation write $u = 0$ and observe that

$$Q_0Q_3{}^2 = \frac{Q_0Q_3}{Q_1Q_2}.$$

It follows that

$$\frac{Q_0Q_3}{Q_1Q_2} = \frac{(1-q)(1-q^2)(1-q^3)(1-q^4)\ldots}{(1+q)(1+q^2)(1+q^3)(1+q^4)\ldots} = 1 - 2q + 2q^4 - 2q^9 + 2q^{16} - \cdots.$$

* See Euler, *Introductio in analysin infinit.*, § 323.

From the formulas

$$sn\frac{2\,ku}{\pi}=\frac{2\,q^{\frac{1}{4}}}{\sqrt{k}}\sin u\prod_{m=1}^{m=\infty}\frac{1-2\,q^{2m}\cos 2\,u+q^{4m}}{1-2\,q^{2m-1}\cos 2\,u+q^{4m-2}},$$

$$cn\frac{2\,Ku}{\pi}=2\,q^{\frac{1}{4}}\sqrt{\frac{k'}{k}}\cos u\prod_{m=1}^{m=\infty}\frac{1+2\,q^{2m}\cos 2\,u+q^{4m}}{1-2\,q^{2m-1}\cos 2\,u+q^{4m-2}},$$

$$dn\frac{2\,Ku}{\pi}=\sqrt{k'}\prod_{m=1}^{m=\infty}\frac{1+2\,q^{2m-1}\cos 2\,u+q^{4m-2}}{1-2\,q^{2m-1}\cos 2\,u+q^{4m-2}},$$

it follows that

$$(1)\quad \log sn\frac{2\,Ku}{\pi}=\log\frac{2q^{\frac{1}{4}}\sin u}{\sqrt{k}}+2\sum_{m=1}^{m=\infty}\frac{1}{m}\,\frac{q^m}{1+q^m}\cos 2\,mu,$$

$$(2)\quad \log cn\frac{2\,Ku}{\pi}=\log\left(2\,q^{\frac{1}{4}}\sqrt{\frac{k'}{k}}\cos u\right)+2\sum_{m=1}^{m=\infty}\frac{1}{m}\,\frac{q^m}{1+(-q)^m}\cos 2\,mu,$$

$$(3)\quad \log dn\frac{2\,Ku}{\pi}=\log\sqrt{k'}+4\sum_{m=1}^{m=\infty}\frac{1}{2\,m-1}\,\frac{q^{2m-1}}{1-q^{4m-2}}\cos(4\,m-2)u.$$

From (1) and (2) we have

$$\left[\log\frac{sn\dfrac{2\,Ku}{\pi}}{\sin u}\right]_{u=0}=\log\frac{2\,K}{\pi}=\log\frac{2\,q^{\frac{1}{4}}}{\sqrt{k}}+2\sum_{m=1}^{m=\infty}\frac{1}{m}\,\frac{q^m}{1+q^m},$$

$$\left[\log\frac{cn\dfrac{2\,Ku}{\pi}}{\cos u}\right]_{u=\frac{\pi}{2}}=\log\frac{2\,k'K}{\pi}=\log\left(2\,q^{\frac{1}{4}}\sqrt{\frac{k'}{k}}\right)+2\sum_{m=1}^{m=\infty}\frac{1}{m}\,\frac{(-q)^m}{1+(-q)^m}.$$

We also have from (1), (2) and (3) the formulas

$$(4)\quad \frac{d}{du}\log sn\frac{2\,Ku}{\pi}=\frac{2\,K}{\pi}\,\frac{cn\dfrac{2\,Ku}{\pi}dn\dfrac{2\,Ku}{\pi}}{sn\dfrac{2\,Ku}{\pi}}=\frac{\cos u}{\sin u}-4\sum_{m=1}^{m=\infty}\frac{q^m}{1+q^m}\sin 2\,mu,$$

$$(5)\quad -\frac{d}{du}\log cn\frac{2\,Ku}{\pi}=\frac{2\,K}{\pi}\,\frac{sn\dfrac{2\,Ku}{\pi}du\dfrac{2\,Ku}{\pi}}{cn\dfrac{2\,Ku}{\pi}}=\frac{\sin u}{\cos u}+4\sum_{m=1}^{m=\infty}\frac{q^m}{1+(-q)^m}\sin 2\,mu,$$

$$(6)\quad -\frac{d}{du}\log dn\frac{2\,Ku}{\pi}=\frac{2\,K}{\pi}\,\frac{k^2 sn\dfrac{2\,Ku}{\pi}cn\dfrac{2\,Ku}{\pi}}{dn\dfrac{2\,Ku}{\pi}}=8\sum_{m=1}^{m=\infty}\frac{q^{2m-1}}{1-q^{4m-2}}\sin(4\,m-2)u.$$

... we put $\frac{\pi}{2}-u$ for u, we have

$$\frac{cn\frac{2Ku}{\pi}}{sn\frac{2Ku}{\pi}dn\frac{2Ku}{\pi}}=\frac{\sin u}{\cos u}-4\sum_{m=1}^{m=\infty}\frac{(-1)^{m-1}q^m}{1+q^m}\sin 2mu,$$

$$\frac{cn\frac{2Ku}{\pi}}{sn\frac{2Ku}{\pi}dn\frac{2Ku}{\pi}}=\frac{\cos u}{\sin u}+4\sum_{m=1}^{m=\infty}\frac{(-1)^{m-1}q^m}{1+(-q)^m}\sin 2mu.$$

add the equations of Art. 231

$$\frac{1}{sn\frac{2Ku}{\pi}}=\frac{1}{\sin u}+4\sum_{m=1}^{m=\infty}\frac{q^{2m-1}}{1-q^{2m-1}}\sin(2m-1)u,$$

$$\frac{1}{cn\frac{2Ku}{\pi}}=\frac{1}{\cos u}+4\sum_{m=1}^{m=\infty}(-1)^m\frac{q^{2m-1}}{1+q^{2m-1}}\cos(2m-1)u;$$

ations of Art. 228

$$\frac{2kK}{\pi}sn\frac{2Ku}{\pi}=4q^{\frac{1}{2}}\sum_{m=1}^{m=\infty}\frac{q^{m-1}}{1-q^{2m-1}}\sin(2m-1)u,$$

$$\frac{2kK}{\pi}cn\frac{2Ku}{\pi}=4q^{\frac{1}{2}}\sum_{m=1}^{m=\infty}\frac{q^{m-1}}{1+q^{2m-1}}\cos(2m-1),$$

$$\frac{2K}{\pi}dn\frac{2Ku}{\pi}=1+4\sum_{m=1}^{m=\infty}\frac{q^m}{1+q^{2m}}\cos 2mu.$$

2) write $u=0$ and in Equa. (9) put $u=\frac{\pi}{2}$; it follows that

$$=\frac{2K}{\pi}=1+4\sum_{m=1}^{m=\infty}\frac{q^m}{1+q^{2m}}=1+4\sum_{m=1}^{m=\infty}(-1)^{m-1}\frac{q^{2m-1}}{1-q^{2m-1}}.$$

writing $u=\frac{\pi}{2}$ in (11) and $u=0$ in (12), we have

$$=\frac{2kK}{\pi}=4q^{\frac{1}{2}}\sum_{m=1}^{m=\infty}(-1)^{m-1}\frac{q^{m-1}}{1-q^{2m-1}}=4q^{\frac{1}{2}}\sum_{m=1}^{m=\infty}\frac{q^{m-1}}{1+q^{2m-1}}.$$

(13) we put $\frac{\pi}{2}-u$ for u we have

$$\frac{k'K}{\pi}\frac{1}{dn\frac{2Ku}{\pi}}=1+4\sum_{m=1}^{m=\infty}(-1)^m\frac{q^m}{1+q^{2m}}\cos 2mu;$$

and substituting $u = 0$ in (10) and $u = 0$ in the equation just written, it is seen that

$$(16)\quad \vartheta_0^2(0) = \frac{2k'K}{\pi} = 1 + 4\sum_{m=1}^{m=\infty}(-1)^m \frac{q^{2m-1}}{1+q^{2m-1}} = 1 + 4\sum_{m=1}^{m=\infty}\frac{(-1)^m q^m}{1+q^{2m}}.$$

If further we differentiate (8) with regard to u and then put $u = \frac{\pi}{2}$, we have

$$(17)\qquad \left(\frac{2K}{\pi}\right)^2 = \vartheta_3^4(0) = 1 + 8\sum_{m=1}^{m=\infty}\frac{mq^m}{1+(-q)^m};$$

and if Equa. (7) be differentiated with regard to u, it becomes for $u = 0$

$$(18)\qquad \left(\frac{2k'K}{\pi}\right)^2 = \vartheta^4(0) = 1 + 8\sum_{m=1}^{m=\infty}\frac{(-1)^m mq^m}{1+q^m}.$$

Subtracting (18) from (17) we have

$$(19)\qquad \left(\frac{2kK}{\pi}\right)^2 = \vartheta_2^4(0) = 16\sum_{m=1}^{m=\infty}\frac{(2m-1)q^{2m-1}}{1-q^{4m-2}}.$$

Jacobi (Werke, I, pp. 159, *et seq.*) has given forty-seven such formulas as those above.

ART. 344. In Art. 89 mention was made of the fact that many of the properties of the Θ-functions had been recognized by Poisson. For example, in the 12th volume of the *Journal de l'École Polytechnique*, p. 420 (1823), he established by means of definite integrals the formula

$$\sqrt{\frac{1}{x}} = \frac{1 + 2e^{-\pi x} + 2e^{-4\pi x} + 2e^{-9\pi x} + 2e^{-16\pi x} + \cdots}{1 + 2e^{-\pi/x} + 2e^{-4\pi/x} + 2e^{-9\pi/x} + 2e^{-16\pi/x} + \cdots}.$$

To verify this formula by means of the elliptic functions, let $x = \frac{K'}{K}$. Instead of k we take the complementary modulus $k' = \sqrt{1-k^2}$, the quantity x becoming $\frac{1}{x} = \frac{K}{K'}$. Hence if in the formula

$$\begin{aligned}\sqrt{\frac{2K}{\pi}} &= 1 + 2q + 2q^4 + 2q^9 + 2q^{16} + \cdots\\ &= 1 + 2e^{-\pi x} + 2e^{-4\pi x} + 2e^{-9\pi x} + \cdots,\end{aligned}$$

we change k to k', we have

$$\sqrt{\frac{2K'}{\pi}} = 1 + 2e^{-\frac{\pi}{x}} + 2e^{-\frac{4\pi}{x}} + \cdots,$$

and consequently the formula of Poisson.*

* In this connection see a remark by Abel, *Crelle*, Bd. 4, p. 93.

ART. 345. In Arts. 260 and 320 we derived the relations

$$\begin{aligned}
\sigma u &= \beta e^{2\eta\omega v^2}\vartheta_1(v), \qquad [u = 2\omega v]\\
\sigma_1 u &= \beta_1 e^{2\eta\omega v^2}\vartheta_2(v),\\
\sigma_2 u &= \beta_3 e^{2\eta\omega v^2}\vartheta_3(v),\\
\sigma_3 u &= \beta_3 e^{2\eta\omega v^2}\vartheta_0(v),
\end{aligned}$$

where

$$\beta = \frac{2\omega}{\vartheta_1'(0)}, \quad \beta_1 = \frac{1}{\vartheta_2(0)}, \quad \beta_2 = \frac{1}{\vartheta_3(0)}, \quad \beta_3 = \frac{1}{\vartheta_0(0)}.$$

Noting that

$$K = \omega\sqrt{e_1 - e_3}, \quad k^2 = \frac{e_2 - e_3}{e_1 - e_3}, \quad k'^2 = \frac{e_1 - e_2}{e_1 - e_3},$$

we have, if we put

$$G = (e_1 - e_2)^2(e_1 - e_3)^2(e_2 - e_3)^2,$$

$$\beta\sqrt[8]{G} = \sqrt{\frac{\pi}{2\omega}}; \qquad \sqrt[8]{G} = \frac{\pi}{2\omega}\sqrt{\frac{\pi}{2\omega}}\, 2q^{\frac{1}{4}}Q_0{}^3,$$

$$\beta_1 = \sqrt{\frac{\pi}{2\omega}}\,\frac{1}{\sqrt[4]{e_2 - e_3}} = \frac{1}{\vartheta_2(0)}; \quad \sqrt[4]{e_2 - e_3} = \sqrt{\frac{\pi}{2\omega}}\, 2q^{\frac{1}{4}}Q_0 Q_1{}^2,$$

$$\beta_2 = \sqrt{\frac{\pi}{2\omega}}\,\frac{1}{\sqrt[4]{e_1 - e_3}} = \frac{1}{\vartheta_3(0)}; \quad \sqrt[4]{e_1 - e_3} = \sqrt{\frac{\pi}{2\omega}}\, Q_0 Q_2{}^2,$$

$$\beta_3 = \sqrt{\frac{\pi}{2\omega}}\,\frac{1}{\sqrt[4]{e_1 - e_2}} = \frac{1}{\vartheta_0(0)}; \quad \sqrt[4]{e_1 - e_2} = \sqrt{\frac{\pi}{2\omega}}\, Q_0 Q_3{}^2.$$

If follows immediately that

$$e_1 = \frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2[\vartheta_3{}^4(0) + \vartheta_0{}^4(0)] = \frac{\pi^2}{12\omega^2}Q_0{}^4[Q_2{}^8 + Q_3{}^8],$$

$$e_2 = \frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2[\vartheta_2{}^4(0) - \vartheta_0{}^4(0)] = \frac{\pi^2}{12\omega^2}Q_0{}^4[16qQ_1{}^8 - Q_3{}^8],$$

$$e_3 = -\frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2[\vartheta_2{}^4(0) + \vartheta_3{}^4(0)] = -\frac{\pi^2}{12\omega^2}Q_0{}^4[16qQ_1{}^8 + Q_2{}^8];$$

$$\sqrt{\frac{2\omega}{\pi}} = \frac{\vartheta_2(0)}{\sqrt[4]{e_2 - e_3}} = \frac{2h^{\frac{1}{4}} + 2h^{\frac{9}{4}} + 2h^{\frac{25}{4}} + \cdots}{\sqrt[4]{e_2 - e_3}},$$

$$\sqrt{\frac{2\omega}{\pi}}\left[\sqrt[4]{e_1 - e_3} - \sqrt[4]{e_1 - e_2}\right] = \vartheta_3(0) - \vartheta_0(0),$$

or

$$\sqrt{\frac{2\omega}{\pi}} = \frac{2}{\sqrt[4]{e_1 - e_3} - \sqrt[4]{e_1 - e_2}}[2h + 2h^9 + 2h^{25} + \cdots].$$

We also have

$$\sqrt{\frac{2\omega}{\pi}} = \frac{1 + 2h + 2h^4 + 2h^9 + \cdots}{\sqrt[4]{e_1 - e_3}} = \frac{2(1 + 2h^4 + 2h^{16} + \cdots)}{\sqrt[4]{e_1 - e_3} + \sqrt[4]{e_1 - e_2}}.$$

$$\sqrt{k} = + \frac{\sqrt[4]{e_2 - e_3}}{\sqrt[4]{e_1 - e_3}} = \frac{\vartheta_2(0)}{\vartheta_3(0)} = 2q^{\frac{1}{4}}\frac{Q_1^2}{Q_2^2},$$

$$\sqrt{k'} = + \frac{\sqrt[4]{e_1 - e_2}}{\sqrt[4]{e_1 - e_3}} = \frac{\vartheta_0(0)}{\vartheta_3(0)} = \frac{Q_3^2}{Q_2^2},$$

$$Q_2^8 - Q_3^8 = 16\,qQ_1^8.$$

It is further seen that

$$\left(\frac{\pi}{2\omega}\right)^4[\vartheta_0^8(0) + \vartheta_2^8(0) + \vartheta_3^8(0)] = (e_3 - e_2)^2 + (e_2 - e_1)^2 + (e_1 - e_3)^2$$
$$= 2(e_1 + e_2 + e_3)^2 - 6(e_1e_2 + e_2e_3 + e_3e_1) = \tfrac{3}{2}\,g_2,$$

or
$$g_2 = \frac{2}{3}\left(\frac{\pi}{2\omega}\right)^4[\vartheta_0^8(0) + \vartheta_2^8(0) + \vartheta_3^8(0)],$$

and similarly

$$g_3 = \frac{4}{27}\left(\frac{\pi}{2\omega}\right)^6[\vartheta_2^4(0) + \vartheta_3^4(0)][\vartheta_3^4(0) + \vartheta_0^4(0)][\vartheta_0^4(0) - \vartheta_2^4(0)] = 4\,e_1e_2e_3,$$

and

$$G = (e_3 - e_2)^2(e_2 - e_1)^2(e_1 - e_3)^2 = \frac{\pi^4}{(2\omega)^{12}}\vartheta_1'^8(0) = q^2\frac{\pi^{12}}{16}\frac{Q_0^{24}}{\omega^{12}}.$$

ART. 346. The formulas of the preceding Article may be written

(1) $$e^{2\eta\omega v^2}\vartheta_1(v, \tau) = \sqrt{\frac{2\omega}{\pi}}\,\sqrt[8]{G}\,\sigma u = \frac{\pi}{\omega}Q_0^3 q^{\frac{1}{4}}\sigma u,$$

(2) $$e^{2\eta\omega v^2}\vartheta_2(v, \tau) = \sqrt{\frac{2\omega}{\pi}}\,\sqrt[4]{e_2 - e_3}\,\sigma_1 u = 2\,Q_0Q_1^2q^{\frac{1}{4}}\sigma_1 u,$$

(3) $$e^{2\eta\omega v^2}\vartheta_3(v, \tau) = \sqrt{\frac{2\omega}{\pi}}\,\sqrt[4]{e_1 - e_3}\,\sigma_2 u = Q_0Q_2^2\sigma_2 u,$$

(4) $$e^{2\eta\omega v^2}\vartheta_0(v, \tau) = \sqrt{\frac{2\omega}{\pi}}\,\sqrt[4]{e_1 - e_2}\,\sigma_3 u = Q_0Q_3^2\sigma_3 u;$$

or,

(5) $$\sigma u = 2\omega\frac{\vartheta_1(v)}{\vartheta_1'(0)}e^{2\eta\omega v^2},$$

(6) $$\sigma_\lambda u = \frac{\vartheta_{\lambda+1}(v)}{\vartheta_{\lambda+1}(0)}e^{2\eta\omega v^2} \qquad (\lambda = 1, 2, 3;\ \vartheta_4 = \vartheta_0).$$

Noting that the coefficient of u^3 in σu is zero, and that the coefficient of u^2 in $\sigma_\lambda u$ is $-\frac{1}{2}e_\lambda$, it follows by a comparison of the coefficients on the right-hand side of equations (5) and (6) that

$$(7) \qquad 2\,\eta\omega = -\frac{1}{6}\frac{\vartheta_1'''(0)}{\vartheta_1'(0)},$$

$$(8) \qquad 2\,\eta\omega = -2\,e_\lambda\omega^2 - \frac{1}{2}\frac{\vartheta''_{\lambda+1}(0)}{\vartheta_{\lambda+1}(0)} \qquad (\lambda = 1, 2, 3;\ \vartheta_4 = \vartheta_0).$$

From (7) and (8) we have at once the relation of Art. 339,

$$\frac{\vartheta_1'''(0)}{\vartheta_1'(0)} = \frac{\vartheta_0''(0)}{\vartheta_0(0)} + \frac{\vartheta_2''(0)}{\vartheta_2(0)} + \frac{\vartheta_3''(0)}{\vartheta_3(0)}.$$

ART. 347. The formulas of Art. 329, in virtue of the relations just derived, may be written

$$\sigma\omega = \frac{e^{\frac{1}{2}\eta\omega}}{\sqrt[4]{e_1-e_3}\,\sqrt[4]{e_1-e_2}} = \frac{2\,\omega}{\pi}\frac{Q_1{}^2}{Q_0{}^2}e^{\frac{1}{2}\eta\omega},$$

$$\sigma\omega'' = \frac{\sqrt{i}e^{\frac{1}{2}\eta''\omega''}}{\sqrt[4]{e_2-e_3}\,\sqrt[4]{e_1-e_2}} = \sqrt{i}\,\frac{2\,\omega}{\pi}\frac{Q_2{}^2}{Q_0{}^2}\frac{e^{\frac{1}{2}\eta''\omega''}}{2\,q^{\frac{1}{4}}},$$

$$\sigma\omega' = \frac{ie^{\frac{1}{2}\eta'\omega'}}{\sqrt[4]{e_2-e_3}\,\sqrt[4]{e_1-e_3}} = i\,\frac{2\,\omega}{\pi}\frac{Q_3{}^2}{Q_0{}^2}\frac{e^{\frac{1}{2}\eta'\omega'}}{2\,q^{\frac{1}{4}}}.$$

The six formulas of Art. 328 thus offer a means of deriving the values of the functions σ_1, σ_2, σ_3, having as arguments the quantities ω, ω'', ω'.

The results as set forth in the Table of Formulas, XLIV, should be verified. We have for example

$$\sigma_1\omega'' = \sqrt{e_2-e_1}\,\sigma\omega'' = -\,i\sqrt{e_1-e_2}\,\sigma\omega'' = -\,i\sqrt{i}\,\frac{Q_3{}^2}{Q_1{}^2}\frac{e^{\frac{\eta''\omega''}{2}}}{2\,q^{\frac{1}{4}}}.$$

Such formulas may also be had as follows:

Since $z = e^{\pi i v}$, where $u = 2\,\omega v$, when u takes the values

$$\omega, \qquad \omega'', \qquad \omega',$$

the values of z are $\quad i, \quad iq^{\frac{1}{4}}, \quad q^{\frac{1}{4}};$

and since
$$\eta\omega'' - \eta''\omega = \frac{\pi i}{2},$$

we have
$$\frac{\eta\omega''^2}{2\,\omega} = \frac{\eta''\omega''}{2} + \frac{\pi i}{4}(1+\tau),$$

so that when u takes the values

$$\omega, \qquad \omega'', \qquad \omega',$$

$e^{2\eta\omega v^2}$ becomes $\quad e^{\frac{\eta\omega}{2}}, \quad e^{\frac{\eta''\omega''}{2}}\sqrt{i}q^{\frac{1}{4}}, \quad e^{\frac{\eta'\omega'}{2}}q^{\frac{1}{4}}.$

If for example for u in the formula for σu (in Art. 291) we write $u = \omega''$, we have

$$\sigma\omega'' = \sqrt{i}e^{\frac{\eta''\omega''}{2}} q^{\frac{1}{4}} \frac{2\omega}{\pi} \frac{iq^{\frac{1}{4}} - \frac{1}{iq^{\frac{1}{4}}}}{2i} \prod_{n=1}^{n=\infty} \frac{1+q^{2n-1}}{1-q^{2n}} \prod_{n=1}^{n=\infty} \frac{1+q^{2n+1}}{1-q^{2n}}$$

$$= \sqrt{i}e^{\frac{\eta''\omega''}{2}} \frac{2\omega}{\pi} \frac{1}{2q^{\frac{1}{4}}} \frac{Q_2^2}{Q_0^2}.$$

The formulas expressing $\sigma_1 u$, $\sigma_2 u$, $\sigma_3 u$ through infinite products are given in Art. 321.

EXAMPLES

1. Show that

$$\vartheta(0, q^2) = \sqrt[4]{k'}\sqrt{\frac{2K}{\pi}},$$

$$\vartheta_2(0, \sqrt{q}) = \sqrt{2}\sqrt[4]{k}\sqrt{\frac{2K}{\pi}},$$

$$\vartheta_2(0, i\sqrt{q}) = \sqrt{2}\sqrt[4]{ikk'}\sqrt{\frac{2K}{\pi}},$$

$$2\vartheta_1'(0, q^2) = k\sqrt[4]{k'}\sqrt{\left(\frac{2K}{\pi}\right)^3},$$

$$\vartheta_1'(0, \sqrt{q}) = \sqrt{2}\,k'\sqrt[4]{k}\sqrt{\left(\frac{2K}{\pi}\right)^3},$$

$$\vartheta_1'(0, i\sqrt{q}) = \sqrt{2}\sqrt[4]{ikk'}\sqrt{\left(\frac{2K}{\pi}\right)^3}.$$

(Jacobi, Werke, II, p. 431.)

2. Through a comparison of the coefficients in Formula (6) of Art. 346 show that

$$2\eta^2\omega^2 = \left(\frac{g_2}{3} - 2e_\lambda^2\right)\omega^4 - \eta\omega\frac{\vartheta''_{\lambda+1}}{\vartheta_{\lambda+1}} - \frac{1}{24}\frac{\vartheta^{(4)}_{\lambda+1}}{\vartheta_{\lambda+1}}$$

$$(\lambda = 1, 2, 3;\ \vartheta_4 = \vartheta_0).$$

3. Show that

$$\frac{g_2}{(e_1 - e_3)^2} = \frac{4}{3}(1 - k^2 + k^4)$$

$$\frac{g_3}{(e_1 - e_3)^3} = \frac{4}{27}(1 + k^2)(2 - k^2)(1 - 2k^2),$$

$$\frac{G}{(e_1 - e_3)^6} = \frac{g_2^3 - 27g_3^2}{16(e_1 - e_3)^6} = k^4k'^4.$$

4. Verify all the formulas given in the Table of Formulas, XLI and XLII.

5. Show that

$$\frac{(1 - k^2 + k^4)^3}{k^4(1 - k^2)^2} = \frac{27g_2^3}{4(g_2^3 - 27g_3^2)} = \frac{1}{8}\frac{[\vartheta_2^8(0) + \vartheta_3^8(0) + \vartheta_0^8(0)]^3}{\vartheta_2^8(0)\vartheta_3^8(0)\vartheta_0^8(0)}.$$

CHAPTER XIX

ELLIPTIC INTEGRALS OF THE THIRD KIND

ARTICLE 348. In Chapter VIII we saw that the elliptic integrals of the third kind in the normal forms of Legendre and of Weierstrass were

$$\int \frac{d\mathbf{z}}{(\mathbf{z}^2 - \beta^2)\sqrt{(1 - \mathbf{z}^2)(1 - k^2\mathbf{z}^2)}} \quad \text{and} \quad \int \frac{dt}{(t - b)\sqrt{4\,t^3 - g_2 t - g_3}}.$$

In the neighborhood of the point $\mathbf{z} = \beta$, if β is *not* a root of

$\mathbf{s}^2 = \mathsf{Z}(\mathbf{z}) = (1 - \mathbf{z}^2)(1 - k^2\mathbf{z}^2) = 0$, the expansion of $\dfrac{1}{\sqrt{(1 - \mathbf{z}^2)(1 - k^2\mathbf{z}^2)}}$ is by Taylor's Theorem

$$\frac{1}{\sqrt{(1 - \mathbf{z}^2)(1 - k^2\mathbf{z}^2)}} = A + a_0(\mathbf{z} - \beta) + a_1(\mathbf{z} - \beta)^2 + \cdots$$

where

$$A = \frac{1}{\sqrt{(1 - \beta^2)(1 - k^2\beta^2)}} = \frac{1}{\sqrt{\mathsf{Z}(\beta)}}.$$

It is evident that Legendre's normal integral becomes logarithmically infinite for $\mathbf{z} = \beta$ in both leaves of the Riemann surface as the two quantities

$$\frac{1}{2\beta} A \log(\mathbf{z} - \beta) \quad \text{and} \quad -\frac{1}{2\beta} A \log(\mathbf{z} - \beta);$$

and for $\mathbf{z} = -\beta$ in both leaves as

$$-\frac{1}{2\beta} A \log(\mathbf{z} + \beta) \quad \text{and} \quad \frac{1}{2\beta} A \log(\mathbf{z} + \beta).$$

If β is a root of $(1 - \mathbf{z}^2)(1 - k^2\mathbf{z}^2) = 0$, say $\beta = 1$, then at the point $\beta = 1$ the integral becomes algebraically infinite of the one-half order.

The integral of the third kind in Weierstrass's normal form becomes logarithmically infinite at the point $t = b$ in both leaves of the Riemann surface as

$$\frac{1}{\sqrt{4\,b^3 - g_2 b - g_3}} \log(t - b) \quad \text{and} \quad -\frac{1}{\sqrt{4\,b^3 - g_2 b - g_3}} \log(t - b).$$

ART. 349. Let us next form the simplest integral of the third kind which becomes logarithmically infinite at only two points of the Riemann surface. There must be at least two such points α_1 and α_2, say, since the sum of the residues of the integrand must be zero.

We may write the integrand in the form

$$\frac{A_0 + A_1\mathbf{z} + A_2\sqrt{\mathbf{Z}(\mathbf{z})}}{(\mathbf{z}-\alpha_1)(\mathbf{z}-\alpha_2)\sqrt{\mathbf{Z}(\mathbf{z})}} = I(\mathbf{z}, \mathbf{s}), \text{ say.}$$

We shall choose the points $[\alpha_1, \sqrt{\mathbf{Z}(\alpha_1)}]$, $[\alpha_2, \sqrt{\mathbf{Z}(\alpha_2)}]$ in the *upper* leaf of the Riemann surface and we must determine the constants A_0, A_1, A_2 so that the integral does not become infinite at the two corresponding points $[\alpha_1, -\sqrt{\mathbf{Z}(\alpha_1)}]$, $[\alpha_2, -\sqrt{\mathbf{Z}(\alpha_2)}]$ in the lower leaf.

Accordingly we must have

$$\text{(1)} \qquad \begin{cases} A_0 + A_1\alpha_1 - A_2\sqrt{\mathbf{Z}(\alpha_1)} = 0, \\ A_0 + A_1\alpha_2 - A_2\sqrt{\mathbf{Z}(\alpha_2)} = 0. \end{cases}$$

In the neighborhood of the point $\mathbf{z} = \alpha_1$ we have by Taylor's Theorem

$$\frac{A_0+A_1\mathbf{z}+A_2\sqrt{\mathbf{Z}(\mathbf{z})}}{(\mathbf{z}-\alpha_2)\sqrt{\mathbf{Z}(\mathbf{z})}} = \frac{A_0+A_1\alpha_1+A_2\sqrt{\mathbf{Z}(\alpha_1)}}{(\alpha_1-\alpha_2)\sqrt{\mathbf{Z}(\alpha_1)}} + c_1(\mathbf{z}-\alpha_1)+c_2(\mathbf{z}-\alpha_1)^2+\cdots;$$

and consequently

$$\frac{A_0+A_1\mathbf{z}+A_2\sqrt{\mathbf{Z}(\mathbf{z})}}{(\mathbf{z}-\alpha_1)(\mathbf{z}-\alpha_2)\sqrt{\mathbf{Z}(\mathbf{z})}} = \frac{A_0+A_1\alpha_1+A_2\sqrt{\mathbf{Z}(\alpha_1)}}{(\mathbf{z}-\alpha_1)(\alpha_1-\alpha_2)\sqrt{\mathbf{Z}(\alpha_1)}} + c_1 + c_2(\mathbf{z}-\alpha_1) + \cdots.$$

It follows that

$$\operatorname*{Res}_{\mathbf{z}=\alpha_1} I(\mathbf{z}, \mathbf{s}) = \frac{A_0 + A_1\alpha_1 + A_2\sqrt{\mathbf{Z}(\alpha_1)}}{(\alpha_1-\alpha_2)\,\mathbf{Z}(\alpha_1)},$$

[which owing to equations (1)] $= \dfrac{2A_2}{\alpha_1-\alpha_2}$.

In a similar manner we have

$$\operatorname*{Res}_{\mathbf{z}=\alpha_2} I(\mathbf{z}, \mathbf{s}) = \frac{A_0 + A_1\alpha_2 + A_2\sqrt{\mathbf{Z}(\alpha_2)}}{(\alpha_2-\alpha_1)\sqrt{\mathbf{Z}(\alpha_2)}} = \frac{2A_2}{\alpha_2-\alpha_1}.$$

Eliminating A_1 from the equations (1), we have

$$A_0 = \frac{A_2\{\alpha_2\sqrt{\mathbf{Z}(\alpha_1)} - \alpha_1\sqrt{\mathbf{Z}(\alpha_2)}\}}{\alpha_2-\alpha_1},$$

and eliminating A_0 from the same equations, we have

$$A_1 = \frac{A_2\{\sqrt{\mathbf{Z}(\alpha_2)} - \sqrt{\mathbf{Z}(\alpha_1)}\}}{\alpha_2-\alpha_1}.$$

It follows that

$$I(\mathbf{z}, \mathbf{s}) = \frac{A_2}{\alpha_2 - \alpha_1} \frac{\alpha_2\sqrt{Z(\alpha_1)} - \alpha_1\sqrt{Z(\alpha_2)} + \left(\sqrt{Z(\alpha_2)} - \sqrt{Z(\alpha_1)}\right)\mathbf{z} + (\alpha_2 - \alpha_1)\sqrt{Z(\mathbf{z})}}{(\mathbf{z} - \alpha_1)(\mathbf{z} - \alpha_2)\sqrt{Z(\mathbf{z})}}$$

$$= \frac{A_2}{\alpha_2 - \alpha_1}\left[-\frac{\sqrt{Z(\alpha_1)} + \sqrt{Z(\mathbf{z})}}{(\mathbf{z} - \alpha_1)\sqrt{Z(\mathbf{z})}} + \frac{\sqrt{Z(\alpha_2)} + \sqrt{Z(\mathbf{z})}}{(\mathbf{z} - \alpha_2)\sqrt{Z(\mathbf{z})}}\right].$$

When $I(\mathbf{z}, \mathbf{s})$ has this form, the integral $\int I(\mathbf{z}, \mathbf{s})d\mathbf{z}$ is of the third kind, being logarithmically infinite at the points $\left(\alpha_1, \sqrt{Z(\alpha_1)}\right)$, $\left(\alpha_2, \sqrt{Z(\alpha_2)}\right)$.

This integral may be considered the *fundamental* integral of the third kind and written

$$\Pi\left(\mathbf{z}, \sqrt{Z(\mathbf{z})};\ \alpha_1, \sqrt{Z(\alpha_1)};\ \alpha_2, \sqrt{Z(\alpha_2)}\right) \text{ or more simply } \Pi(\mathbf{z};\ \alpha_1;\ \alpha_2).$$

In a similar manner, as was proved in the case of the integrals of the second kind, we have a general integral of the third kind with the two logarithmic infinities α_1, α_2 if we add integrals of the first kind to $\Pi(\mathbf{z}; \alpha_1; \alpha_2)$.

ART. 350. Take three points $\alpha_1, \sqrt{Z(\alpha_1)}$; $\alpha_2, \sqrt{Z(\alpha_2)}$; $\alpha_3, \sqrt{Z(\alpha_3)}$ on the Riemann surface of Chapter VI and form the integrals

$$\Pi(\mathbf{z};\ \alpha_1;\ \alpha_2), \qquad \Pi(\mathbf{z};\ \alpha_2;\ \alpha_3), \qquad \Pi(\mathbf{z};\ \alpha_3;\ \alpha_1).$$

Further, let A_2, $A_2^{(1)}$ and $A_3^{(1)}$ be the constants that correspond to A_2 above.

We may so choose the constants λ, μ, ν that the expression

$$(1) \qquad \lambda\,\Pi(\mathbf{z};\ \alpha_1;\ \alpha_2) + \mu\,\Pi(\mathbf{z};\ \alpha_2;\ \alpha_3) + \nu\,\Pi(\mathbf{z};\ \alpha_3;\ \alpha_1)$$

does not become infinite at any point of the Riemann surface and is consequently an integral of the first kind or a constant.

We note that in the neighborhood of the point α_1 the expression becomes infinite as

$$\frac{2\,\lambda A_2}{\alpha_1 - \alpha_2}\log(\mathbf{z} - \alpha_1) + \frac{2\,\nu A_2^{(2)}}{\alpha_1 - \alpha_3}\log(\mathbf{z} - \alpha_1)$$

and consequently remains finite at α_1 if

$$\frac{\lambda A_2}{\alpha_1 - \alpha_2} + \frac{\nu A_2^{(2)}}{\alpha_1 - \alpha_3} = 0.$$

Similarly the expression remains finite at α_2 and α_3 if

$$\frac{\lambda A_2}{\alpha_2 - \alpha_1} + \frac{\mu A_2^{(1)}}{\alpha_2 - \alpha_3} = 0 \quad \text{and} \quad \frac{\mu A_2^{(1)}}{\alpha_3 - \alpha_2} + \frac{\nu A_2^{(2)}}{\alpha_3 - \alpha_1} = 0.$$

The third equation is a consequence of the first two. If the ratios of λ, μ, ν have been determined from these equations the integral (1) is an integral of the first kind * or a constant.

ART. 351. We have seen in Chapters VII and XIII that the integral of the first kind has in common with the integral of the second kind the property of being a one-valued function of position on the Riemann surface T'. This is not true of the integral of the third kind; for consider in the Riemann surface the fundamental integral above. In the neighborhood of the point $\mathbf{z} = \alpha_1$ we saw that the integrand could be written in the form

$$\frac{2A_2}{\alpha_1 - \alpha_2} \frac{1}{\mathbf{z} - \alpha_1} + P(\mathbf{z} - \alpha_1).$$

It follows that the integral over a small circle including the point α_1 as center is

$$\frac{2A_2}{\alpha_1 - \alpha_2} 2\pi i,$$

while the integral over a small circle including the point $\mathbf{z} = \alpha_2$ is

$$\frac{2A_2}{\alpha_2 - \alpha_1} 2\pi i.$$

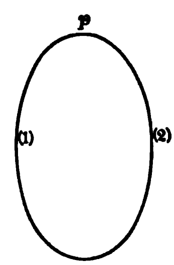


Fig. 73.

If then two paths of integration (1) and (2) starting from the point p_0 include both or neither of the points α_1 and α_2, we come to the point p with the same value along either path.

Hence to construct a Riemann surface upon which the fundamental integral of the third kind will be one-valued we draw small circles around α_1 and α_2 and join these circles by a canal so as to form a connected curve. To make the surface simply connected we join this canal with the canal a, say in T' (of Art. 142), by another canal AB. The new surface we denote by T''.

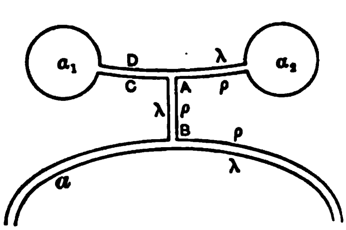


Fig. 74.

Denote the difference in values of the integral $\Pi(\mathbf{z}; \alpha_1; \alpha_2)$ taken on the left and right banks of the canals in T'' by $\Pi(\lambda) - \Pi(\rho)$. It is seen that for the canal AB any path of integration must encircle both or neither of the points α_1 and α_2 to get from the left to the right bank. It follows that along the canal AB we have $\Pi(\lambda) - \Pi(\rho) = 0$.

* See Clebsch und Gordan, *Theorie der Abel'schen Functionen*, p. 118; or Klein-Fricke, *Theorie der elliptischen Modulfunctionen*, Bd. I, p. 518.

To go from the point D to the point C in the figure we must encircle either α_1 or α_2. In either case we have

$$\Pi(\lambda) - \Pi(\rho) = \frac{2A_2}{\alpha_1 - \alpha_2} 2\pi i.$$

This difference may be made $-2\pi i$ if in the fundamental integral we give to the arbitrary constant A_2 a value such that

$$\frac{A_2}{\alpha_2 - \alpha_1} = \frac{1}{2}.$$

ART. 352. Let us consider next the *elementary* integral of the third kind in the Weierstrassian notation

$$\Pi\left(t, \sqrt{S(t)};\ \alpha, \sqrt{S(\alpha)};\ \infty\right) = \int^{t,\sqrt{S(t)}} \frac{\sqrt{S(t)} + \sqrt{S(\alpha)}}{2(t-\alpha)} \frac{dt}{\sqrt{S(t)}},$$

where $$S(t) = 4t^3 - g_2 t - g_3.$$

Writing $\beta = \sqrt{S(\alpha)}$ we note that in the neighborhood of the point (α, β) we have

$$\frac{\sqrt{S(t)} + \sqrt{S(\alpha)}}{2} = \beta + \frac{1}{2}\frac{d\beta}{d\alpha}(t-\alpha) + \frac{1}{4}\frac{d^2\beta}{d\alpha^2}(t-\alpha)^2 + \cdots,$$

$$\frac{1}{\sqrt{S(t)}} = \frac{1}{\beta} - \frac{1}{\beta^2}\frac{d\beta}{d\alpha}(t-\alpha) + \left[\frac{1}{\beta^3}\left(\frac{d\beta}{d\alpha}\right)^2 - \frac{1}{2\beta^2}\frac{d^2\beta}{d\alpha^2}\right](t-\alpha)^2 + \cdots,$$

so that in the neighborhood of $t = \alpha$

$$\Pi(t;\ \alpha;\ \infty) = \log(t-\alpha) - \frac{1}{2\beta}\frac{d\beta}{d\alpha}(t-\alpha) + \cdots$$

and that the residue corresponding to the point $t = \alpha$ is $+2\pi i$.

In the neighborhood of the point at infinity we have

$$\begin{aligned}\sqrt{S(t)} + \beta &= 2(\sqrt{t})^3\sqrt{1 - \frac{g_2 t + g_3}{4t^3}} + \beta \\ &= 2(\sqrt{t})^3 + \beta - \frac{1}{4}\frac{g_2}{\sqrt{t}} - \frac{1}{4}\frac{g_3}{\sqrt{t^3}} - \cdots,\end{aligned}$$

$$\frac{1}{t-\alpha} = \frac{1}{t} + \frac{\alpha}{t^2} + \frac{\alpha^2}{t^3} + \cdots,$$

$$\frac{1}{\sqrt{S(t)}} = \frac{1}{2\sqrt{t^3}} + \frac{g_2}{16}\frac{1}{\sqrt{t^7}} + \frac{g_3}{16\sqrt{t^9}} + \cdots,$$

and consequently in the neighborhood of infinity

$$\Pi(t;\ \alpha;\ \infty) = \log\sqrt{t} - \frac{\alpha}{2}\frac{1}{t} - \frac{\alpha}{6}\frac{1}{t^3}\cdots.$$

Further, if we put $$t = re^{i\theta}$$

and $$v = \sqrt{\frac{1}{t}} = \rho e^{i\phi},$$

we have $$\rho = \sqrt{\frac{1}{r}}, \qquad \phi = -\tfrac{1}{2}\theta,$$

so that a double circle about the point at infinity in the t-plane corresponds in the opposite direction to a single circle taken around the origin in the v-plane. Hence (see Art. 120) the residue corresponding to the point at infinity is $-2\pi i$.

ART. 353. It is also seen that if in the T'-surface we draw canals from the points α_1 and infinity to the canal a, say, we form another simply connected surface T'' in which the integral $\Pi(t;\ \alpha;\ \infty)$ is one-valued. On the first of these canals we have

$$\Pi(\lambda) - \Pi(\rho) = 2\pi i$$

and on the second $$\Pi(\lambda) - \Pi(\rho) = -2\pi i.$$

If the point α coincides with one of the branch-points e_1, say, then in the neighborhood of $t = e_1$ the integral $\Pi(t;\ e_1;\ \infty)$ becomes infinite as $\log\sqrt{t - e_1}$; while in the neighborhood of $t = \infty$ this integral becomes infinite as $\log\sqrt{t}$.

Further, if we put

$$\begin{aligned}\Pi(t;\ \alpha_2;\ \alpha_1) &= \Pi(t;\ \alpha_2;\ \infty) - \Pi(t;\ \alpha_1;\ \infty)\\ &= \int\frac{\sqrt{S(t)} + \sqrt{S(\alpha_2)}}{2(t-\alpha_2)}\frac{dt}{\sqrt{S(t)}} - \int\frac{\sqrt{S(t)} + \sqrt{S(\alpha_1)}}{2(t-\alpha_1)}\frac{dt}{\sqrt{S(t)}},\end{aligned}$$

it follows from Art. 349 that $\Pi(t;\ \alpha_2;\ \alpha_1)$ becomes logarithmically infinite at the arbitrary points α_2, α_1 but has a definite value for $t = \infty$. If here the point α_1 is in the lower leaf directly under α_2, so that $\alpha_2 = \alpha_1$, $\sqrt{S(\alpha_2)} = -\sqrt{S(\alpha_1)}$, then the above integral

$$\Pi(t, \sqrt{S(t)};\ \alpha_2, \sqrt{S(\alpha_2)};\ \alpha_2, -\sqrt{S(\alpha_2)}) = \sqrt{S(\alpha_1)}\int\frac{dt}{(t-\alpha_1)\sqrt{S(t)}},$$

which is the integral cited at the beginning of this Chapter.

ART. 354. To study the moduli of periodicity of the integrals of the first, second and third kinds, Riemann * took two functions u and v and considered the integral

$$\int u\frac{dv}{dz}dz.$$

When u and v are integrals of the first and second kinds the integrand $u\dfrac{dv}{dz}$ is one-valued in the Riemann surface T'; when one of these functions is an

* Riemann, *Theorie der Abel'schen Functionen;* see also Appell et Goursat, *Fonctions algébriques*, Chap. III.

integral of the third kind, the integrand is one-valued in the surface T''. Riemann's mode of procedure is essentially the following: The integration is taken first over the entire boundary of the simply connected surface in which the integrand is one-valued, and secondly over a curve which gives the same value of the integral as the first curve; for example, the circle or double circle around the point at infinity. Since the latter curve contains in general no discontinuities of the integrand, the associated integral is zero.

Consider the two integrals

$$I_1 = \int u \, d\Pi(t; \alpha; \infty),$$

$$I_2 = \int \zeta u \, d\Pi(t; \alpha; \infty),$$

where u and ζu are integrals of the first and second kinds respectively and where the integration is over the complete boundary of the surface T' taken in the positive direction.

Let the moduli of periodicity of $\Pi(t; \alpha; \infty)$ on the canal $\boldsymbol{a}$ be $\Pi(\lambda) - \Pi(\rho) = 2\,\upsilon$, and on the canal $\boldsymbol{b}$ let $\Pi(\rho) - \Pi(\lambda) = 2\,\upsilon'$. Further, note that the integrands of both I_1 and I_2 are continuous at the point $t = \infty$.

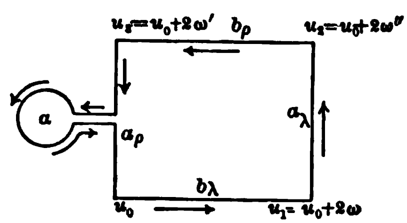


Fig. 75.

ART. 355. The Riemann surface T'' projected on the u-plane is (see Art. 197) represented in the figure.

It is evident that

$$I_1 = \int_{u_0}^{u_1} u \, d\Pi + \int_{u_1}^{u_2} u \, d\Pi + \int_{u_2}^{u_3} u \, d\Pi + \int_{u_3}^{u_0} u \, d\Pi + \int_{\text{around } \alpha} u \, d\Pi,$$

or

$$I_1 = \int_{u_0}^{u_1} [u d\Pi - (u + 2\,\omega')d\Pi] + \int_{u_1}^{u_2} [(u + 2\,\omega)d\,\Pi - u\,d\Pi] + 2\,\pi i u(\alpha)$$

$$= -\,2\,\omega' \int_{u_0}^{u_1} d\Pi + 2\,\omega \int_{u_1}^{u_2} d\Pi + 2\,\pi i u(\alpha)$$

$$= -\,2\,\omega'[\underset{\text{on } a}{\Pi(\lambda) - \Pi(\rho)}] + 2\,\omega[\underset{\text{on } b}{\Pi(\rho) - \Pi(\lambda)}] + 2\,\pi i u(\alpha).$$

This integral around the circle at infinity is zero. It follows that

$$\omega' \upsilon - \omega \upsilon' = \frac{\pi i}{2} u(\alpha),$$

and similarly from I_2,

$$\eta' \upsilon - \eta \upsilon' = \frac{\pi i}{2} \zeta(\alpha).$$

Noting that $$\eta\omega' - \omega\eta' = \frac{\pi i}{2},$$

it is seen * that
$$\upsilon = \eta u(\alpha) - \omega\zeta(\alpha),$$
$$\upsilon' = \eta' u(\alpha) - \omega'\zeta(\alpha).$$

The quantities υ and υ', the moduli of periodicity of the integral $\Pi(t; \alpha; \infty)$, have values that are the negative of those given, if the canals $\boldsymbol{a}$ and $\boldsymbol{b}$ are crossed in the opposite direction, or what is the same thing, if the direction of integration around these two canals is taken in the opposite direction.

EXAMPLES

1. Derive the results analogous to those given above for the integral $\Pi(\mathbf{z}; \alpha_1; \alpha_2)$, the surface T'' being that given in Art. 351 (see Forsyth, *Theory of Functions*, § 238).

2. Let $u = \Pi(\mathbf{z}; \alpha_1; \alpha_2)$ and $v = \Pi(\mathbf{z}; \alpha_3; \alpha_4)$ and discuss the moduli of periodicity in the associated Riemann surface (see Koenigsberger, *Ellip. Funct.*, p. 278).

ART. 356. We wrote (see Art. 196) $t = \wp u$, $\sqrt{S(t)} = - \wp' u$; it follows, if $\alpha = \wp u_0$, $\sqrt{S(\alpha)} = - \wp' u_0$, that

$$\int^{t, \sqrt{S(t)}} \frac{\sqrt{S(t)} + \sqrt{S(\alpha)}}{2(t - \alpha)} \frac{dt}{\sqrt{S(t)}} = \frac{1}{2}\int^{u} \frac{\wp' u + \wp' u_0}{\wp u - \wp u_0} du.$$

The quantity u_0 must not be congruent to the origin. In Art. 299 we saw that

$$\frac{1}{2}\frac{\wp' u + \wp' u_0}{\wp u - \wp u_0} = \frac{\sigma'(u - u_0)}{\sigma(u - u_0)} - \frac{\sigma' u}{\sigma u} + \frac{\sigma' u_0}{\sigma u_0}.$$

Through integration it follows that

$$\frac{1}{2}\int^{u} \frac{\wp' u + \wp' u_0}{\wp u - \wp u_0} du = \log \frac{\sigma(u_0 - u)}{\sigma u \sigma u_0} + u\frac{\sigma' u_0}{\sigma u_0} + C.$$

The constant C is to be so determined that in the development (see Art. 300)

$$\frac{1}{2}\int^{u} \frac{\wp' u + \wp' u_0}{\wp u - \wp u_0} du = - \log u + C - \frac{u^2}{2}\wp u_0 + \cdots,$$

C is zero.

We then have (see Schwarz, *loc. cit.*, § 56)

$$\Pi(t; \alpha; \infty) = \log \frac{\sigma(u_0 - u)}{\sigma u \sigma u_0} + u\frac{\sigma' u_0}{\sigma u_0}.$$

* See Schwarz, *loc. cit.*, § 59.

It follows at once, if m is an integer, that

$$\Pi(t;\alpha;\infty)-\Pi(\alpha;t;\infty)=u\frac{\sigma' u_0}{\sigma u_0}-u_0\frac{\sigma' u}{\sigma u}+(2m+1)\pi i,$$

a result which corresponds to the interchange of argument and parameter in the Legendre-Jacobi theory of Art. 258.

ART. 357. Legendre *Traité des fonctions elliptiques*, t. I, p. 18, represented the elliptic integral of the third kind in the form

$$\Pi(n,k,\phi)=\int\frac{d\phi}{(1+n\sin^2\phi)\,\Delta\phi}\quad\text{[see Art. 167]},$$

where the parameter n may be positive or negative, real or imaginary. This integral may be written

$$\Pi(n,k,u)=\int\frac{du}{1+n\,sn^2u}.$$

It follows that

$$\Pi(n,k,u)-u=\int\frac{-n\;sn^2u}{1+n\,sn^2u}du,$$

where u is an elliptic integral of the first kind. Jacobi [Werke, I, p. 197] made a further change in notation by writing [see also Legendre, *loc. cit.*, p. 70]

$$n=-k^2sn^2a,$$

where a being susceptible of both real and imaginary values, leaves n arbitrary.

Multiplying the right-hand side by $\dfrac{k^2cna\,dna}{sna}$, the form of the elliptic integral of the third kind adopted by Jacobi is

$$\Pi(u,a,k)=\int_0^u\frac{k^2sna\;cna\;dna\;sn^2u}{1-k^2sn^2a\;sn^2u}du.$$

ART. 358. In Art. 294 the following equation was derived:

$$\frac{\Theta^2(0)\,\Theta(u+a)\,\Theta(u-a)}{\Theta^2(u)\,\Theta^2(a)}=1-k^2sn^2u\;sn^2a.$$

If we differentiate logarithmically with regard to a, we have

$$\frac{2\,k^2sna\;cna\;dna\;sn^2u}{1-k^2sn^2u\;sn^2a}=\frac{\Theta'(u-a)}{\Theta(u-a)}-\frac{\Theta'(u+a)}{\Theta(u+a)}+2\frac{\Theta'(a)}{\Theta(a)},$$

from which it follows at once that

$$\Pi(u,a)=\frac{1}{2}\log\frac{\Theta(u-a)}{\Theta(u+a)}+u\frac{\Theta'(a)}{\Theta(a)}.$$

Interchanging u and a we further have

$$\Pi(a, u) = \frac{1}{2}\log\frac{\Theta(a-u)}{\Theta(a+u)} + a\frac{\Theta'(u)}{\Theta(u)},$$

from which it is seen that

$$\Pi(u, a) - \Pi(a, u) = uE(a) - aE(u).$$

We note that this equation remains unchanged when the argument u and the parameter a are interchanged (see Legendre, *loc. cit.*, pp. 132 *et seq.*).

ART. 359. It is evident from the integral above through which $\Pi(u, a)$ is defined, that

(1) $$\Pi(u, a) = -\Pi(-u, a) \quad \text{and}$$

(2) $$\Pi(0, a) = 0.$$

Further, since $sn K = 1$, $cn\, K = 0$, $dn K = k'$, we have

(3) $$\Pi(u, K) = 0.$$

For $a = iK'$ we have $sn\, a = \infty = cn\, a = dn\, a$, so that

(4) $$\Pi(u, iK') = \infty;$$

and since

$$sn(K \pm iK') = \frac{1}{k}, \quad cn(K \pm iK') = \mp\frac{ik'}{k}, \quad dn(K \pm iK') = 0,$$

it follows that

(5) $$\Pi(u, K \pm iK') = 0.$$

From the formula expressing the interchange of argument and parameter we have

(6) $$\Pi(K, a) = KE(a) - aE = KZ(a) \quad \text{[Legendre]}.$$

These formulas follow also directly from the expression of $\Pi(u, a)$ through the theta-functions, as do also the formulas

(7) $$\Pi(K + iK', a) = (K + iK')\,Z(a) + \frac{\pi i a}{2K},$$

(8) $$\Pi(2iK', a) = 2iK'Z(a) + \frac{\pi i a}{2K},$$

(9) $$\Pi(u + 2K, a) = \Pi(u, a) + 2KZ(a),$$

(10) $$\Pi(u, a + 2K) = \Pi(u, a) = \Pi(u, a + 2iK'),$$

(11) $$\begin{aligned}\Pi(u + 2iK', a) &= \Pi(u, a) + 2\,\Pi(K + iK', a) - 2\,\Pi(K, a)\\ &= \Pi(u, a) + 2iK'Z(a) + \frac{\pi i a}{K}.\end{aligned}$$

From the equations (9) and (11) it is seen that the moduli of periodicity of $\Pi(u, a)$ are respectively

$$2KZ(a) \quad \text{and} \quad 2iK'Z(a) + \frac{\pi i a}{K}.$$

ART. 360. From the definition of $\Pi(u, a)$ given in Art. 357 we have

$$\frac{d\,\Pi(u,a)}{du}=\frac{k^2 sn\,a\,cn\,a\,dn\,a\,sn^2u}{1-k^2sn^2a\,sn^2u}$$
$$=Z(a)+\tfrac{1}{2}Z(u-a)-\tfrac{1}{2}Z(u+a)\quad\text{[Art. 297].}$$

We therefore have the theorem: *The derivative of an elliptic integral of the third kind with regard to an elliptic integral of the first kind may be expressed through elliptic integrals of the second kind.*

Interchanging u and a, we also have

$$\frac{k^2 sn\,u\,cn\,u\,dn\,u\,sn^2a}{1-k^2sn^2a\,sn^2u}=Z(u)-\tfrac{1}{2}Z(u-a)-\tfrac{1}{2}Z(u+a).$$

The addition of these two equations gives

$$Z(u)+Z(a)-Z(u+a)=k^2 sn\,u\,sn\,a\,sn(u+a),$$

which is the addition-theorem of the Z-function (see Art. 297).

ART. 361. From the formula

$$\Theta(u+K)=\frac{dn\,u}{\sqrt{k'}}\Theta(u)$$

we have by writing iu in the place of u

$$\Theta(iu+K)=\frac{dn\,iu}{\sqrt{k'}}\Theta(iu),$$

or, (see Arts. 204 and 220)

$$\frac{\Theta(iu+K)}{\Theta(0)}=\frac{1}{\sqrt{k'}}e^{\frac{\pi u^2}{4KK'}}dn(u,k')\,\frac{\Theta(u,k')}{\Theta(0,k')}$$
$$=\sqrt{\frac{k}{k'}}\,e^{\frac{\pi u^2}{4KK'}}\frac{\Theta(u+K',k')}{\Theta(0,k')}.$$

If we take the logarithmic derivative of this equation, we have

$$iZ(iu+K)=\frac{\pi u}{2KK'}+Z(u+K',k').$$

If these expressions are written in the formula

$$\Pi(iu,ia+K)=iuZ(ia+K)+\frac{1}{2}\log\frac{\Theta(ia-iu+K)}{\Theta(ia+iu+K)},$$

we have

$$\Pi(iu,ia+K)=uZ(a+K',k')+\frac{1}{2}\log\frac{\Theta(a-u+K',k')}{\Theta(a+u+K',k')};$$

or

$$\Pi(iu,ia+K)=\Pi(u,a+K',k').$$

If a is changed into ia, it follows that

$$\Pi(iu, a + K) = -\Pi(u, ia + K', k').$$

These results may be derived directly by a consideration of the integral which defines $\Pi(u, a)$ [see Jacobi, Werke I, p. 220].

ART. 362. In Art. 227 we saw that

$$\frac{1}{2}\log\Theta\left(\frac{2Ku}{\pi}\right) = \text{const.} - \frac{q\cos 2u}{1-q^2} - \frac{q^2\cos 4u}{2(1-q^4)} - \frac{q^3\cos 6u}{3(1-q^6)} - \frac{q^4\cos 8u}{4(1-q^8)} - \cdots.$$

It follows directly from the formula

$$\Pi(u, a) = u\frac{\Theta'(a)}{\Theta(a)} + \frac{1}{2}\log\frac{\Theta(u-a)}{\Theta(u+a)}$$

that

$$\begin{aligned}\Pi\left(\frac{2Ku}{\pi}, \frac{2Ka}{\pi}\right) = \frac{2Ku}{\pi}\,\frac{\Theta'\left(\frac{2Ka}{\pi}\right)}{\Theta\left(\frac{2Ka}{\pi}\right)} &+ \frac{q\cos 2(u+a)}{1-q^2} + \frac{q^2\cos 4(u+a)}{2(1-q^4)} + \cdots \\ &- \frac{q\cos 2(u-a)}{1-q^2} - \frac{q^2\cos 4(u-a)}{2(1-q^4)} - \cdots \\ = \frac{2Ku}{\pi}\,\frac{\Theta'\left(\frac{2Ka}{\pi}\right)}{\Theta\left(\frac{2Ka}{\pi}\right)} - 2\Bigg[\frac{q\sin 2a\sin 2u}{1-q^2} &+ \frac{q^2\sin 4a\sin 4u}{2(1-q^4)} \\ &+ \frac{q^3\sin 6a\sin 6u}{3(1-q^6)} + \cdots\Bigg].\end{aligned}$$

THE OMEGA-FUNCTION.

ART. 363. Jacobi (Werke, I, p. 300) put

$$\int_0^u E(u)du = \log\Omega(u).$$

If we integrate the formula of Art. 297

$$E(u+a) + E(u-a) = 2E(u) - \frac{2k^2 sn^2 a\, sn u\, cn u\, dn u}{1 - k^2 sn^2 a\, sn^2 u}$$

from $u = 0$ to $u = u$, we have at once

$$\log\frac{\Omega(u+a)}{\Omega(a)} + \log\frac{\Omega(u-a)}{\Omega(a)} = 2\log\Omega(u) + \log(1 - k^2 sn^2 a\, sn^2 u),$$

or

$$\frac{\Omega(u+a)\,\Omega(u-a)}{\Omega^2(u)\,\Omega^2(a)} = 1 - k^2 sn^2 a\, sn^2 u.$$

Further, if u and a are interchanged in the above formula, it becomes

$$E(u+a)-E(u-a)=2E(a)-\frac{2k^2 sn\,a\, cn\,a\, dn\,a\, sn^2u}{1-k^2sn^2a\, sn^2u},$$

which integrated from $u=0$ to $u=u$ is

$$\log\frac{\Omega(u+a)}{\Omega(u-a)}-2uE(a)=-2\,\Pi(u,a),$$

or

$$\Pi(u,a)=uE(a)+\frac{1}{2}\log\frac{\Omega(u-a)}{\Omega(u+a)}.$$

In Art. 251 the following formula was derived:

$$E(iu)=i\,[tn(u,k')\,dn(u,k')+u-E(u,k')].$$

We have at once

$$\log\Omega(iu)=\log cn(u,k')-\frac{u^2}{2}+\log\Omega(u,k').$$

or

$$\Omega(iu)=e^{-\frac{u^2}{2}}cn(u,k')\,\Omega(u,k').$$

ART. 364. From the formula $E(u+2mK)=E(u)+2mE$ we have

$$\log\frac{\Omega(u+2mK)}{\Omega(2mK)}=u\cdot 2mE+\log\Omega(u),$$

or

$$\frac{\Omega(u+2mK)}{\Omega(2mK)}=e^{u\cdot 2mE}\Omega(u).$$

If we put $u=-2mK$ in this formula, and note that

$$\Omega(-u)=\Omega(u),\quad \Omega(0)=1,$$

we have

$$\Omega(2mK)=e^{2m^2EK},$$

and also

$$\Omega(u+2mK)=e^{2mE(u+mK)}\Omega(u),$$

or

$$e^{-\frac{E}{2K}(u+2mK)^2}\Omega(u+2mK)=e^{-\frac{Eu^2}{2K}}\Omega(u).$$

This formula shows that the function $e^{-\frac{Eu^2}{2K}}\Omega(u)$ remains unchanged when the argument is increased by the real period $2K$.

Further, if in the formula

$$\Omega(iu)=e^{-\frac{u^2}{2}}cn(u,k')\Omega(u,k'),$$

we write $u+2nK'$ in the place of u, we have

$$\Omega(iu+2niK')=(-1)^n e^{-\frac{(u+2nK')^2}{2}}cn(u,k')\,\Omega(u+2nK',k'),$$

or

$$e^{-\frac{E}{2K'}(u+2nK')^2}\Omega(iu+2niK')=(-1)^n e^{-\frac{(u+2nK')^2}{2}}cn(u,k')e^{-\frac{u^2E'}{2K'}}\Omega(u,k').$$

It follows that

$$e^{\frac{K'-E'}{2K'}(u+2nK')^2}\Omega(iu+2niK')=(-1)^n e^{-\frac{u^2E'}{2K'}}cn(u,k')\,\Omega(u,k')$$
$$=(-1)^n e^{\frac{(K'-E')}{2K'}u^2}\Omega(iu).$$

If in this expression we put $-iu$ for u or u for iu, we have

$$e^{-\frac{K'-E'}{2K'}(u+2niK')^2}\Omega(u+2niK')=(-1)^n e^{-\frac{K'-E'}{2K'}u^2}\Omega(u),$$

from which formula it is seen that the expression

$$e^{-\frac{K'-E'}{2K'}u^2}\Omega(u)$$

remains unchanged when u is changed * into $u+4niK'$.

ART. 365. We derived in Art. 263 the formula

$$E(\sqrt{e_1-e_3}\cdot u)=\frac{1}{\sqrt{e_1-e_3}}\left(\frac{\sigma_3'u}{\sigma_3u}+e_1u\right),$$

from which we have at once through logarithmic integration

$$\Omega(\sqrt{e_1-e_3}\cdot u)=e^{\frac{1}{2}e_1u^2}\sigma_3u.$$

Writing these values in the formula

$$\Pi(u,a)=uE(a)+\frac{1}{2}\log\frac{\Omega(u-a)}{\Omega(u+a)},$$

it is seen that

$$\Pi(\sqrt{e_1-e_3}\cdot u,\ \sqrt{e_1-e_3}\cdot a)=\frac{1}{2}\log\frac{\Omega[\sqrt{e_1-e_3}(u-a)]}{\Omega[\sqrt{e_1-e_3}(u+a)]}$$
$$+\sqrt{e_1-e_3}\,u\,E(\sqrt{e_1-e_3}\cdot a)$$
$$=\frac{1}{2}\log\frac{\sigma_3(u-a)}{\sigma_3(u+a)}+u\frac{\sigma_3'a}{\sigma_3a}.$$

[See Schwarz, *loc. cit.*, p. 52.]

ART. 366. The following relations may be derived from the addition-theorems of the theta-functions given in Art. 211, formulas [C]:

$$\frac{\Theta^2(0)\mathrm{H}(u+a)\mathrm{H}(u-a)}{k\,\Theta^2(u)\,\Theta^2(a)}=sn^2u-sn^2a,$$

$$\frac{k'\Theta^2(0)\,\Theta_1(u+a)\,\Theta_1(u-a)}{\Theta^2(u)\,\Theta^2(a)}=k^2cn^2u\,cn^2a+k'^2,$$

$$\frac{kk'\Theta^2(0)\mathrm{H}_1(u+a)\mathrm{H}_1(u-a)}{\Theta^2(u)\,\Theta^2(a)}=dn^2u\,dn^2a-k'^2.$$

* See Jacobi, Werke, I, p. 309.

If as in Art. 358 these expressions are differentiated logarithmically with regard to a and integrated with regard to u, the variable in the first equation being less than the parameter a, we have

$$\int_0^u \frac{sn\, a\, cn\, a\, dn\, a}{sn^2 u - sn^2 a}\, du = \frac{1}{2}\log\frac{\mathrm{H}(a-u)}{\mathrm{H}(a+u)} + u\frac{\Theta'(a)}{\Theta(a)},$$

$$\int_0^u \frac{k^2 sn\, a\, cn\, a\, dn\, a\, cn^2 u}{k^2 cn^2 u\, cn^2 a + k'^2}\, du = \frac{1}{2}\log\frac{\Theta_1(u-a)}{\Theta_1(u+a)} + u\frac{\Theta'(a)}{\Theta(a)},$$

$$\int_0^u \frac{k^2 sn\, a\, cn\, a\, dn\, a\, dn^2 u}{dn^2 u\, dn^2 a - k'^2}\, du = \frac{1}{2}\log\frac{\mathrm{H}_1(u-a)}{\mathrm{H}_1(u+a)} + u\frac{\Theta'(a)}{\Theta(a)}.$$

These integrals * may all be expressed through the integral $\Pi(u, a)$ and an elliptic integral of the first kind; for example

$$\int_0^u \frac{sn\, a\, cn\, a\, dn\, a}{sn^2 u - sn^2 a}\, du = \Pi(u, a + iK') - \frac{u\, cn\, a\, dn\, a}{sn\, a}.$$

Addition-Theorems for the Integrals of the Third Kind.

Art. 367. The addition-theorem for the elliptic integral of the third kind follows directly from the equation of Art. 358 in the form

$$\Pi(u, a) + \Pi(v, a) - \Pi(u + v, a) = \frac{1}{2}\log\frac{\Theta(u-a)\,\Theta(v-a)\,\Theta(u+v+a)}{\Theta(u+a)\,\Theta(v+a)\,\Theta(u+v-a)}.$$

For brevity we shall put

$$\frac{\Theta(u-a)\,\Theta(v-a)\,\Theta(u+v+a)}{\Theta(u+a)\,\Theta(v+a)\,\Theta(u+v-a)} = F(u, v, a),$$

and we shall derive several different forms for $F(u, v, a)$ which are due to Legendre and Jacobi.†

From the formula

$$\Theta^2(0)\,\Theta(\mu+\nu)\,\Theta(\mu-\nu) = \Theta^2(\mu)\,\Theta^2(\nu)\left\{1 - k^2 sn^2\mu\, sn^2\nu\right\}$$

we have at once

$$\Theta^2(0)\Theta(u-a)\Theta(v-a) = \Theta^2\left(\frac{u-v}{2}\right)\Theta^2\left(\frac{u+v}{2}-a\right)\left\{1 - k^2 sn^2\frac{u-v}{2}\, sn^2\left(\frac{u+v}{2}-a\right)\right\},$$

$$\Theta^2(0)\Theta(u+a)\Theta(v+a) = \Theta^2\left(\frac{u-v}{2}\right)\Theta^2\left(\frac{u+v}{2}+a\right)\left\{1 - k^2 sn^2\frac{u-v}{2}\, sn^2\left(\frac{u+v}{2}+a\right)\right\},$$

$$\Theta^2(0)\Theta(a)\Theta(u+v-a) = \Theta^2\left(\frac{u+v}{2}\right)\Theta^2\left(\frac{u+v}{2}-a\right)\left\{1 - k^2 sn^2\frac{u+v}{2}\, sn^2\left(\frac{u+v}{2}-a\right)\right\},$$

$$\Theta^2(0)\Theta(a)\Theta(u+v+a) = \Theta^2\left(\frac{u+v}{2}\right)\Theta^2\left(\frac{u+v}{2}+a\right)\left\{1 - k^2 sn^2\frac{u+v}{2}\, sn^2\left(\frac{u+v}{2}+a\right)\right\};$$

* See *note by Hermite in* Serret's *Calcul*, t. II, p. 840.

† Legendre, *Fonct. Ellipt.*, t. I, Chap. XV; Jacobi, Werke, I, pp. 207 *et seq.*

and by taking the product of the first and fourth of these equations divided by that of the second and third we have

$$F(u, v, a) = \frac{\left\{1 - k^2 sn^2\left(\frac{u-v}{2}\right) sn^2\left(\frac{u+v}{2} - a\right)\right\}\left\{1 - k^2 sn^2 \frac{u+v}{2} sn^2\left(\frac{u+v}{2} + a\right)\right\}}{\left\{1 - k^2 sn^2\left(\frac{u-v}{2}\right) sn^2\left(\frac{u+v}{2} + a\right)\right\}\left\{1 - k^2 sn^2 \frac{u+v}{2} sn^2\left(\frac{u+v}{2} - a\right)\right\}}.$$

From the formula

$$sn(\mu + \nu)\, sn(\mu - \nu) = \frac{sn^2\mu - sn^2\nu}{1 - k^2 sn^2\mu\, sn^2\nu}$$

we further have

$$\left\{1 - k^2 sn^2 \frac{u+v}{2} sn^2 \frac{u-v}{2}\right\} sn\, u\, sn\, v = sn^2 \frac{u+v}{2} - sn^2 \frac{u-v}{2},$$

$$\left\{1 - k^2 sn^2 \frac{u+v}{2} sn^2\left(\frac{u+v}{2} - a\right)\right\} sn\, a\, sn(u+v-a) = sn^2 \frac{u+v}{2} - sn^2\left(\frac{u+v}{2} - a\right).$$

Taking the products of these two equations each multiplied by $-k^2$ and adding a common term on either side, we have *

$$\left\{1 - k^2 sn^2 \frac{u+v}{2} sn^2 \frac{u-v}{2}\right\}\left\{1 - k^2 sn^2 \frac{u+v}{2} sn^2\left(\frac{u+v}{2} - a\right)\right\}$$

multiplied by $\{1 - k^2 sn\, a\, sn\, u\, sn\, v\, sn(u+v-a)\}$

$$= \left\{1 - k^2 sn^2 \frac{u+v}{2} sn^2 \frac{u-v}{2}\right\}\left\{1 - k^2 sn^2 \frac{u+v}{2} sn^2\left(\frac{u+v}{2} - a\right)\right\}$$

$$- k^2\left\{sn^2 \frac{u+v}{2} - sn^2 \frac{u-v}{2}\right\}\left\{sn^2 \frac{u+v}{2} - sn^2\left(\frac{u+v}{2} - a\right)\right\}$$

$$= 1 + k^4 sn^4 \frac{u+v}{2} sn^2 \frac{u-v}{2} sn^2\left(\frac{u+v}{2} - a\right) - k^2 sn^4 \frac{u+v}{2} - k^2 sn^2 \frac{u-v}{2} sn^2\left(\frac{u+v}{2} - a\right)$$

$$= \left(1 - k^2 sn^4 \frac{u+v}{2}\right)\left\{1 - k^2 sn^2 \frac{u-v}{2} sn^2\left(\frac{u+v}{2} - a\right)\right\}.$$

Writing $-a$ for a in this equation, we have a second equation, which divided by the first gives

$$\frac{\left\{1 - k^2 sn^2 \frac{u+v}{2} sn^2\left(\frac{u+v}{2} - a\right)\right\}\left\{1 - k^2 sn^2 \frac{u-v}{2} sn^2\left(\frac{u+v}{2} + a\right)\right\}}{\left\{1 - k^2 sn^2 \frac{u+v}{2} sn^2\left(\frac{u+v}{2} + a\right)\right\}\left\{1 - k^2 sn^2 \frac{u-v}{2} sn^2\left(\frac{u+v}{2} - a\right)\right\}}$$

$$= \frac{1 + k^2 sn\, a\, sn\, u\, sn\, v\, sn(u + v + a)}{1 - k^2 sn\, a\, sn\, u\, sn\, v\, sn(u + v - a)}.$$

* See Cayley, *Elliptic Functions*, p. 159.

If a is changed to $-a$ in this expression, it is seen that

$$F(u, v, a) = \frac{1 - k^2 sn\, a\, sn\, u\, sn\, v\, sn(u + v - a)}{1 + k^2 sn\, a\, sn\, u\, sn\, v\, sn(u + v + a)}.$$

ART. 368. It follows also from the expressions given in the preceding Article that

$$\Theta^2(u - a)\Theta^2(v - a) = \Theta^2(0)\frac{\Theta(u - v)\Theta(u + v - 2a)}{1 - k^2 sn^2(u - a)\, sn^2(v - a)},$$

$$\Theta^2(u + a)\Theta^2(v + a) = \Theta^2(0)\frac{\Theta(u - v)\Theta(u + v + 2a)}{1 - k^2 sn^2(u + a)\, sn^2(v + a)},$$

$$\Theta^2(a)\Theta^2(u + v - a) = \Theta^2(0)\frac{\Theta(u + v)\Theta(u + v - 2a)}{1 - k^2 sn^2 a\, sn^2(u + v - a)},$$

$$\Theta^2(a)\Theta^2(u + v + a) = \Theta^2(0)\frac{\Theta(u + v)\Theta(u + v + 2a)}{1 - k^2 sn^2 a\, sn^2(u + v + a)}.$$

From these equations we have

$$F(u, v, a) = \left[\frac{\{1 - k^2 sn^2(u + a)\, sn^2(v + a)\}\{1 - k^2 sn^2 a\, sn^2(u + v - a)\}}{\{1 - k^2 sn^2(u - a)\, sn^2(v - a)\}\{1 - k^2 sn^2 a\, sn^2(u + v + a)\}}\right]^{\frac{1}{2}}.$$

ART. 369. Since $\Pi(u, a) - \Pi(a, u) = uZ(a) - aZ(u)$, we have

$$\begin{aligned}\Pi(u, a) + \Pi(u, b) - \Pi(u, a + b)& \\ &= \Pi(a, u) + \Pi(b, u) - \Pi(a + b, u) + u\{Z(a) + Z(b) - Z(a + b)\}\\ &= \tfrac{1}{2}\log F(a, b, u) + u k^2 sn\, a\, sn\, b\, sn(a + b),\end{aligned}$$

which is a theorem for the addition of the parameters.

ART. 370. In the formula (see Table (B) of Art. 211)

$$\begin{aligned}\vartheta_0(0)\vartheta(y + z)\vartheta(x + z)\vartheta(x + y) &= \vartheta(x + y + z)\vartheta(x)\vartheta(y)\vartheta(z)\\ &+ \vartheta_1(x + y + z)\vartheta_1(x)\vartheta_1(y)\vartheta_1(z)\end{aligned}$$

write $x = \frac{2Ku}{\pi}$, $y = \frac{2Kv}{\pi}$ and $z = -\frac{2Ka}{\pi}$ and $+\frac{2Ka}{\pi}$ respectively.

Divide the first result by the second and we have

$$\frac{\Theta(u - a)\Theta(v - a)\Theta(u + v + a)}{\Theta(u + a)\Theta(v + a)\Theta(u + v - a)} = \frac{1 - \dfrac{H(a)H(u)H(v)H(u + v - a)}{\Theta(a)\Theta(u)\Theta(v)\Theta(u + v - a)}}{1 + \dfrac{H(a)H(u)H(v)H(u + v + a)}{\Theta(a)\Theta(u)\Theta(v)\Theta(u + v + a)}},$$

or

$$F(u, v, a) = \frac{1 - k^2 sn a\, sn u\, sn v\, sn(u + v - a)}{1 + k^2 sn a\, sn u\, sn v\, sn(u + v + a)}.$$

Remark. — By writing as we have done

$$n = -k^2 \sin^2\theta,$$

and allowing θ to take imaginary values, the expression on the right-hand side of the addition-theorems above is always a logarithm. Legendre *

* Legendre, *Traité des fonctions elliptiques*, t. III, p. 138.

considered the following two cases, to the one or the other of which by means of real transformations the parameter n may always be reduced:

$$(1)\ n = -k^2\sin^2\theta, \quad (2)\ n = 1 + k'^2\sin^2\theta,$$

where θ is real in both cases.

Owing to the fact that

$$\tan^{-1} it = \frac{1}{2} i \log\frac{1+t}{1-t},$$

the inverse tangent appears in the second case instead of the logarithm.*

ART. 371. If we put

$$\begin{array}{llll} t_0 = \wp u_0, & t_1 = \wp u_1, & t_2 = \wp u_2, & t_3 = \wp u_3, \\ -\sqrt{S(t_0)} = \wp' u_0, & -\sqrt{S(t_1)} = \wp' u_1, & -\sqrt{S(t_2)} = \wp' u_2, & -\sqrt{S(t_3)} = \wp' u_3, \end{array}$$

we have from Art. 355

$$\Pi(t_1;\ t_0;\ \infty) = \log\frac{\sigma(u_0 - u_1)}{\sigma u_1 \sigma u_0} + u_1\zeta u_0,$$

$$\Pi(t_2;\ t_0;\ \infty) = \log\frac{\sigma(u_0 - u_2)}{\sigma u_2 \sigma u_0} + u_2\zeta u_0,$$

$$\Pi(t_3;\ t_0;\ \infty) = \log\frac{\sigma(u_0 - u_3)}{\sigma u_3 \sigma u_0} + u_3\zeta u_0.$$

If $u_3 = u_1 + u_2$, it follows that

$$\Pi(t_1;\ t_0;\ \infty) + \Pi(t_2;\ t_0;\ \infty) = \Pi(t_3;\ t_0;\ \infty) - \log f(u_3, u_2, u_1, u_0),$$

where

$$\begin{aligned} f(u_3, u_2, u_1, u_0) &= \frac{\sigma(u_0 - u_3)\sigma u_2 \sigma u_1 \sigma u_0}{\sigma u_3 \sigma(u_0 - u_2)\sigma(u_0 - u_1)} \\ &= \frac{1}{2}\frac{1}{\wp u_1 - \wp u_2}\left\{\frac{\wp' u_2 + \wp' u_0}{\wp u_2 - \wp u_0} - \frac{\wp' u_1 + \wp' u_0}{\wp u_1 - \wp u_0}\right\}. \end{aligned}$$

The last formula is verified by using the equation (see Art. 335, [B.])

$$\sigma w \sigma(u + v + w)\sigma_\lambda(u - v) = \sigma(u + w)\sigma(v + w)\sigma_\lambda u\, \sigma_\lambda v - \sigma_\lambda(u + w)\sigma_\lambda(v + w)\sigma u \sigma v,$$

and the formulas given in the Table of Formulas, No. LXII, combined with the formula

$$\wp u - e_\lambda = \frac{\sigma_\lambda^2 u}{\sigma^2 u}.$$

It follows that

$$\begin{aligned} \Pi(t_1;\ t_0;\ \infty) + \Pi(t_2;\ t_0;\ \infty) &= \Pi(t_3;\ t_0;\ \infty) \\ &\quad - \log\left[\frac{1}{2}\frac{1}{t_1 - t_2}\left\{\frac{\sqrt{S(t_1)} + \sqrt{S(t_0)}}{t_1 - t_0} - \frac{\sqrt{S(t_2)} + \sqrt{S(t_0)}}{t_2 - t_0}\right\}\right]. \end{aligned}$$

[See Schwarz, *loc. cit.*, p. 90.]

* As an application of Abel's Theorem, Professor Forsyth (*Phil. Trans.*, 1883, p. 344) has given a very elegant method for the addition of the elliptic integrals of the third kind. See also a paper by Rowe (*Phil. Trans.*, 1881, p. 713).

EXAMPLES

1. Show that

$$\Pi(u+\tfrac{1}{2}K, \tfrac{1}{2}K) = \tfrac{1}{2}(1-k')(u+\tfrac{1}{2}K) - \tfrac{1}{2}\log dn\, u + \tfrac{1}{2}\log\sqrt{k'},$$

$$\Pi(u+\tfrac{1}{2}iK', \tfrac{1}{2}iK') = \tfrac{1}{2}i(1+k)(u+\tfrac{1}{2}iK') - \tfrac{1}{2}\log sn\, u + \tfrac{1}{2}\log\left(\frac{-i}{\sqrt{k}}\right),$$

$$\Pi(u+\tfrac{1}{2}K+\tfrac{1}{2}iK', \tfrac{1}{2}K+\tfrac{1}{2}iK')$$
$$= \tfrac{1}{2}(k+ik')(u+\tfrac{1}{2}K+\tfrac{1}{2}iK') - \tfrac{1}{2}\log cn\, u + \tfrac{1}{2}\log\sqrt{\frac{-ik'}{k}}.$$

2. Show that

$$\Pi(u+K, a) = \Pi(u, a) + KZ(a) + \tfrac{1}{2}\log\frac{dn(u-a)}{dn(u+a)},$$

$$\Pi(u, a+K) = \Pi(u, a) - k^2 sn\, a \sin \text{coam}\, a \cdot u + \tfrac{1}{2}\log\frac{dn(u-a)}{dn(u+a)}.$$

3. Verify the formulas

$$\int_0^u \frac{dn\, a \cot am\, a\, du}{1-k^2 sn^2 a\, sn^2 u} = u\frac{d\log \mathrm{H}(a)}{da} - \frac{1}{2}\log\frac{\Theta(u+a)}{\Theta(u-a)},$$

$$\int_u^K \frac{sn\, a\, cn\, a\, dn\, a\, du}{sn^2 u - sn^2 a} = (K-u)\frac{d\log\Theta(a)}{da} + \frac{1}{2}\log\frac{\mathrm{H}(u+a)}{\mathrm{H}(u-a)},$$

$$\int_0^u \frac{k^2 sn(ia) cn(ia) dn(ia) sn^2 u\, du}{i[1-k^2 sn^2(ia) sn^2 u]} = -u\frac{d\log\Theta(ia)}{da} - \frac{1}{2i}\log\frac{\Theta(u+ia)}{\Theta(u-ia)}.$$

4. Show that

$$\Pi(u, a) + \Pi(v, a) - \Pi(u+v, a)$$
$$= \frac{1}{2}\log\frac{\Omega(u-a)\,\Omega(v-a)\,\Omega(u+v+a)}{\Omega(u+a)\,\Omega(v+a)\,\Omega(u+v-a)};$$

and that

$$\frac{\Omega(u-a)\,\Omega(v-a)\,\Omega(u+v)}{\Omega(u)\,\Omega(v)\,\Omega(a)\,\Omega(u+v-a)} = 1 - k^2 sn\, a\, sn\, u\, sn\, v\, sn(u+v-a),$$

$$\frac{\Omega(u+a)\,\Omega(v+a)\,\Omega(u+v)}{\Omega(u)\,\Omega(v)\,\Omega(a)\,\Omega(u+v+a)} = 1 + k^2 sn\, a\, sn\, u\, sn\, v\, sn(u+v+a).$$

CHAPTER XX

METHODS OF REPRESENTING ANALYTICALLY DOUBLY PERIODIC FUNCTIONS OF ANY ORDER WHICH HAVE EVERYWHERE IN THE FINITE PORTION OF THE PLANE THE CHARACTER OF INTEGRAL OR (FRACTIONAL) RATIONAL FUNCTIONS

ARTICLE 372. We have seen that the simplest doubly periodic functions, which in the finite portion of the plane have everywhere the character of integral or (fractional) rational functions, are the functions $\wp u$, $\operatorname{sn} u$, etc. We shall show in the present Chapter that all other doubly periodic functions which have the properties just mentioned may be expressed in terms of these simpler functions.

We shall study in particular five kinds of representations:

(1) *Representation as a sum of terms each of which is a complete derivative.*

(2) *Representation as a rational function of, say, $\wp u$ and $\wp' u$* [*Liouville's Theorem*].

(3) *Representation in the form of a quotient of two products of theta-functions or sigma-functions.*

(4) *Representation in the form of a sum of rational functions.*

(5) *Representation in the form of a sum of rational functions of an exponential function.*

ART. 373. The first representation mentioned above and due to Hermite has been made fundamental throughout this treatise; upon it, as already stated, the other representations all depend. We shall produce it again in a somewhat different form so that the dependence upon it of the other representations may be more readily seen. In Art. 87 Hermite's intermediary function of the first order was denoted by $\mathrm{X}(u)$ and was defined through the equation

$$\mathrm{X}(u) = \sum_{m=-\infty}^{m=+\infty} Q^{m^2} e^{\frac{2\pi i m u}{a}}, \text{ where } Q = e^{\pi i \frac{b}{a}}.$$

We saw that this function satisfied the functional equations

$$(1)\quad \mathrm{X}(u+a) = \mathrm{X}(u),$$

$$(2)\quad \mathrm{X}(u+b) = e^{-\frac{\pi i}{a}(2u+b)}\mathrm{X}(u).$$

We also saw that this function vanished on the point $\frac{a+b}{2} = c$ and on all congruent points, but nowhere else.

By writing $X(u+c) = X_1(u)$ we formed in $X_1(u)$ a function that vanished for $u = 0$ and congruent points. It is seen that

$$X_1(u+a) = X_1(u),$$

$$X_1(u+b) = e^{-\frac{\pi i}{a}(2u+a+2b)} X_1(u).$$

We next wrote (Art. 96)

$$Z_0(u) = \frac{X_1'(u)}{X_1(u)} = \frac{X'(u+c)}{X(u+c)},$$

and we saw (Art. 98) that every one-valued doubly periodic function $F(u)$ with periods a and b and which had everywhere in the finite portion of the plane the character of an integral or (fractional) rational function could be expressed in the form

$$F(u) = C + \sum_{k=1}^{k=n} \left[b_{k,1} Z_0(u-u_k) - \frac{b_{k,2}}{1!} Z_0'(u-u_k) \right.$$
$$\left. + \frac{b_{k,3}}{2!} Z_0''(u-u_k) - \cdots \pm \frac{b_{k,\lambda_k}}{(\lambda_k-1)!} Z_0^{(\lambda_k-1)}(u-u_k) \right],$$

where k extends over the n infinities u_k of $F(u)$ that are situated within a period-parallelogram, the order of these infinities being λ_k respectively; C is an arbitrary constant, while $b_{k,\nu}$ is the coefficient of $\dfrac{1}{(u-u_k)^\nu}$ in the expansion of $F(u)$ in the neighborhood of the infinities $u = u_k$ $(k = 1, 2, \ldots n)$.

If r is the order of the function $F(u)$ (see Art. 92), then $r = \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n$.

The function $Z_0(u)$ is infinite of the first order for $u = 0$. We may next write

$$\psi(u) = e^{\lambda u^2 + \mu u} X_1(u),$$

where λ, μ are constants.

It follows that

$$\frac{\psi_1'(u)}{\psi(u)} = \frac{X_1'(u)}{X_1(u)} + 2\lambda u + \mu = Z_1(u), \quad \text{say.}$$

We therefore have

$$Z_0(u) + 2\lambda u + \mu = Z_1(u),$$
$$Z_0'(u) + 2\lambda = Z_1'(u),$$
$$Z_0''(u) = Z_1''(u), \text{ etc.}$$

The formula above becomes

$$F(u) = C + \sum_{k=1}^{k=n} \left[b_{k,1} \left\{ Z_1(u-u_k) - 2\lambda(u-u_k) - \mu \right\} \right.$$
$$- \frac{b_{k,2}}{1!} \left\{ Z_1'(u-u_k) - 2\lambda \right\} + \frac{b_{k,3}}{2!} Z_1''(u-u_k)$$
$$\left. \cdots \pm \frac{b_{k,\lambda_k}}{(\lambda_k-1)!} Z_1^{(\lambda_k-1)}(u-u_k) \right].$$

The constants $\quad \sum_{k=1}^{k=n} b_{k,1}(2\,\lambda u_k - \mu)$ and $\sum_{k=1}^{k=n} b_{k,2} 2\,\lambda$

may be embodied in the constant C, making, say, C_1. We also note that

$$\sum_{k=1}^{k=n} b_{k,1} = \sum \text{Res}\, F(u) = 0.$$

It follows * that

$$F(u) = C_1 + \sum_{k=1}^{k=n}\Big[b_{k,1}\,\mathrm{Z}_1(u-u_k) - \frac{b_{k,2}}{1!}\mathrm{Z}_1'(u-u_k)$$
$$+ \frac{b_{k,3}}{2!}\mathrm{Z}_1''(u-u_k) - \cdots \pm \frac{b_{k,\lambda_k}}{(\lambda_k-1)!}\mathrm{Z}_1^{(\lambda_k-1)}(u-u_k)\Big].$$

ART. 374. To introduce the Jacobi Theory write

$$a = 2\,K \quad \text{and} \quad b = 2\,iK'.$$

It follows at once that

$$\mathrm{X}_1(u + 2\,K) = \mathrm{X}_1(u)$$

and $\quad \mathrm{X}_1(u + 2\,iK') = -e^{-\frac{\pi i}{K}(u+2\,iK')}\mathrm{X}_1(u).$

If we make $\lambda = 0$, $\mu = \dfrac{\pi i}{2\,K}$, we have from above

$$\psi(u) = e^{\frac{\pi i}{2\,K}u}\,\mathrm{X}_1(u)$$

and also

$$\psi(u + 2\,K) = -\,\psi(u),$$

$$\psi(u + 2\,iK') = -\,e^{-\frac{\pi i}{K}(u+iK')}\psi(u).$$

On the other hand we had

$$\mathrm{H}(u + 2\,K) = -\,\mathrm{H}(u),$$

$$\mathrm{H}(u + 2\,iK') = -\,e^{-\frac{\pi i}{K}(u+iK')}\mathrm{H}(u).$$

We may therefore write in the formulas above $A\mathrm{H}(u)$ instead of $\psi(u)$, where A is an arbitrary constant, and $\mathrm{Z}_1(u) = \dfrac{\mathrm{H}'(u)}{\mathrm{H}(u)}$.

It follows that we may express every doubly periodic function $F(u)$ with the characteristics required above through the function $\dfrac{\mathrm{H}'(u)}{\mathrm{H}(u)}$.

* See Hermite, *Ann. de Toulouse*, t. II (1888), pp. 1–12.

ART. 375. To introduce the theory of Weierstrass write

$$a = 2\omega \quad \text{and} \quad b = 2\omega',$$

so that
$$X_1(u + 2\omega) = X_1(u)$$
and
$$X_1(u + 2\omega') = -e^{-\frac{\pi i}{\omega}(u+2\omega')} X_1(u).$$

We shall so choose the constants λ, μ that instead of the function $\psi(u)$ we may employ the function σu. We have the relations

$$\sigma(u + 2\omega) = -e^{2\eta(u+\omega)}\sigma u,$$
$$\sigma(u + 2\omega') = -e^{2\eta'(u+\omega')}\sigma u.$$

We further have
$$\psi(u) = e^{\lambda u^2 + \mu u} X_1(u)$$
$$\psi(u + 2\omega) = e^{\lambda(u+2\omega)^2 + \mu(u+2\omega)} X_1(u + 2\omega).$$

It follows that
$$\psi(u + 2\omega) = e^{4\lambda\omega u + 4\lambda\omega^2 + 2\mu\omega}\psi(u).$$

Comparing this result with

$$\sigma(u + 2\omega) = -e^{2\eta(u+\omega)}\sigma u,$$

it is seen that we must write

$$4\lambda\omega = 2\eta \quad \text{and} \quad 4\lambda\omega^2 + 2\mu\omega = 2\eta\omega + \pi i,$$

where πi has been added to change the sign.

We have at once

$$\lambda = \frac{\eta}{2\omega} \quad \text{and} \quad \mu = \frac{\pi i}{2\omega},$$

and consequently also

$$\psi(u) = e^{\frac{\eta}{2\omega}u^2 + \frac{\pi i}{2\omega}u} X_1(u).$$

This function satisfies the first of the functional equations which σu satisfies.

We have further

$$\psi(u + 2\omega') = -e^{2\eta\frac{\omega'}{\omega}u + 2\eta\frac{\omega'^2}{\omega} + \pi i\frac{\omega'}{\omega}} e^{-\frac{\pi i}{\omega}u - 2\pi i\frac{\omega'}{\omega}}\psi(u);$$

or, since
$$\eta\omega' - \eta'\omega = \frac{\pi i}{2},$$

we have
$$\psi(u + 2\omega') = -e^{2\eta'(u+\omega')}\psi(u).$$

It is thus proved that $\psi(u)$ satisfies also the second functional equation satisfied by σu. We may therefore put

$$\psi(u) = B\sigma u,$$

where B is an arbitrary constant, and

$$Z_1(u) = \frac{\sigma' u}{\sigma u} = \zeta u.$$

ART. 376. It is evident from above that we may write $F(u)$ in the form *

$$F(u) = C_1 + \sum_k B_k^{(1)}\zeta(u - u_k) + \sum_k \sum_\nu \frac{(-1)^\nu}{\nu!} B_k^{(\nu+1)} \frac{d^\nu}{du^\nu}\zeta(u - u_k)$$

$$\begin{pmatrix} k = 1, 2, \dots, n; & \\ \nu = 1, 2, \dots, \lambda_k - 1; & B_k^{(\nu+1)} = b_{k,\nu+1} \end{pmatrix}.$$

We here have $F(u)$ *expressed as a sum of terms each of which is a complete derivative.*

This formula is therefore especially useful in all applications of the elliptic functions that involve integration. The constant C_1 may be determined if we know the value of $F(u)$ for any value of the argument different from the quantities u_k.

ART. 377. We saw in Art. 299 that

$$\zeta(u - u_k) = \zeta(u) - \zeta(u_k) + \frac{1}{2}\frac{\wp' u + \wp' u_k}{\wp u - \wp u_k},$$

where we assumed that u_k is *not* congruent to a period; otherwise ζu_k and $\wp u_k$ would be infinite. We therefore first exclude in this discussion all the quantities u_k which are congruent to periods and attach a star to the summation sign to call attention to this fact. We have accordingly, if we note the formulas of the preceding Article,

$$\sum{}^* B_k^{(1)}\zeta(u - u_k) = \sum{}^* B_k^{(1)}\zeta u - \sum{}^* B_k^{(1)}\zeta u_k + \frac{1}{2}\sum{}^* B_k^{(1)}\frac{\wp' u + \wp' u_k}{\wp u - \wp u_k}.$$

We note that the second summation on the right is a constant. Two cases are possible:

(1) None of the quantities u_k is congruent to a period; or

(2) Some of the quantities u_k are congruent to periods.

In the first case we may remove the star from the summations. We then have $\zeta u \sum B_k^{(1)} = 0$. It then follows at once that $\sum B_k^{(1)}\zeta(u - u_k)$ is rationally expressed in terms of $\wp u$ and $\wp' u$. In the second case only one of the quantities u_k can be congruent to a period and therefore also to zero, since the quantities u_k form by hypothesis a complete system of *incongruent* infinities. This infinity may be transformed to the origin.

We must consequently add $B_k^{(1)}\zeta u$ to $\sum{}^* B_k^{(1)}\zeta(u - u_k)$ that we may have $\sum_k B_k^{(1)}\zeta(u - u_k)$. But here also it is seen that

$$\sum{}^* B_k^{(1)}\zeta u + B_k^{(1)}\zeta u = 0, \quad \text{since} \quad \sum B_k^{(1)} = 0.$$

Thus without exception it is seen that $\sum_k B_k^{(1)}\zeta(u - u_k)$ is rationally expressible through $\wp u$ and $\wp' u$.

* See Kiepert, *Crelle's Journ.*, Bd. 76, pp. 21 *et seq.*

Further, since the derivatives of $\zeta(u-u_k)$ are all rationally expressible through $\wp u$ and $\wp' u$, it follows that

$$F(u)=R(\wp u, \wp' u),$$

where R denotes a rational function of its arguments. This theorem is due to Liouville (see Art. 155).

Corollary.—If a doubly periodic function has the property of being infinite only at the point $u=0$ and congruent points, then this function $F(u)$, say, is an integral function of $\wp u$ and $\wp' u$. To prove this note that since $u=0$ is the only infinity within the first period-parallelogram we have $k=1$ and $u_1=0$. Further, since $\sum B_k^{(1)}=0$, it follows that $B_1^{(1)}=0$. We thus have

$$F(u)=C_1+\sum_m \frac{(-1)^m}{m!} B_1^{(m+1)} \frac{d^m}{du^m}\zeta u.$$

By definition we had *

$$\frac{d}{du}\zeta u=-\wp u,$$

and consequently

$$\frac{d^2}{du^2}\zeta u=-\wp' u,$$

$$\frac{d^3}{du^3}\zeta u=-\wp'' u=-6\wp^2 u+\tfrac{1}{2}g_2,$$

$$\wp''' u=6(\wp u\wp' u+\wp' u\wp u),$$

$$\wp^{(\text{IV})}=6(\wp\wp''+2\wp'\wp+\wp''\wp),$$

$$\wp^{(\text{V})}=6(\wp\wp'''+3\wp'\wp''+3\wp''\wp'+\wp'''\wp),$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$\wp^{(n+2)}=6\left(\wp\wp^{(n)}+n\wp'\wp^{(n-1)}+\frac{n\cdot n-1}{2!}\wp''\wp^{(n-2)}+\cdots\right).$$

It follows that $F(u)$ is an integral function of $\wp(u)$ and $\wp'(u)$.

ART. 378. Let $F(u)$ be a doubly periodic function of the second sort so that

$$F(u+a)=\nu F(u),$$
$$F(u+b)=\nu' F(u).$$

The logarithmic derivative of $F(u)$,

$$\frac{F'(u)}{F(u)}=\phi(u), \text{ say,}$$

is a doubly periodic function of the first sort. The function $\phi(u)$, as seen in Art. 4, becomes infinite on the zeros and on the infinities of $F(u)$. Let $u_1^0, u_2^0, \ldots, u_m^0$ be the zeros of $F(u)$; and at u_i^0 let $F(u)$ be zero of the

* See Kiepert, *Dissertation* (*De curvis quarum arcus*, etc., Berlin, 1870).

λ_i order ($i = 1, 2, \ldots, m$). Let $u_1, u_2, \ldots, u_n$ be the infinities of $F(u)$; and at u_j let $F(u)$ be infinite of the μ_j order ($j = 1, 2, \ldots, n$). We may therefore write

$$F(u) = (u - u_i^0)^{\lambda_i} F_i(u) \quad (i = 1, 2, \ldots, m),$$

where $F_i(u)$ is neither zero nor infinite for $u = u_i^0$.

It follows that

$$\phi(u) = \frac{F'(u)}{F(u)} = \frac{\lambda_i}{u - u_i^0} + \frac{F_i'(u)}{F_i(u)},$$

and consequently

$$\operatorname*{Res}_{u=u_i^0} \phi(u) = \lambda_i;$$

and similarly

$$\operatorname*{Res}_{u=u_j} \phi(u) = -\mu_j.$$

It is thus seen that $\phi(u)$ has only infinities of the first order. It was seen in the previous Article that if the development of $\phi(u)$ in the neighborhood of its infinities is given, we may express $\phi(u)$ through the ζ-functions.

It follows also here that the quantities $B_k^{(\nu+1)}$ are all zero, and consequently

$$\begin{aligned}\phi(u) = C_1 + \lambda_1\zeta(u - u_1^0) + \lambda_2\zeta(u - u_2^0) + \cdots + \lambda_m\zeta(u - u_m^0)\\ - \mu_1\zeta(u - u_1) - \mu_2\zeta(u - u_2) - \cdots - \mu_m\zeta(u - u_n).\end{aligned}$$

Also, since

$$\phi(u) = \frac{F'(u)}{F(u)} \quad \text{and} \quad \zeta u = \frac{\psi'(u)}{\psi(u)},$$

it is seen that

$$\frac{F'(u)}{F(u)} = C_1 + \sum_{i=1}^{i=m} \lambda_i \frac{\psi'(u - u_i^0)}{\psi(u - u_i^0)} - \sum_{j=1}^{j=n} \mu_j \frac{\psi'(u - u_j)}{\psi(u - u_j)}.$$

Through integration it follows that

$$F(u) = e^{C_1 u + C'} \frac{\psi(u - u_1^0)^{\lambda_1}\psi(u - u_2^0)^{\lambda_2} \cdots \psi(u - u_m^0)^{\lambda_m}}{\psi(u - u_1)^{\mu_1}\psi(u - u_2)^{\mu_2} \cdots \psi(u - u_n)^{\mu_n}}.$$

Every doubly periodic function of the second sort and consequently also every doubly periodic function of the first sort may be expressed in this manner. This representation corresponds to the decomposition of a rational function into its linear factors (see Arts. 12 and 26). Instead of the function $\psi(u)$ we may write either $\mathrm{H}(u)$ or σu.

Further, since the sum of the residues of a doubly periodic function of the first sort (Art. 99) is zero, we have

$$\Sigma \operatorname{Res} \phi(u) = \Sigma\lambda - \Sigma\mu = 0,$$

or

$$\Sigma\lambda = \Sigma\mu = r,$$

where r is the order of the function $F(u)$. It follows also that a doubly periodic function of the second sort $F(u)$ has as many zeros of the first order as it has infinities of the first order, a zero or infinity of the νth order counting as ν zeros or infinities of the first order.

ART. 379. We may write

$$\text{(A)} \qquad F(u) = e^{cu+c'} \frac{\sigma(u-u_1^0)\sigma(u-u_2^0)\cdots\sigma(u-u_r^0)}{\sigma(u-u_1)\sigma(u-u_2)\cdots\sigma(u-u_r)},$$

where some of the quantities $u_1^0, u_2^0, \ldots, u_r^0$ may be equal and some of the quantities $u_1, u_2, \ldots, u_r$ may be equal. This representation of a doubly periodic function is very convenient when all the zeros and infinities are known.

We have assumed that the points u_i^0 and u_j all lie within the same period-parallelogram. This assumption, however, is not necessary; for if 2ω be added to or subtracted from the argument of one of the σ-functions which enters in the expression above, then only the factor before the fraction is changed.

For example,

$$\sigma(u-u_\rho-2\omega) = -e^{-2\eta(u-u_\rho-\omega)}\sigma(u-u_\rho),$$

or

$$\sigma(u-u_\rho) = -e^{2\eta(u-u_\rho-\omega)}\sigma[u-(u_\rho+2\omega)].$$

It follows that every elliptic function of the rth degree may be expressed in the above form in an infinite number of ways.

ART. 380. It we write $u+2\omega$ in the place of u, then $\sigma(u-u')$ becomes $-e^{2\eta(u-u'+\omega)}\sigma(u-u')$, where $u'=u_1^0, u_2^0, \ldots, u_r^0; u_1, u_2, \ldots, u_r$. Hence, since $F(u+2\omega)=F(u)$ [if we suppose that $F(u)$ is a doubly periodic function of the first sort], it follows that

$$\text{(B)} \qquad F(u) = e^{c(u+2\omega)+c'} \frac{e^{-2\eta\sum\limits_{i=1}^{i=r}u_i^0}\,\sigma(u-u_1^0)\sigma(u-u_2^0)\cdots\sigma(u-u_r^0)}{e^{-2\eta\sum\limits_{i=1}^{i=r}u_i}\,\sigma(u-u_1)\sigma(u-u_2)\cdots\sigma(u-u_r)}.$$

The two expressions (A) and (B) must be equal. We must consequently have

$$e^{cu+c'} = e^{c(u+2\omega)+c'}e^{-2\eta\left(\sum\limits_{i=1}^{i=r}u_i^0-\sum\limits_{i=1}^{i=r}u_i\right)},$$

or

$$e^{2c\omega+2\eta\left(\sum\limits_{i=1}^{i=r}u_i-\sum\limits_{i=1}^{i=r}u_i^0\right)} = 1,$$

and similarly

$$e^{2c\omega'+2\eta'\left(\sum\limits_{i=1}^{i=r}u_i-\sum\limits_{i=1}^{i=r}u_i^0\right)} = 1.$$

In virtue of these relations we also have

$$\text{(1)} \qquad 2c\omega + 2\eta\left(\sum_{i=1}^{i=r}u_i - \sum_{i=1}^{i=r}u_i^0\right) = 2M\pi i,$$

$$\text{(2)} \qquad 2c\omega' + 2\eta'\left(\sum_{i=1}^{i=r}u_i - \sum_{i=1}^{i=r}u_i^0\right) = 2M'\pi i,$$

where M and M' are integers (positive or negative, including zero).

From the two equations just written it follows that

$$2c(\eta\omega' - \omega\eta') = 2\pi i(M'\eta - M\eta').$$

But since $\eta\omega' - \omega\eta' = \frac{1}{2}\pi i$, it is seen that

$$c = 2M'\eta - 2M\eta'.$$

If c is eliminated from (1) and (2), we have

$$\sum_{i=1}^{i=r} u_i - \sum_{i=1}^{i=r} u_i^0 = 2M\omega' - 2M'\omega.$$

For the sake of greater simplicity we may write $-m'$ for M and m for M'. We then have

$$c = 2m\eta + 2m'\eta',$$

$$\sum_{i=1}^{i=r} u_i^0 - \sum_{i=1}^{i=r} u_i = 2m\omega + 2m'\omega',$$

where m, m' are positive or negative integers or zero. This theorem is due to Liouville.*

From the latter relation it is seen that if the r infinities of a doubly periodic function of the rth order have been chosen, then only $r-1$ of the zeros are arbitrary.

As we saw above, we may write for a zero another zero that is congruent to it. We may therefore increase u_r^0 by $u_r^0 + 2m\omega + 2m'\omega'$. If this is done, then for the new system of zeros and infinities we have $m = 0 = m'$ and consequently

$$\sum_{i=1}^{i=r} u_i^0 = \sum_{i=1}^{i=r} u_i \quad \text{and} \quad c = 0.$$

We then have

$$F(u) = C\frac{\sigma(u-u_1^0)\sigma(u-u_2^0)\ \cdots\ \sigma(u-u_r^0)}{\sigma(u-u_1)\sigma(u-u_2)\ \cdots\ \sigma(u-u_r)}.$$

It is thus seen that $F(u)$ depends upon the quantities 2ω, $2\omega'$, C; $u_1, u_2, \ldots, u_r$; and upon $r-1$ of the quantities u_i^0 (we note in particular that of the r quantities u_i^0 there are only $r-1$ arbitrary). It follows that the function $F(u)$ depends upon $2r+2$ constants.†

* Liouville (*Lectures delivered in* 1847, published by Borchardt, *Crelle*, Bd. 88, p. 277, or Liouville, *Comptes Rendus*, t. 32, p. 450) proves this important theorem and also the two fundamental theorems already given, viz.: *a doubly periodic function of the nth order may be expressed rationally through an elliptic function of the second order and its derivative; a doubly periodic function must become infinite at least twice within a period-parallelogram.* Prof. Osgood, *Lehrbuch der Funktiontheorie*, p. 412, uses these three theorems as the foundation of his treatment of the doubly periodic functions.

† See Schwarz, *loc. cit.*, p. 20, or Kiepert, *Crelle*, Bd. 76, p. 21; or Appell et Lacour, *Fonct. Ellip.*, p. 48.

The expansion of the function $F(u)$ through $\mathrm{H}(u)$ in the place of σu may be derived in a similar manner (see Riemann-Stahl, *Elliptische Functionen*, p. 110).

Corollary I. — We note that the function $F(-u)$ is an elliptic function of the same nature as the function $F(u)$ considered above. It is also evident that

$\frac{1}{2}[F(u) + F(-u)] = \psi_0(u)$, say, is an even function, and that

$\frac{1}{2}[F(u) - F(-u)] = \psi_1(u)$ is an odd elliptic function.

That every elliptic function my be expressed as a sum of an even and an odd elliptic function is seen from the identity

$$F(u) = \tfrac{1}{2}[F(u) + F(-u)] + \tfrac{1}{2}[F(u) - F(-u)],$$

or

$$F(u) = \psi_0(u) + \psi_1(u).$$

Corollary II. — We may next prove that every even elliptic function of order say $2r$ may be rationally expressed through $\wp u$. Such a function may be represented in the form

$$\psi_0(u) = C\frac{\sigma(u-u_1^0)\sigma(u-u_2^0)\cdots\sigma(u-u_r^0)\sigma(u+u_1^0)\sigma(u+u_2^0)\cdots\sigma(u+u_r^0)}{\sigma(u-u_1)\sigma(u-u_2)\cdots\sigma(u-u_r)\sigma(u+u_1)\sigma(u+u_2)\cdots\sigma(u+u_r)}.$$

We may also write

$$\frac{\sigma(u-u_i^0)\sigma(u+u_i^0)}{\sigma(u-u_i)\sigma(u+u_i)} = \frac{\dfrac{\sigma(u-u_i^0)\sigma(u+u_i^0)}{\sigma^2u\sigma^2u_i^0}}{\dfrac{\sigma(u-u_i)\sigma(u+u_i)}{\sigma^2u\sigma^2u_i}}\cdot\frac{\sigma^2u_i^0}{\sigma^2u_i}$$

$$= \frac{\wp u - \wp u_i^0}{\wp u - \wp u_i}\cdot\frac{\sigma^2u_i^0}{\sigma^2u_i}.$$

We therefore have

$$\psi_0(u) = C\frac{\prod_{i=1}^{i=r}\sigma^2u_i^0\prod_{i=1}^{i=r}(\wp u - \wp u_i^0)}{\prod_{i=1}^{i=r}\sigma^2u_i\prod_{i=1}^{i=r}(\wp u - \wp u_i)},$$

a formula by which it is shown that $\psi_0(u)$ is rationally expressed through $\wp u$.

We may therefore write

$$\psi_0(u) = R_0(\wp u),$$

where R_0 denotes a rational function of its argument. Further, if $\psi_1(u)$ is an odd elliptic function, then, since $\wp' u$ is also an odd elliptic function,

$$\frac{\psi_1(u)}{\wp' u} \text{ is an even elliptic function} = R_1(\wp u), \text{ say,}$$

so that

$$\psi_1(u) = \wp' u\, R_1(\wp u),$$

where R_1 denotes a rational function of its argument.

ART. 381. As an interesting application of the above representation of an elliptic function we note the following:

In determinantal form we write the formula

$$\wp(u) - \wp v = -\frac{\sigma(u+v)\sigma(u-v)}{\sigma^2 u \sigma^2 v} = -\begin{vmatrix} 1, & \wp u \\ 1, & \wp v \end{vmatrix}.$$

We may also express through sigma-quotients such expressions as

$$\begin{vmatrix} 1, & \wp u, & \wp' u \\ 1, & \wp v, & \wp' v \\ 1, & \wp w, & \wp' w \end{vmatrix} = \Delta(u), \text{ say.}$$

The infinities of $\wp u$ and $\wp' u$ are congruent to the origin, $\wp u$ being infinite of the second and $\wp' u$ of the third order for $u = 0$. The determinant is a doubly periodic function of the third order in u with zeros $u_1{}^0 = v$, $u_2{}^0 = w$ and $u_3{}^0 = -v - w$. Further, $u_1{}^0 + u_2{}^0 + u_3{}^0 = 0 = \Sigma$ (infinities), the infinities being the triple pole zero.

It follows then that the determinant must be of the form*

$$C\frac{\sigma(u+v+w)\sigma(u-v)\sigma(u-w)\sigma(v-w)}{\sigma^3 u \sigma^3 v \sigma^3 w} = \Delta u.$$

Multiply both sides of this expression by u^3 and then make $u = 0$, and we have

$$C\frac{\sigma(v+w)\sigma(v-w)}{\sigma^2 v \sigma^2 w} = -2\begin{vmatrix} 1, & \wp v \\ 1, & \wp w \end{vmatrix},$$

so that $C = -2$. It follows that

$$\frac{\sigma(u+v+w)\sigma(u-v)\sigma(u-w)\sigma(v-w)}{\sigma^3 u \sigma^3 v \sigma^3 w} = -\frac{1}{2}\begin{vmatrix} 1, & \wp u, & \wp' u \\ 1, & \wp v, & \wp' v \\ 1, & \wp w, & \wp' w \end{vmatrix}.$$

Appell and Lacour (*loc. cit.*, p. 63, Ex. 2) give an incorrect value to the constant C.

Further, since $\wp'\omega = 0 = \wp'\omega'$, if we write in the expression above $v = \omega$ and $w = \omega'$, it becomes

$$\frac{\sigma(u+\omega+\omega')\sigma(u-\omega)\sigma(u-\omega')\sigma(\omega-\omega')}{\sigma^3 u \sigma^3 \omega \sigma^3 \omega'} = -\frac{1}{2}\wp' u\begin{vmatrix} 1, & \wp\omega \\ 1, & \wp\omega' \end{vmatrix},$$

or

$$\wp' u = -\wp'(-u) = -2\frac{\sigma(u+\omega+\omega')\sigma(u-\omega)\sigma(u-\omega')}{\sigma^3 u \sigma\omega \sigma\omega' \sigma(\omega+\omega')},$$

and consequently

$$(\wp' u)^2 = -4\frac{\sigma(u+\omega+\omega')\sigma(u-\omega-\omega')}{\sigma^2 u \sigma^2(\omega+\omega')} \cdot \frac{\sigma(u+\omega)\sigma(u-\omega)}{\sigma^2 u \sigma^2 \omega} \cdot \frac{\sigma(u+\omega')\sigma(u-\omega')}{\sigma^2 u \sigma^2 \omega'}$$

$$= 4[\wp u - \wp(\omega+\omega')][\wp u - \wp\omega][\wp u - \wp\omega']$$

$$= 4(\wp u - e_2)(\wp u - e_1)(\wp u - e_3).$$

* See Daniels, *Am. Journ. Math.*, Vol. VI, p. 266.

ART. 382. The fourth method of the representation of the doubly periodic functions is as follows:

We had in Art. 376

$$F(u)=C_1+\sum_k B_k^{(1)}\zeta(u-u_k)+\sum_k B_k^{(2)}\wp(u-u_k)$$
$$-\sum_k B_k^{(3)}\frac{1}{2!}\wp'(u-u_k)+\sum_k\frac{B_k^{(4)}}{3!}\wp''(u-u_k)-\cdots$$
$$+(-1)^{\lambda_k}\sum\frac{B_k^{(\lambda_k)}}{(\lambda_k-1)!}\wp^{(\lambda_k-2)}(u-u_k).$$

In Art. 272 we saw that

$$-\int\wp u\,du=\zeta u=\frac{1}{u}+\sum_w{}'\left\{\frac{1}{u-w}+\frac{1}{w}+\frac{u}{w^2}\right\},$$

and consequently

$$B_k^{(1)}\zeta(u-u_k)=\frac{B_k^{(1)}}{u-u_k}+\sum_w{}'\left\{\frac{B_k^{(1)}}{u-u_k-w}+\frac{B_k^{(1)}}{w}+\frac{B_k^{(1)}(u-u_k)}{w^2}\right\}.$$

If we take the summation over this expression with regard to k and note that the summations with regard to w and with regard to k may be interchanged, we have

$$\sum_k B_k^{(1)}\zeta(u-u_k)=\sum_k\frac{B_k^{(1)}}{u-u_k}+\sum_k\sum_w{}'\left\{\frac{B_k^{(1)}}{u-u_k-w}-\frac{B_k^{(1)}u_k}{w^2}\right\}.$$

We further note that

$$\wp(u-u_k)=\frac{1}{(u-u_k)^2}+\sum_w{}'\left\{\frac{1}{(u-u_k-w)^2}-\frac{1}{w^2}\right\},$$
$$\wp'(u-u_k)=-2!\sum_w\frac{1}{(u-u_k-w)^3},$$
$$\wp''(u-u_k)=3!\sum_w\frac{1}{(u-u_k-w)^4},\text{ etc.}$$

It follows at once that

$$F(u)=C_1+\sum_k\frac{B_k^{(1)}}{u-u_k}+\sum_k\sum_w{}'\left\{\frac{B_k^{(1)}}{u-u_k-w}-\frac{B_k^{(1)}u_k}{w^2}\right\}$$
$$+\sum_k\frac{B_k^{(2)}}{(u-u_k)^2}+\sum_k\sum_w{}'\left\{\frac{B_k^{(2)}}{(u-u_k-w)^2}-\frac{B_k^{(2)}}{w^2}\right\}$$
$$+\sum_k\sum_w\frac{B_k^{(3)}}{(u-u_k-w)^3}+\sum_k\sum_w\frac{B_k^{(4)}}{(u-u_k-w)^4}$$
$$+\cdots+\sum_k\sum_w\frac{B_k^{(\lambda_k)}}{(u-u_k-w)^{\lambda_k}}.$$

If for brevity we put

$$f(u)=\sum_k\left\{\frac{B_k^{(1)}}{u-u_k}+\frac{B_k^{(2)}}{(u-u_k)^2}+\cdots+\frac{B_k^{(\lambda_k)}}{(u-u_k)^{\lambda_k}}\right\},$$

the above formula may be written

$$F(u)=C_1+f(u)+\sum_w{}'\left\{f(u-w)-\sum_k\frac{B_k^{(1)}u_k+B_k^{(2)}}{w^2}\right\}.$$

ART. 383. We may next consider the fifth kind of representation of the doubly periodic function $F(u)$.

We saw in Art. 287 that

$$\zeta u=\frac{\eta}{\omega}u+\frac{\pi i}{2\omega}\left\{\frac{z+z^{-1}}{z-z^{-1}}+\sum_{m=1}^{m=\infty}\frac{2h^{2m}z^{-2}}{1-h^{2m}z^{-2}}-\sum_{m=1}^{m=\infty}\frac{2h^{2m}z^2}{1-h^{2m}z^2}\right\},$$

where $z^2=t=e^{\frac{u\pi i}{\omega}}$.

We have at once

$$\frac{z+z^{-1}}{z-z^{-1}}=\frac{t+1}{t-1}.$$

If we write

$$e^{(u-u_k)\frac{\pi i}{\omega}}=\frac{t}{t_k},\quad\text{where } t_k=e^{u_k\frac{\pi i}{\omega}},$$

it follows that

$$\zeta(u-u_k)=\frac{\eta}{\omega}(u-u_k)+\frac{\pi i}{2\omega}\left\{\frac{t+t_k}{t-t_k}+\sum_m\left(\frac{th^{-2m}+t_k}{th^{-2m}-t_k}-1\right)\right.$$
$$\left.+\sum_m\left(\frac{th^{2m}+t_k}{th^{2m}-t_k}+1\right)\right\}.$$

Next let

$$f_1(t)=\sum_k B_k^{(1)}\frac{\pi i}{2\omega}\frac{t+t_k}{t-t_k},$$

and observe that $f_1(t)=0$ for $t=0$ and for $t=\infty$.

We may then write the formula for $F(u)$ in the form

$$F(u)=C_1-\frac{\eta}{\omega}\sum_k B_k^{(1)}u_k+\sum_{m=-\infty}^{m=+\infty}f_1(h^{2m}t)$$
$$+\sum_k\sum_\nu(-1)^\nu\frac{B_k^{(\nu+1)}}{\nu!}\frac{d^\nu}{du^\nu}\zeta(u-u_k)$$
$$\begin{pmatrix}k=1,2,\ldots,n\\ \nu=1,2,\ldots,\lambda_k-1\end{pmatrix}.$$

We have the following expansion (Art. 286):

$$\wp u=-\frac{\eta}{\omega}-\left(\frac{\pi}{\omega}\right)^2\left\{\frac{1}{(z-z^{-1})^2}+\sum_{m=1}^{m=\infty}\frac{h^{2m}z^{-2}}{(1-h^{2m}z^{-2})^2}+\sum_{m=1}^{m=\infty}\frac{h^{2m}z^2}{(1-h^{2m}z^2)^2}\right\}.$$

It is further seen that

$$\frac{z}{z - z_k} = \frac{1}{1 - z_k z^{-1}} \quad \text{and} \quad \frac{\frac{z}{z_k}}{\frac{z}{z_k} - 1} = \frac{z^{-1} z_k}{1 - z_k z^{-1}}.$$

Next let

$$f_1(z) = -\frac{\pi}{\omega i}\sum_k B_k \frac{z_k}{z - z_k}.$$

It is evident that $f_1(0) = 0 = f_1(\infty)$.

The terms in $F(u)$ which correspond to $\lambda = 1$ are

$$-\frac{1}{\omega}\sum_k B_k u - \sum_{n=-\infty}^{n=\infty} f_1(zq^{2n}).$$

The terms in $F(u)$ which correspond to $\lambda = 2$ are

$$\sum_k \frac{B_k^{(1)}}{1!}\wp(u - u_k).$$

If we differentiate the formula above for $\wp u$ we have a suitable expression for $\wp(u)$ in the form of an infinite summation, which may be written

$$\sum_{n=-\infty}^{n=\infty} f_2(zq^{2n}),$$

where $f_2(z)$ is a rational function in z having the property that

$$f_2(0) = 0 = f_2(\infty).$$

We continue this process and finally write

$$f(z) = f_1(z) + f_2(z) + \cdots + f_\lambda(z),$$

the function $f(z)$ being a rational function in z such that

$$f(0) = 0 = f(\infty).$$

We therefore have

$$\begin{aligned} F(u) = C_0 - \frac{1}{\omega}\sum_k B_k^{(1)} u_k - B_k u + f(z) \\ + f(zq^2) + f(zq^4) + \cdots \\ + f(zq^{-2}) + f(zq^{-4}) + \cdots. \end{aligned}$$

Since z has the period 2ω, it is evident that $F(u)$ has the period 2ω. Also noting that z becomes zq^2 when u is increased by $2\omega'$, it is seen that $2\omega'$ is also a period of $F(u)$, provided the above series is convergent.

ART. 384. We may establish the convergence of the series in the previous Article as follows: Since $f(t) = 0$ for $t = 0$, we observe that $t = 0$ is a root of $f(t) = 0$, so that we may write

$$f(t) = t\,\frac{a_1 + a_2 t + \cdots + a_{\rho-1}t^{\rho-2}}{1 + b_1 t + b_2 t^2 + \cdots + b_\rho t^\rho}.$$

It is always possible to choose t so small that

$$|b_1 t| + |b_2 t^2| + \cdots + |b_\rho t^\rho| < \tfrac{1}{2}.$$

It follows that the denominator in the fraction above is greater than $\frac{1}{2}$, while the numerator is finite. We may therefore write

$$f(t) < At,$$

where A is a finite quantity. It is further seen that

$$\begin{aligned} f(h^2 t) &< Ah^2 t, \\ f(h^4 t) &< Ah^4 t, \\ &\cdots\cdots \end{aligned}$$

It follows that the series $f(t) + f(h^2 t) + f(h^4 t) + \cdots$ is convergent; and in the same way it may be shown that $f(h^{-2}t) + f(h^{-4}t) + \cdots$ is convergent. We have therefore established the convergence of the series expressing $F(u)$.

ART. 385. We may also express $F(u)$ in the form of an infinite product whose factors are rational functions of t.

In Art. 380 we derived the formula

$$F(u) = C\,\frac{\sigma(u - u_1^0)\,\sigma(u - u_2^0)\,\ldots\,\sigma(u - u_r^0)}{\sigma(u - u_1)\,\sigma(u - u_2)\,\ldots\,\sigma(u - u_r)},$$

where $u_1^0 + u_2^0 + \cdots + u_r^0 = u_1 + u_2 + \cdots + u_r$.

In Art. 291 we saw that

$$\sigma u = e^{2\eta\omega v^2}\,\frac{2\omega}{\pi}\,\frac{z - z^{-1}}{2i}\prod_n \frac{1 - h^{2n}z^{-2}}{1 - h^{2n}}\prod_n \frac{1 - h^{2n}z^2}{1 - h^{2n}}.$$

If for brevity we write

$$e^{\frac{u\pi i}{\omega}} = t,\quad e^{\frac{u_k^0\pi i}{\omega}} = t_k^0,\quad e^{\frac{u_k\pi i}{\omega}} = t_k,$$

it follows that

$$\sigma(u - u_k^0) = e^{2\eta\omega\left(\frac{u - u_k^0}{2\omega}\right)^2}\,\frac{2\omega}{\pi}\,\frac{\dfrac{t}{t_k^0} - 1}{2i\sqrt{\dfrac{t}{t_k^0}}}\prod_n \frac{1 - h^{2n}\dfrac{t_k^0}{t}}{1 - h^{2n}}\prod_n \frac{1 - h^{2n}\dfrac{t}{t_k^0}}{1 - h^{2n}}$$

with corresponding formulas for $\sigma(u - u_k)$.

We next write

$$f_1(t) = \frac{\prod_{k=1}^{k=r}(t - t_k^0)}{\prod_{k=1}^{k=r}(t - t_k)}$$

and note that $f_1(0) = 1 = f_1(\infty)$.

We have at once

$$F(u) = Ce^{\frac{\eta}{2\omega}\sum_k (u_k^{0\,2} - u_k^2)} \prod_{m=-\infty}^{m=+\infty} f_1(h^{2m}t).$$

That the product on the right-hand side is absolutely convergent may be proved by writing

$$f_1(t) = 1 + f_0(t),$$

where $f_0(t) = 0 = f_0(\infty)$; it then follows by Art. 17 that the above product is absolutely convergent if

$$\sum_{m=-\infty}^{m=+\infty} f_0(h^{2m}t)$$

is absolutely convergent. The convergence of this series is easily established by using a geometric progression whose ratio is h^2.

ART. 386. We saw in Art. 377 that every one-valued doubly periodic function which has everywhere in the finite portion of the plane the character of an integral or (fractional) rational function may be expressed rationally through $\wp u$ and $\wp' u$, say

$$\phi(u) = R_1(\wp u, \wp' u),$$

where R_1 denotes a rational function of its arguments. It follows that

$$\phi'(u) = \frac{\partial R_1}{\partial \wp u}\wp' u + \frac{\partial R_1}{\partial \wp' u}\wp'' u.$$

Writing for $\wp'' u$ its value $\wp'' u = 6\,\wp^2 u - \frac{1}{2}\,g_2$, it is seen that $\phi'(u)$ may be rationally expressed through $\wp u$, $\wp' u$. We therefore write

$$\phi'(u) = R_2(\wp u, \wp' u),$$

where R_2 is a rational function of its arguments.

Any rational function of $\wp u$ and $\wp' u$ may be written in the form

$$R_1(\wp u, \wp' u) = \frac{G_1(\wp u, \wp' u)}{G_2(\wp u, \wp' u)},$$

where G_1 and G_2 are integral functions.

Further, since

$$(\wp' u)^2 = 4\wp^3 u - g_2 \wp u - g_3,$$

it is evident that we may write

$$R_1(\wp u, \wp' u) = \frac{S + T\wp' u}{W},$$

where S, T and W are integral functions of $\wp u$; or finally

$$R_1(\wp u, \wp' u) = U + V\wp' u,$$

where U and V are rational functions of $\wp u$. We have accordingly

(1) $$\phi(u) = U(\wp u) + V(\wp u)\wp' u$$

and similarly

(2) $$\phi'(u) = U_1(\wp u) + V_1(\wp u)\wp' u,$$

where U_1 and V_1 are rational functions of $\wp u$. We note that V and V_1 cannot be simultaneously zero; for $U(\wp u)$ and $U_1(\wp u)$ are both even functions of u, while if $\phi(u)$ is even $\phi'(u)$ must be odd and *vice versa.*

From (1) and (2) it follows that

$$(3)\quad \wp' u = \frac{\phi(u) - U}{V} \quad \text{and} \quad (4)\quad \wp' u = \frac{\phi'(u) - U_1}{V_1}.$$

In general both of these equations (and always one of them) have definite forms, since V and V_1 cannot both be simultaneously zero. If then the values $\phi(u)$ and $\wp u$ are known, then $\wp' u$ is uniquely determined.

If in the equations (1) and (2) neither V nor V_1 is zero, by eliminating $\wp' u$, we have

(I) $$g\{\wp u, \phi(u), \phi'(u)\} = 0,$$

where g denotes an integral function of its arguments. If further we square the equation (3) and give to $\wp' u^2$ its value in terms of $\wp u$, we have

(II) $$g_1\{\wp u, \phi(u)\} = 0,$$

where g_1 is an integral function.

On the other hand if V, say, is zero, we have from (1) the equation

(I′) $$\bar{g}\{\wp u, \phi(u)\} = 0, \text{ and from (4)}$$

(II′) $$\bar{g}_1\{\wp u, \phi'(u)\} = 0,$$

where $\bar{g}$ and $\bar{g}_1$ are integral functions. We thus always have two algebraic equations among the three functions $\wp u$, $\phi(u)$, $\phi'(u)$.

Under the assumption that the pair of primitive periods 2ω, $2\omega'$ of $\wp u$ are at the same time a primitive pair of periods of $\phi(u)$ it may be shown that the two equations (I) and (II) or (I′) and (II′) have only one common root in $\wp u$.

The following indirect proof is due, I believe, to Weierstrass: Suppose that a pair of values belonging to $\phi(u)$ and $\phi'(u)$ has been chosen and suppose that the equations (I) and (II) have two common roots, say

$$\wp u = s_1 \quad \text{and} \quad \wp u = s_2.$$

Suppose that u_1 is the value of u which satisfies the equation

$$\wp u_1 = s_1.$$

Then also, since $\wp u$ is an even function, the value $-u_1$ satisfies the same equation.

From the equation (3) above we have

$$\wp' u_1 = \frac{\phi(u) - U(s_1)}{V(s_1)}, \tag{a}$$

or

$$4 s_1^3 - g_2 s_1 - g_3 = \left[\frac{\phi(u) - U(s_1)}{V(s_1)}\right]^2.$$

The two values that are had through the extraction of the root are $+\wp' u$ and $-\wp' u$ and there is only a choice of u between $+u_1$ and $-u_1$. We shall suppose that $+u_1$ gives

$$+\wp'(u_1) = \frac{\phi(u_1) - U(s_1)}{V(s_1)}. \tag{b}$$

By a comparison of (a) and (b) it is seen that

$$\phi(u) = \phi(u_1).$$

Next suppose that u_2 is the value of u which satisfies the equation

$$\wp u_2 = s_2,$$

then also $-u_2$ satisfies the same equation.

In the same way as the equations (a) and (b) were formed, we have

$$\wp' u_2 = \frac{\phi(u) - U(s_2)}{V(s_2)}$$

and

$$\wp' u_2 = \frac{\phi(u_2) - U(s_2)}{V(s_2)}.$$

It follows that $\phi(u) = \phi(u_2)$, and consequently corresponding to $\phi(u)$ to which a definite value was given at the outset, we have shown that

$$\phi(u) = \phi(u_1) = \phi(u_2). \tag{i}$$

In the same way from the value of $\phi'(u)$ which was chosen at the outset we have

$$\phi'(u) = \phi'(u_1) = \phi'(u_2). \tag{ii}$$

In Art. 37a it was seen that if the relations (i) and (ii) are true, then $u_1 - u_2$ is a period of $\phi(u)$. It follows that, if 2ω and $2\omega'$ are a pair of primitive periods of this function,

$$u_1 - u_2 = 2m\omega + 2m'\omega',$$

where m and m' are integers. We have thus shown that *the two equations (I) and (II) have only one common root.* The method to be followed is the same if we take the equations (I′) and (II′).

It may be shown * that if two algebraic equations have only one root in common, then this root may be expressed rationally in terms of the coefficients of the two equations, so that therefore here

$$\wp u = R_3[\phi(u), \phi'(u)],$$

where R_3 is a rational function of its arguments. In this connection note the proof due to Briot and Bouquet in Art. 156.

It follows then as was shown in Art. 158 that *every transcendental one-valued analytic function which has an algebraic addition-theorem is necessarily a simply or a doubly periodic function.*

ART. 387. It follows from Art. 376 that

$$\int F(u)du = C_0 + C_1 u + \sum_k B_k^{(1)} \log \sigma(u - u_k) - \sum_k B_k^{(2)} \zeta(u - u_k)$$
$$+ \sum_k \sum_\nu \frac{(-1)^{\nu+1}}{\nu!} B_k^{(\nu+1)} \wp^{(\nu-2)}(u - u_k)$$
$$[\wp^{(0)}(u - u_k) = \wp(u - u_k);\ \nu = 2, 3, \ldots, \lambda_k - 1].$$

Since $\sum B_k^{(1)} = 0$, we may write

$$\sum B_k^{(1)} \log \sigma(u - u_k) = \sum_k B_k^{(1)} \log \frac{\sigma(u_k - u)}{\sigma u \sigma u_k} + \text{Constant}.$$

We also saw in Art. 299 that

$$-\sum_k B_k^{(2)} \zeta(u - u_k) = -\zeta u \sum_k B_k^{(2)} + \textit{an elliptic function of } u.$$

It follows that

$$\int F(u)du = C_1 u - \zeta u \sum_k B_k^{(2)} + \sum_k B_k^{(1)} \log \frac{\sigma(u_k - u)}{\sigma u \sigma u_k} + \phi_1(u),$$

where $\phi_1(u)$ is a doubly periodic function with periods 2ω, $2\omega'$. Further, since (Art. 356)

$$\log \frac{\sigma(u_k - u)}{\sigma u \,\sigma u_k} = \frac{1}{2} \int \frac{\wp' u + \wp' u_k}{\wp u - \wp u_k} du + u \zeta u_k,$$

* See Baltzer, *Theorie der Determinanten*, p. 109.

we have

$$\int F(u)du = u\Big[C_1 + \sum B_k^{(1)}\zeta u_k\Big] - \zeta u \sum_k B_k^{(2)}$$
$$+ \sum B_k^{(1)} \int \frac{\wp' u + \wp' u_k}{2(\wp u - \wp u_k)} du + \phi_1(u).$$

The moduli of periodicity of the general integral $\int F(u)du$ are therefore had at once; at the same time it is seen that this general integral may be expressed through (see also Chapter VIII):

1. An elliptic integral of the first kind;
2. An elliptic integral of the second kind;
3. A finite number of elliptic integrals of the third kind;
4. A rational function of $\wp u$ and $\wp' u$.

EXAMPLES

1. Show that any integral function of $\wp u$ and $\wp' u$ may be written in the form

$$\Phi(u) = A\frac{\sigma(u-u_1^0)\,\sigma(u-u_2^0)\,\ldots\,\sigma(u-u_n^0)}{\sigma(u)^n},$$

where $u_1^0, u_2^0, \ldots, u_n^0$ are the zeros of the function.

2. Show that any rational function of $\wp u$ and $\wp' u$ may be written

$$F(u) = A\frac{a_0 + a_1\wp u + a_2\wp' u + \cdots + a_n\wp^{(n-1)}u}{b_0 + b_1\wp u + b_2\wp' u + \cdots + b^n\wp^{(n-1)}u}.$$

3. Write

$$\Delta(u, u_1, u_2, \ldots, u_n) = \begin{vmatrix} 1, & \wp u, & \wp' u, & \ldots, & \wp^{(n-1)}u \\ 1, & \wp u_1, & \wp' u_1, & \ldots, & \wp^{(n-1)}u_1 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 1, & \wp u_n, & \wp' u_n, & \ldots, & \wp^{(n-1)}u_n \end{vmatrix}.$$

Show that

$$\Delta(u, u_1, \ldots, u_n) = C\frac{\sigma(u_1-u)\,\sigma(u_2-u)\,\ldots\,\sigma(u_n-u)\sigma(u+u_1+\ldots+u_n)}{\sigma^{n+1}u},$$

where C is independent of u.

Multiply both sides of this expression by u^{n+1} and determine C.

4. Express $F(u)$ through the function $Z_1(u)$ of Art. 374, and derive the expression corresponding to the one of Art. 387 for the integral $\int F(u)du$ in terms of $Z_1(u)$ and the theta-functions.

CHAPTER XXI

THE DETERMINATION OF ALL ANALYTIC FUNCTIONS WHICH HAVE ALGEBRAIC ADDITION-THEOREMS

ARTICLE 388. The problem of this Chapter has already been solved for the case of the *one-valued* functions. Weierstrass * has also solved it for the *many-valued* functions by making use of the principles which we shall attempt to give in the sequel. Using a method due to him (see references in Chapter II) we must first show that a function $\phi(u)$ which has an algebraic addition-theorem may be extended by analytic continuation over an arbitrarily large portion of the plane without ceasing to have the character of an algebraic function; that is, in the neighborhood of any given point the function may be developed in a convergent series according to powers of a certain quantity which may stand under a root-sign, and in which series the number of negative exponents is finite. We assume that the function may be defined in the neighborhood of a certain region about the origin and we choose a point u_0 such that one branch of the function $\phi(u)$ has the character of an integral function at the point u_0.

We may therefore write

$$(1) \qquad \phi(u) = \phi(u_0) + \frac{u - u_0}{1!}\phi'(u_0) + \frac{(u - u_0)^2}{2!}\phi''(u_0) + \cdots.$$

Next put

$$u = u_0 + u',$$
$$v = u_0 + v', \quad u_0 \text{ being a constant.}$$

Since $\phi(u)$ has by hypothesis an algebraic addition-theorem, we have an equation of the form

$$G\{\phi(u), \phi(v), \phi(u+v)\} = 0,$$

where G denotes an integral function of its arguments.

We therefore have

$$G\{\phi(u_0 + u'), \phi(u_0 + v'), \phi(2u_0 + u' + v')\} = 0.$$

Further, if we write

$$u = u_0,$$
$$v = u_0 + u' + v',$$

it is seen that

$$G\{\phi(u_0), \phi(u_0 + u' + v'), \phi(2u_0 + u' + v')\} = 0.$$

* See Forsyth, *Theory of Functions*, Chap. XIII; or Phragmen, *Acta Math.*, Bd. 7, p. 33; I wish to mention in particular the Berlin lectures of Prof. H. A. Schwarz, which have been used freely in the preparation of this Chapter.

If $\phi(2u_0 + u' + v')$ be eliminated from these two equations, there results an algebraic equation of the form

$$H\{\phi(u_0),\ \phi(u_0 + u'),\ \phi(u_0 + v'),\ \phi(u_0 + u' + v')\} = 0.$$

We may consider $\phi(u_0)$ as a new constant.

Writing

$$\phi(u_0 + u') = \phi_1(u'),$$
$$\phi(u_0 + v') = \phi_1(v'),$$
$$\phi(u_0 + u' + v') = \phi_1(u' + v'),$$

we see that

$$H\{\phi_1(u'),\ \phi_1(v'),\ \phi_1(u' + v')\} = 0.$$

If in equation (1) we write $u_0 + u$ instead of u, we have

$$\phi(u_0 + u') = \phi_1(u') = \phi(u_0) + \frac{\phi'(u_0)}{1!}u' + \frac{\phi''(u_0)}{2!}u'^2 + \cdots,$$

from which it follows that by a change of the origin the function $\phi(u)$ may be changed into the function $\phi_1(u')$ in such a way that the function $\phi_1(u')$ has the character of an integral function at the point $u' = 0$ in the branch of the function under consideration.

Hence without limiting the generality of the given function $\phi(u)$, we may assume that the point $u = 0$ in the branch in question of the function $\phi(u)$ is a point at which $\phi(u)$ has the character of an integral function. Making this assumption suppose next that ρ is the radius of the circle of convergence of the series expressing $\phi(u)$ in the neighborhood of $u = 0$.

If then $|u| < \rho$, the function $\phi(u)$ has the character of an integral function in the branch considered.

If $|u| < \frac{1}{2}\rho$, $|v| < \frac{1}{2}\rho$, then is $|u + v| < \rho$, and we have

$$G\{\phi(u), \phi(v), \phi(u + v)\} = 0$$

for the region considered.

If in this equation v is put $= u$ it follows that

$$G\{\phi(u), \phi(u), \phi(2u)\} = 0,$$

which is an algebraic equation between $\phi(u)$ and $\phi(2u)$ with constant coefficients. We may write this equation

$$G_1\{\phi(u), \phi(2u)\} = 0. \tag{2}$$

If in this equation the value of u is limited so that $|u| < \frac{1}{2}\rho$, then within this region $\phi(2u)$ has the character of an integral function, since $|2u| < \rho$.

Suppose that for $\phi(u)$ its expression as a power series in terms of u is written in equation (2) which is then solved with respect to $\phi(2u)$. We know that one root of this equation represents the branch of $\phi(2u)$ under consideration if $|u| < \frac{1}{2}\rho$. But the coefficients of this equation may be analytically continued throughout the whole region of the circle with the radius ρ. In this extended region with the radius ρ the function $\phi(2u)$ retains the character of an algebraic function. Hence the definition of the function may be extended to a wider region than the original and indeed to a region with the radius 2ρ.

By writing $2u$ for u in the equation (2) we have

$$G_1\{\phi(2u), \phi(2^2u)\} = 0.$$

Eliminate $\phi(2u)$ from this equation and equation (2) and we have an algebraic equation of the form

$$G_2\{\phi(u), \phi(4u)\} = 0.$$

If the variable u be limited to values such that $|u| < \frac{\rho}{2}$, then by repeating the above process it is seen that the function may be continued to the region of a circle with radius 4ρ.

By repetition of this process we come finally to an algebraic equation

$$G_m\{\phi(u), \phi(2^m u)\} = 0,$$

from which it is seen that the original functional element may be continued over an arbitrarily large portion of the plane without the function $\phi(u)$ ceasing to have the character of an algebraic function.

It is also easily shown that by this continuation of the function the addition-theorem is true for the extended region (see Art. 51) and that all the properties originally ascribed to the function remain true throughout the analytical continuation.

ART. 389. Suppose that the equation which expresses the addition-theorem

$$G\{\phi(u), \phi(v), \phi(u+v)\} = 0$$

is developed in powers of $\phi(u+v)$. It takes, say, the form

$$(3)\quad \phi^{m_1}(u+v) + P_{1,1}[\phi(u), \phi(v)]\phi^{m_1-1}(u+v) + P_{1,2}[\phi(u), \phi(v)]\phi^{m_1-2}(u+v)$$
$$+ \cdots + P_{1,m_1}[\phi(u), \phi(v)] = 0,$$

where the P's are rational functions of $\phi(u)$, $\phi(v)$.

In this equation write

$$u + k_1 \text{ for } u$$

and

$$v - k_1 \text{ for } v,$$

where k_1 is a variable quantity which may be limited to small values.

By this substitution $u + v$ remains unchanged, and the above equation becomes

(4) $\phi^{m_1}(u + v) + P_{1,1}[\phi(u + k_1), \phi(v - k_1)]\phi^{m_1-1}(u + v) + \cdots = 0.$

The equations (3) and (4) are algebraic and have at least one root in common, viz., $\phi(u + v)$ which belongs to the branch of the function in question.

Through a finite number of essentially rational operations we may by Euler's Method derive the greatest common divisor of the two equations and thus form a new algebraic equation whose degree is less than the degree of either of the original equations unless these equations have all roots in common. This we suppose is not the case.

Let the form of the new equation be

$$\begin{aligned}\phi^{m_2}(u + v) &+ P_{2,1}[\phi(u), \phi(u + k_1), \phi(v), \phi(v - k_1)]\phi^{m_2-1}(u + v)\\ &+ \cdots + P_{2,m_2}[\phi(u), \phi(u + k_1), \phi(v), \phi(v - k_1)] = 0,\end{aligned}$$

where $m_2 < m_1$.

We write in the above equation

$$u + k_2 \text{ instead of } u$$

and

$$v - k_2 \text{ instead of } v.$$

That equation then becomes

$$\begin{aligned}\phi^{m_2}(u + v) + P_{2,1}[\phi(u + k_2), \phi(u + k_1 + k_2), \phi(v - k_2),\\ \phi(v - k_1 - k_2)]\phi^{m_2-1}(u + v) + \cdots = 0.\end{aligned}$$

It may happen that for every value k_2 this equation has all its roots the same as those of the previous equation, and consequently its coefficients do not depend upon k_2. If this is not the case the two equations have a common divisor, and when we derive this divisor we have a new equation of the form

$$\phi^{m_3}(u + v) + P_{3,1}\begin{bmatrix}\phi(u), \phi(u + k_1), \phi(u + k_2), \phi(u + k_1 + k_2),\\ \phi(v), \phi(v - k_1), \phi(v - k_2), \phi(v - k_1 - k_2)\end{bmatrix}\phi^{m_3-1}(u + v)$$
$$+ \cdots = 0,$$

where $m_3 < m_2$.

This process may be continued. Each following m_k is less than the preceding. Finally we must either have $m_k = 1$, or the two equations through which a further reduction is made possible have all their roots common.

We thus derive an equation of the form

$$\phi^{\mu}(u + v) + P_1\begin{bmatrix}\phi(u), \phi(u + k_1), \phi(u + k_2), \ldots, \phi(u + k_r), \phi(u + k_1 + k_2),\\ \phi(v), \phi(v - k_1), \phi(v - k_2), \ldots, \phi(v - k_r), \phi(v - k_1 - k_2),\end{bmatrix}$$
$$\begin{bmatrix}\ldots, \phi(u + k_1 + \cdots + k_r),\\ \ldots, \phi(v - k_1 - \cdots - k_r)\end{bmatrix}\phi^{\mu-1}(u + v) + \cdots + P_{\mu}[\text{same arguments}] = 0,$$

the P's being rational functions of their arguments. We may assume that the degree of this equation cannot be decreased by the above process. It

follows that all the coefficients of the equation remain unaltered when u is increased by a certain quantity k, and v diminished by the same quantity k. Some of the coefficients of the above equation may be constants, but they cannot all be constant, for in that case $\phi(u + v)$ would be a constant.

Suppose that P_ν is one of the variable coefficients, which is therefore a function of both u and v.

We may write
$$P_\nu = f(u, v),$$
and will show that P_ν is a function of $u + v$.

We know that $P_\nu = f(u, v)$ has the property that
$$f(u + k, v - k) = f(u, v).$$
We may choose k so small that
$$\frac{\partial f}{\partial u} k + \frac{\partial f}{\partial v}(-k) = 0,$$
or
$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial v}.$$
It follows that f is a function of $u + v$.

We shall put $f(u + v) = \psi_\nu(u + v)$ and shall show that ψ_ν is a *one-valued* function, while $\phi(u)$ may be an arbitrarily many-valued function.

Draw a circle about $u = 0$ with a radius R, where R may be taken as large as we wish. If we then succeed in showing that $\psi_\nu(u + v)$ is one-valued within this circle with radius R, the theorem may be considered proved, since R may be taken arbitrarily large. We know that in the neighborhood of $u = 0$, the function $\phi(u)$ has the character of an integral function. We shall seek to cut out of the circle two narrow strips that are perpendicular to each other and which have the property that for all points within this cross the branch of the function $\phi(u)$ under consideration has everywhere the character of an integral function. This may be done as follows: We suppose that all the branch-points of $\phi(u)$, or of the analytic continuation of the branch of $\phi(u)$ under consideration, are known. This number of branch-points is finite, since the circle is finite and the function has the character of an integral function. A straight line is drawn connecting each of these points with the origin, and at the origin a straight line is drawn perpendicular to each of these lines. We next choose a direction from the origin which coincides with none of these lines or with the perpendiculars to them. The perpendicular to this direction through the origin does not coincide with any of the straight lines or the perpendiculars to them.

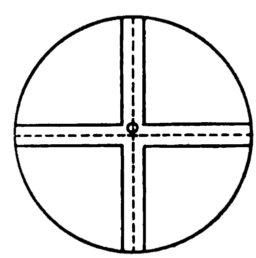


Fig. 76.

We thus have two straight lines perpendicular to each other through the origin which within the circle pass through no branch-point of the function $\phi(u)$.

Through all the branch-points which lie within the circle we draw parallels to the two lines, and among all these parallels we choose those which lie nearest the two lines. The two pairs of parallel lines which have thus been chosen form a cross-shaped figure within which no branch-point is situated, excepting always the origin, which in the leaf under consideration of the function is not a branch-point. The functions $\phi(u)$ and $\phi(v)$ are one-valued along the middle lines of the strips which form the cross. We shall now take the k's defined above so small that $|k_1| + |k_2| + \cdots + |k_r|$ is less than half the width of the more narrow of the two strips. Then if u moves along the middle line of one of the strips, while v moves along the middle line of the other, all the arguments which have been used in the formation of P_ν are situated within the cross. If u and v are added geometrically, it is seen that $P_\nu = f(u, v) = \psi_\nu(u + v)$ is a one-valued function for all values of $u + v$ within the square that circumscribes the circle with radius R. It follows, since R is arbitrarily large, that ψ_ν is a one-valued analytic function of its arguments.

ART. 390. If we write $v = 0$, then $\psi_\nu(u + v)$ becomes

$$\psi_\nu(u) = P_\nu\begin{bmatrix}\phi(u), \phi(u + k_1), \ldots, \phi(u + k_1 + \cdots + k_r), \\ \phi(0), \phi(-k_1), \ldots, \phi(-k_1 - \cdots - k_r)\end{bmatrix}.$$

From this it may be shown as follows that $\phi(u)$ and $\psi_\nu(u)$ are connected by an algebraic equation:

The function $\psi_\nu(u)$ is expressed rationally through $\phi(u), \phi(u+k_1), \ldots, \phi(u + k_1 + k_2 + \cdots + k_r)$. By means of the addition-theorem $\phi(u + k_1)$ may be expressed algebraically through $\phi(u)$ and $\phi(k_1)$, and similarly $\phi(u + k_2)$, etc.

We thus have an algebraic equation of the form

$$H[\phi(u), \psi_\nu(u)] = 0. \tag{5}$$

From the four algebraic equations

$$\begin{aligned} &G[\phi(u),\ \phi(v),\ \phi(u + v)] = 0, \\ &H[\phi(u),\ \psi_\nu(u)] = 0, \\ &H[\phi(v),\ \psi_\nu(v)] = 0, \\ &H[\phi(u + v),\ \psi_\nu(u + v)] = 0, \end{aligned}$$

we may eliminate $\phi(u)$, $\phi(v)$, $\phi(u + v)$ and have the algebraic equation

$$g[\psi_\nu(u),\ \psi_\nu(v),\ \psi_\nu(u + v)] = 0.$$

Further, if we differentiate equation (5) we have an algebraic equation

$$(6) \qquad H_1[\phi(u),\ \psi_\nu(u),\ \phi'(u),\ \psi'(u)] = 0.$$

We also have the eliminant equation

$$(7) \qquad E[\phi(u), \phi'(u)] = 0. \qquad \text{(i)}$$

If from the equations (5), (6) and (7) we eliminate $\phi(u)$ and $\phi'(u)$ we have the eliminant equation

$$E_1[\psi_\nu(u),\ \psi_\nu'(u)] = 0. \qquad \text{(ii)}$$

It follows then that $\psi_\nu(u)$ has an algebraic addition-theorem.

Since the algebraic equation (5) exists connecting $\phi(u)$ and $\psi_\nu(u)$, it follows that $\phi(u)$ is an algebraic function of $\psi_\nu(u)$. We have thus solved the problem of determining the function $\phi(u)$ in its greatest generality. *The function $\phi(u)$ is the root of an algebraic equation, whose coefficients are rationally expressed through a one-valued analytic function $\psi_\nu(u)$, which function has an algebraic addition-theorem.* In the Weierstrassian theory the one-valued analytic functions that have algebraic addition-theorems, as shown in Chapter VII, are either

I, rational functions of u, or

II, rational functions of $e^{\frac{u\pi i}{\omega}}$, or

III, rational functions of $\wp u$ and $\wp' u$.

TABLE OF FORMULAS

(The formulas of Jacobi and of Weierstrass in juxtaposition)

I.

$$u = \int_0^{z} \frac{dz}{\sqrt{(1 - z^2)(1 - k^2z^2)}} = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = F(k, \phi). \quad \text{p. 285.}$$

$$z = \sin \phi, \qquad \phi = \mathrm{am}\, u. \quad \text{p. 241.}$$

$$z = sn\, u, \quad \sqrt{1 - z^2} = \cos\phi = cn\, u, \quad \sqrt{1 - k^2z^2} = dn\, u. \quad \text{p. 241.}$$

$$\sqrt{1 - k^2 \sin^2 \phi} = \Delta\phi, \qquad u = F(k, z) = F(k, \phi). \quad \text{p. 285.}$$

$$am\, 0 = 0, \qquad sn\, 0 = 0, \qquad cn\, 0 = 1, \qquad dn\, 0 = 1. \quad \text{p. 245.}$$

$$am(-u) = -am\, u, \quad sn(-u) = -sn\, u, \quad cn(-u) = cn\, u, \quad dn(-u) = dn\, u.$$

$$sn^2u + cn^2u = 1, \qquad k^2sn^2u + dn^2u = 1. \quad \text{p. 247.}$$

II.

$$du = \frac{d\phi}{\Delta\phi}, \quad \frac{d\phi}{du} = \Delta\phi \quad \text{or} \quad \frac{d\, \mathrm{am}\, u}{du} = dn\, u. \quad \text{p. 243.}$$

$$\left(\frac{dz}{du}\right)^2 = (sn'u)^2 = (1 - sn^2u)(1 - k^2sn^2u). \quad \text{p. 247.}$$

$$sn'u = cn\, u\, dn\, u, \quad \text{p. 247.}$$

$$cn'u = -sn\, u\, dn\, u,$$

$$dn'u = -k^2 sn\, u\, cn\, u.$$

$$(sn'u)^2 = (1 - sn^2u)(1 - k^2sn^2u), \quad \text{p. 247.}$$

$$(cn'u)^2 = (1 - cn^2u)(1 - k^2 + k^2cn^2u),$$

$$(dn'u)^2 = (1 - dn^2u)(dn^2u - 1 + k^2).$$

(See also No. LVI).

III.

$$u = \int_{\infty}^{t} \frac{dt}{-\sqrt{4t^3 - g_2 t - g_3}}. \quad . \quad . \quad . \quad . \quad . \quad \text{p. 215.}$$

$$t = \wp u, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{p. 298.}$$

$$\frac{dt}{du} = \wp' u = -\sqrt{4t^3 - g_2 t - g_3}.$$

$$(\wp' u)^2 = 4\wp^3 u - g_2 \wp u - g_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{p. 325.}$$

$$= 4(\wp u - e_1)(\wp u - e_2)(\wp u - e_3).$$

$$4t^3 - g_2 t - g_3 = 4(t - e_1)(t - e_2)(t - e_3). \quad . \quad . \quad \text{pp. 191, 200.}$$

$$e_1 + e_2 + e_3 = 0,$$

$$e_1 e_2 + e_2 e_3 + e_3 e_1 = -\tfrac{1}{2}(e_1^2 + e_2^2 + e_3^2) = -\tfrac{1}{4} g_2,$$

$$e_1 e_2 e_3 = \tfrac{1}{4} g_3.$$

$$\frac{\Delta}{16} = \frac{g_2^3 - 27 g_3^2}{16} = (e_1 - e_2)^2 (e_2 - e_3)^2 (e_3 - e_1)^2 = G. \quad . \quad \text{p. 408.}$$

IV.

$$K = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\Delta\phi} = F\left(k, \frac{\pi}{2}\right). \quad . \quad \text{p. 212.}$$

$$F(k, \pi) = \int_0^{\pi} \frac{d\phi}{\Delta\phi} = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\Delta\phi} + \int_{\frac{\pi}{2}}^{\pi} \frac{d\phi}{\Delta\phi} = 2K. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{p. 285.}$$

$$F(k, n\pi + \rho) = 2nK + F(k, \rho).$$

$$\operatorname{am} K = \frac{\pi}{2}, \quad \operatorname{am} 2K = \pi = 2 \operatorname{am} K, \quad \operatorname{am}(\rho \pm 2nK) = \operatorname{am} \rho \pm n\pi. \quad \text{p. 241.}$$

V.

$$\operatorname{sn} K = 1, \quad \operatorname{cn} K = 0, \quad \operatorname{dn} K = k'. \quad . \quad . \quad . \quad . \quad . \quad \text{p. 245.}$$

$$k^2 + k'^2 = 1. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{p. 213.}$$

VI.

$$K' = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k'^2z^2)}} = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1-k'^2\sin^2\phi}} = F\left(k', \frac{\pi}{2}\right). \quad . \quad . \quad \text{p. 213.}$$

$$\int_0^{\frac{1}{k}} \frac{dz}{\sqrt{Z}} = \int_0^1 \frac{dz}{\sqrt{Z}} + i\int_1^{\frac{1}{k}} \frac{dz}{\sqrt{Z}} = K + iK', \quad . \quad . \quad . \quad \text{p. 289.}$$

$$Z = (1-z^2)(1-k^2z^2).$$

VII.

$$\operatorname{sn}(K + iK') = \frac{1}{k}, \quad \operatorname{cn}(K + iK') = -\frac{ik'}{k}, \quad \operatorname{dn}(K + iK') = 0. \quad \text{p. 246.}$$

VIII.

$$\omega = \int_{e_1}^{\infty} \frac{dt}{\sqrt{4t^3 - g_2 t - g_3}} = \int_{e_1}^{\infty} \frac{dt}{2\sqrt{(t-e_1)(t-e_2)(t-e_3)}} \equiv \int_{e_3}^{e_2} \frac{dt}{\sqrt{S}}, \text{ p. 215.}$$

$$S = 4t^3 - g_2 t - g_3.$$

$$\omega' = \int_{e_3}^{\infty} \frac{dt}{\sqrt{S}} \equiv \int_{e_2}^{e_1} \frac{dt}{\sqrt{S}}, \quad . \; . \; . \; . \; . \; . \; . \quad \text{p. 215.}$$

$$\omega'' = \int_{e_2}^{\infty} \frac{dt}{\sqrt{S}} \equiv \int_{e_3}^{e_1} \frac{dt}{\sqrt{S}},$$

where

$$\mathsf{R}\left(\frac{\omega'}{i\omega}\right) > 0. \quad . \; . \; . \; . \; . \; . \; . \quad \text{pp. 93, 384.}$$

$$\omega'' = \omega + \omega'. \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{p. 215.}$$

IX.

$$\wp(\omega) = e_1, \qquad \wp(\omega'') = e_2, \qquad \wp(\omega') = e_3, \quad . \; . \; . \quad \text{p. 216.}$$

$$\wp'\omega = 0, \qquad \wp'\omega'' = 0, \qquad \wp'\omega' = 0. \quad . \quad \text{pp. 315, 355.}$$

X.

$$K = \sqrt{e_1 - e_3}\,\omega, \quad iK' = \sqrt{e_1 - e_3}\,\omega'. \quad . \; . \; . \quad \text{p. 201.}$$

$$k^2 = \frac{e_2 - e_3}{e_1 - e_3}, \qquad k'^2 = \frac{e_1 - e_2}{e_1 - e_3}. \quad . \; . \; . \; . \; . \; . \quad \text{p. 201.}$$

$$\frac{g_2^3}{g_3^2} = \frac{108\,(1 - k^2 + k^4)^3}{[(2 - k^2)(2k^2 - 1)(1 + k^2)]^2}. \quad . \; . \; . \; . \; . \; . \quad \text{p. 201.}$$

XI.

$$\operatorname{sn}(-u) = -\operatorname{sn} u, \quad \ldots\ldots\ldots \quad \text{p. 245.}$$
$$\operatorname{cn}(-u) = \operatorname{cn} u,$$
$$\operatorname{dn}(-u) = \operatorname{dn} u.$$

$$\operatorname{sn}(u+K) = \frac{\operatorname{cn} u}{\operatorname{dn} u}, \quad \ldots\ldots\ldots \quad \text{p. 245.}$$
$$\operatorname{cn}(u+K) = -\frac{k' \operatorname{sn} u}{\operatorname{dn} u},$$
$$\operatorname{dn}(u+K) = \frac{k'}{\operatorname{dn} u}.$$

$$\operatorname{sn}(u+2K) = -\operatorname{sn} u,$$
$$\operatorname{cn}(u+2K) = -\operatorname{cn} u,$$
$$\operatorname{dn}(u+2K) = \operatorname{dn} u.$$

$$\operatorname{sn}(u+iK') = \frac{1}{k \operatorname{sn} u}, \quad \ldots\ldots\ldots \quad \text{p. 246.}$$
$$\operatorname{cn}(u+iK') = -\frac{i \operatorname{dn} u}{k \operatorname{sn} u},$$
$$\operatorname{dn}(u+iK') = -\frac{i \operatorname{cn} u}{\operatorname{sn} u}.$$

$$\operatorname{sn}(u+2iK') = \operatorname{sn} u,$$
$$\operatorname{cn}(u+2iK') = -\operatorname{cn} u,$$
$$\operatorname{dn}(u+2iK') = -\operatorname{dn} u.$$

$$\operatorname{sn}(u+K+iK') = \frac{\operatorname{dn} u}{k \operatorname{cn} u},$$
$$\operatorname{cn}(u+K+iK') = -\frac{ik'}{k \operatorname{cn} u},$$
$$\operatorname{dn}(u+K+iK') = \frac{ik' \operatorname{sn} u}{\operatorname{cn} u}.$$

$$\operatorname{sn}(u+2K+2iK') = -\operatorname{sn} u,$$
$$\operatorname{cn}(u+2K+2iK') = \operatorname{cn} u,$$
$$\operatorname{dn}(u+2K+2iK') = -\operatorname{dn} u.$$

XII.

$$\wp(u \pm 2\omega) = \wp u, \quad \ldots \ldots \quad \text{p. } 317.$$

$$\wp(u \pm 2\omega') = \wp u.$$

$$\wp(u \pm \omega) = e_1 + \frac{(e_1 - e_2)(e_1 - e_3)}{\wp u - e_1}, \quad \ldots \quad \text{pp. } 355, 369.$$

$$\wp(u \pm \omega'') = e_2 + \frac{(e_2 - e_1)(e_2 - e_3)}{\wp u - e_2},$$

$$\wp(u \pm \omega') = e_3 + \frac{(e_3 - e_1)(e_3 - e_2)}{\wp u - e_3}.$$

XIII.

$$\wp u = e_3 + \frac{e_1 - e_3}{sn^2(\sqrt{e_1 - e_3} \cdot u)}, \quad \ldots \quad \text{pp. } 216, 298.$$

$$sn(\sqrt{e_1 - e_3} \cdot u, k) = \frac{\sqrt{e_1 - e_3}}{\sqrt{\wp u - e_3}}, \quad \ldots \ldots \quad \text{p. } 305.$$

$$cn(\sqrt{e_1 - e_3} \cdot u, k) = \frac{\sqrt{\wp u - e_1}}{\sqrt{\wp u - e_3}}, \quad \ldots \ldots \quad \text{p. } 307.$$

$$dn(\sqrt{e_1 - e_3} \cdot u, k) = \frac{\sqrt{\wp u - e_2}}{\sqrt{\wp u - e_3}}. \quad \ldots \ldots \quad \text{p. } 307.$$

(See also formulas LIV.)

XIV.

$$sn(iu, k) = \frac{i\, sn(u, k')}{cn(u, k')}, \qquad sn(u, k') = \frac{1}{i}\frac{sn(iu, k)}{cn(iu, k)}, \quad \text{p. 247.}$$

$$cn(iu, k) = \frac{1}{cn(u, k')}, \qquad cn(u, k') = \frac{1}{cn(iu, k)},$$

$$dn(iu, k) = \frac{dn(u, k')}{cn(u, k')}, \qquad dn(u, k') = \frac{dn(iu, k)}{cn(iu, k)}.$$

$$sn(iu + K, k) = \frac{1}{dn(u, k')}, \quad \text{p. 261.}$$

$$cn(iu + K, k) = -\frac{ik' sn(u, k')}{dn(u, k')},$$

$$dn(iu + K, k) = \frac{k' cn(u, k')}{dn(u, k')};$$

$$sn(iu + iK', k) = \frac{-i\, cn(u, k')}{k\, sn(u, k')},$$

$$cn(iu + iK', k) = \frac{-dn(u, k')}{k\, sn(u, k')},$$

$$dn(iu + iK', k) = \frac{-1}{sn(u, k')}.$$

XV.

p. 246.

Function	Periods
$sn\,u$	$4K$ and $2iK'$
$cn\,u$	$4K$ and $2K + 2iK'$
$dn\,u$	$2K$ and $4iK'$

p. 245.

Function	Zeros	Infinities
$sn\,u$	$2mK + 2niK'$	$2mK + (2n + 1)iK'$
$cn\,u$	$(2m + 1)K + 2niK'$	"
$dn\,u$	$(2m + 1)K + (2n + 1)iK'$	"

(m, n integers including zero.)

XVI.

$$\begin{matrix} sn \\ cn \\ dn \end{matrix}\left\{u + (0, 1, 2, 3)K + (0, 1, 2, 3)iK'\right\}. \quad . \quad . \quad . \quad . \quad \text{p. } 245$$

	$\frac{1}{k\,sn\,u}$ $\frac{i\,dn\,u}{k\,sn\,u}$ $\frac{ik\,cn\,u}{k\,sn\,u}$	$\frac{dn\,u}{k\,cn\,u}$ $\frac{ik'}{k\,cn\,u}$ $\frac{-ikk'sn\,u}{k\,cn\,u}$	$\frac{-1}{k\,sn\,u}$ $\frac{-i\,dn\,u}{k\,sn\,u}$ $\frac{ik\,cn\,u}{k\,sn\,u}$	$\frac{-dn\,u}{k\,cn\,u}$ $\frac{-ik'}{k\,cn\,u}$ $\frac{-ikk'sn\,u}{k\,cn\,u}$
$3iK'$				
	$sn\,u$ $-cn\,u$ $-dn\,u$	$\frac{cn\,u}{dn\,u}$ $\frac{k'sn\,u}{dn\,u}$ $\frac{-k'}{dn\,u}$	$-sn\,u$ $cn\,u$ $-dn\,u$	$\frac{-cn\,u}{dn\,u}$ $\frac{-k'sn\,u}{dn\,u}$ $\frac{-k'}{dn\,u}$
$2iK'$				
	$\frac{1}{k\,sn\,u}$ $\frac{-i\,dn\,u}{k\,sn\,u}$ $\frac{-ik\,cn\,u}{k\,sn\,u}$	$\frac{dn\,u}{k\,cn\,u}$ $\frac{-ik'}{k\,cn\,u}$ $\frac{ikk'sn\,u}{k\,cn\,u}$	$\frac{-1}{k\,sn\,u}$ $\frac{i\,dn\,u}{k\,sn\,u}$ $\frac{-ik\,cn\,u}{k\,sn\,u}$	$\frac{-dn\,u}{k\,cn\,u}$ $\frac{ik'}{k\,cn\,u}$ $\frac{ikk'sn\,u}{k\,cn\,u}$
iK'				
	$sn\,u$ $cn\,u$ $dn\,u$	$\frac{cn\,u}{dn\,u}$ $\frac{-k'sn\,u}{dn\,u}$ $\frac{k'}{dn\,u}$	$-sn\,u$ $-cn\,u$ $dn\,u$	$\frac{-cn\,u}{dn\,u}$ $\frac{k'sn\,u}{dn\,u}$ $\frac{k'}{dn\,u}$
	0	K	$2K$	$3K$

$$sn(u + 2mK + 2m'iK') = (-1)^m sn\,u,$$

$$cn(u + 2mK + 2m'iK') = (-1)^{m+m'} cn\,u,$$

$$dn(u + 2mK + 2m'iK') = (-1)^{m'} dn\,u,$$

(m, m' integers including zero.)

XVII. [See p. 368.]

$u =$		0	$\frac{1}{2}K$	K	$\frac{3}{2}K$	$2K$
$2iK'$	sn	0	$\dfrac{1}{\sqrt{1+k'}}$	1	$\dfrac{1}{\sqrt{1+k'}}$	0
	cn	-1	$\dfrac{-\sqrt{k'}}{\sqrt{1+k'}}$	0	$\dfrac{\sqrt{k'}}{\sqrt{1+k'}}$	1
	dn	-1	$-\sqrt{k'}$	$-k'$	$-\sqrt{k'}$	-1
$\frac{3}{2}iK'$	sn	$\dfrac{-i}{\sqrt{k}}$	$\dfrac{\sqrt{k-ik'}}{\sqrt{k}}$	$\dfrac{1}{\sqrt{k}}$	$\dfrac{\sqrt{k+ik'}}{\sqrt{k}}$	$\dfrac{i}{\sqrt{k}}$
	cn	$\dfrac{-\sqrt{1+k}}{\sqrt{k}}$	$\dfrac{-\sqrt{ik'}}{\sqrt{k}}$	$\dfrac{-i\sqrt{1-k}}{\sqrt{k}}$	$\dfrac{\sqrt{-ik'}}{\sqrt{k}}$	$\dfrac{\sqrt{1+k}}{\sqrt{k}}$
	dn	$-\sqrt{1+k}$	$\sqrt{k'(k'+ik)}$	$-\sqrt{1-k}$	$-\sqrt{k'(k'-ik)}$	$-\sqrt{1+k}$
iK'	sn	$+I$*	$\dfrac{1}{\sqrt{1-k'}}$	$\dfrac{1}{k}$	$\dfrac{1}{\sqrt{1-k'}}$	$-I$
	cn	$-iI$	$\dfrac{-i\sqrt{k'}}{\sqrt{1-k'}}$	$\dfrac{-ik'}{k}$	$\dfrac{-i\sqrt{k'}}{\sqrt{1-k'}}$	$+iI$
	dn	$-ikI$	$-i\sqrt{k'}$	0	$i\sqrt{k'}$	$-ikI$
$\frac{1}{2}iK'$	sn	$\dfrac{i}{\sqrt{k}}$	$\dfrac{\sqrt{k+ik'}}{\sqrt{k}}$	$\dfrac{1}{\sqrt{k}}$	$\dfrac{\sqrt{k-ik'}}{\sqrt{k}}$	$\dfrac{-i}{\sqrt{k}}$
	cn	$\dfrac{\sqrt{1+k}}{\sqrt{k}}$	$\dfrac{\sqrt{-ik'}}{\sqrt{k}}$	$\dfrac{-i\sqrt{1-k}}{\sqrt{k}}$	$\dfrac{-\sqrt{ik'}}{\sqrt{k}}$	$\dfrac{-\sqrt{1+k}}{\sqrt{k}}$
	dn	$\sqrt{1+k}$	$\sqrt{k'(k'-ik)}$	$\sqrt{1-k}$	$\sqrt{k'(k'+ik)}$	$\sqrt{1+k}$
0	sn	0	$\dfrac{1}{\sqrt{1+k'}}$	1	$\dfrac{1}{\sqrt{1+k'}}$	0
	cn	1	$\dfrac{\sqrt{k'}}{\sqrt{1+k'}}$	0	$\dfrac{-\sqrt{k'}}{\sqrt{1+k'}}$	-1
$+$	dn	1	$\sqrt{k'}$	k'	$\sqrt{k'}$	1

* In the table $I = \lim\limits_{u=0} \dfrac{1}{k\,sn\,u}$.

XVIII.

$$\wp\left(\frac{\omega}{2}\right)=e_1+\sqrt{e_1-e_2}\sqrt{e_1-e_3}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{p. } 369.$$

$$\wp'\left(\frac{\omega}{2}\right)=-2(e_1-e_3)\sqrt{e_1-e_2}-2(e_1-e_2)\sqrt{e_1-e_3},$$

$$\wp\left(\frac{\omega}{2}+\omega'\right)=e_1-\sqrt{e_1-e_2}\sqrt{e_1-e_3},$$

$$\wp'\left(\frac{\omega}{2}+\omega'\right)=2(e_1-e_3)\sqrt{e_1-e_2}-2(e_1-e_2)\sqrt{e_1-e_3},$$

$$\wp\left(\frac{\omega'}{2}\right)=e_3-\sqrt{e_2-e_3}\sqrt{e_1-e_3},$$

$$\wp'\left(\frac{\omega'}{2}\right)=-2i(e_1-e_3)\sqrt{e_2-e_3}-2i(e_2-e_3)\sqrt{e_1-e_3},$$

$$\wp\left(\frac{\omega'}{2}+\omega\right)=e_3+\sqrt{e_2-e_3}\sqrt{e_1-e_3},$$

$$\wp'\left(\frac{\omega'}{2}+\omega\right)=2i(e_1-e_3)\sqrt{e_2-e_3}-2i(e_2-e_3)\sqrt{e_1-e_3},$$

$$\wp\left(\frac{\omega''}{2}\right)=e_2-i\sqrt{e_2-e_3}\sqrt{e_1-e_2},$$

$$\wp'\left(\frac{\omega''}{2}\right)=-2(e_1-e_2)\sqrt{e_2-e_3}-2i(e_2-e_3)\sqrt{e_1-e_2},$$

$$\wp\left(\frac{\omega-\omega'}{2}\right)=e_2+i\sqrt{e_2-e_3}\sqrt{e_1-e_2},$$

$$\wp'\left(\frac{\omega-\omega'}{2}\right)=-2(e_1-e_2)\sqrt{e_2-e_3}+2i(e_2-e_3)\sqrt{e_1-e_2}.$$

(Halphen, *Fonct. Ellip.*, Vol. I, p. 54.)

XIX.

$$\sin \pi u = \pi u \prod_m{}' \left\{ \left(1 - \frac{u}{m}\right) e^{\frac{u}{m}} \right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{p. 18.}$$

$$\pi \cot \pi u = \frac{1}{u} + \sum_m{}' \left\{ \frac{1}{u-m} + \frac{1}{m} \right\}, \quad . \quad . \quad . \quad . \quad . \quad \text{p. 20.}$$

$$\frac{\pi^2}{\sin^2 \pi u} = \sum_{m=-\infty}^{m=+\infty} \frac{1}{(u-m)^2}.$$

XX.

$$\underline{q = e^{-\pi \frac{K'}{K}}}$$

$$\Theta_1(u) = \sum_{m=-\infty}^{m=+\infty} q^{m^2} e^{\frac{\pi i m u}{K}} = 1 + 2q \cos \frac{\pi u}{K} + 2q^4 \cos \frac{2\pi u}{K} + 2q^9 \cos \frac{3\pi u}{K} + \cdots$$

p. 220.

$$\Theta_1\left(\frac{2Ku}{\pi}\right) = 1 + 2q \cos 2u + 2q^4 \cos 4u + 2q^9 \cos 6u + \cdots,$$

$$\mathrm{H}_1(u) = \sum_{m=-\infty}^{m=+\infty} q^{\frac{(2m+1)^2}{4}} e^{\frac{(2m+1)\pi i u}{2K}} = 2\sqrt[4]{q} \cos \frac{\pi u}{2K} + 2\sqrt[4]{q^9} \cos \frac{3\pi u}{2K} + 2\sqrt[4]{q^{25}} \cos \frac{5\pi u}{2K} + \cdots,$$

$$\mathrm{H}_1\left(\frac{2Ku}{\pi}\right) = 2\sqrt[4]{q} \cos u + 2\sqrt[4]{q^9} \cos 3u + 2\sqrt[4]{q^{25}} \cos 5u + \cdots,$$

$$\Theta(u) = \Theta_1(K-u), \qquad \mathrm{H}(u) = \mathrm{H}_1(K-u), \quad . \quad . \quad \text{p. 221.}$$

$$\Theta\left(\frac{2Ku}{\pi}\right) = 1 - 2q \cos 2u + 2q^4 \cos 4u - 2q^9 \cos 6u + \cdots,$$

$$\mathrm{H}\left(\frac{2Ku}{\pi}\right) = 2\sqrt[4]{q} \sin u - 2\sqrt[4]{q^9} \sin 3u + 2\sqrt[4]{q^{25}} \sin 5u - \cdots.$$

H is an *odd* function; Θ, Θ_1, H_1 are *even* functions.

XXI.

$$\sigma u = u \prod_w{}' \left\{ \left(1 - \frac{u}{w}\right) e^{\frac{u}{w} + \frac{1}{2}\frac{u^2}{w^2}} \right\} \left[\begin{array}{c} w = 2\,\mu\omega + 2\,\mu'\omega' \\ \mu, \mu' = 0, \pm 1, \pm 2, \cdots \\ w \neq 0 \end{array} \right], \quad . \quad . \quad \text{p. } 319.$$

$$\zeta u = \frac{d}{du} \log \sigma u = \frac{1}{u} + \sum{}' \left\{ \frac{1}{u - w} + \frac{1}{w} + \frac{u}{w^2} \right\}, \quad . \quad . \quad . \quad . \quad \text{pp. } 318, 324.$$

$$\wp u = -\zeta' u = -\frac{d^2}{du^2} \log \sigma u \,[\text{p. } 299] = \frac{1}{u^2} + \sum_w{}' \left\{ \frac{1}{(u - w)^2} - \frac{1}{w^2} \right\} \cdot \text{ p. } 311.$$

$$-\frac{1}{2} \wp' u = \sum_w \frac{1}{(u - w)^3}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{p. } 315.$$

$$\sigma(-u) = -\sigma u, \qquad \zeta(-u) = -\zeta u, \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{p. } 323.$$

$$\wp(-u) = \wp u, \qquad \wp^{(n)}(-u) = (-1)^n \wp^{(n)}(u). \quad . \quad . \quad \text{p. } 298.$$

XXII.

$$g_2 = 2^2 \cdot 3 \cdot 5 \sum_w{}' \frac{1}{w^4}, \qquad g_3 = 2^2 \cdot 5 \cdot 7 \sum{}' \frac{1}{w^6}. \quad . \quad . \quad \text{p. } 324.$$

$$\sigma u = u - \frac{1}{4} u^5 \sum_w{}' \frac{1}{w^4} - \frac{1}{6} u^7 \sum_w{}' \frac{1}{w^6} - \cdots \quad . \quad . \quad . \quad \text{p. } 323.$$

$$\zeta u = \frac{1}{u} - u^3 \sum_w{}' \frac{1}{w^4} - u^5 \sum_w{}' \frac{1}{w^6} - \cdots \quad . \quad . \quad . \quad \text{p. } 323.$$

$$\wp u = \frac{1}{u^2} + 3\,u^2 \sum_w{}' \frac{1}{w^4} + 5\,u^4 \sum_w{}' \frac{1}{w^6} + \cdots \quad . \quad . \quad . \quad \text{p. } 323.$$

XXIII.

$$\zeta(u + 2\,\omega) = \zeta u + 2\,\eta, \quad \zeta(u + 2\,\omega') = \zeta u + 2\,\eta'. \quad \text{pp. } 303, 338.$$

$$\eta = \zeta\omega, \qquad \eta' = \zeta\omega', \qquad \eta'' = \eta + \eta'. \quad . \quad . \quad . \quad . \quad \text{p. } 301.$$

$$\eta\omega' - \omega\eta' = \frac{\pi i}{2}, \text{ if } \mathrm{R}\left(\frac{\omega'}{i\omega}\right) \text{ is } \textit{positive}. \quad . \quad . \quad . \quad . \quad \text{p. } 339.$$

ω' [illegible] 2.70

ω [illegible] $2\left|\frac{\log k}{\pi}\right|$ 70

XXIV.

$$\Theta_1(u+K)=\Theta(u), \qquad \Theta_1(u+iK')=\lambda \mathrm{H}_1(u),$$ pp. 222, 223.

$$\mathrm{H}_1(u+K)=-\mathrm{H}(u), \qquad \mathrm{H}_1(u+iK')=\lambda\Theta_1(u),$$

$$\Theta(u+K)=\Theta_1(u), \qquad \Theta(u+iK')=i\lambda \mathrm{H}(u),$$

$$\mathrm{H}(u+K)=\mathrm{H}_1(u), \qquad \mathrm{H}(u+iK')=i\lambda\Theta(u).$$

$$\lambda=\lambda(u)=e^{-\frac{\pi i}{4K}(2u+iK')}.$$

$$\Theta_1(u+K+iK')=i\lambda \mathrm{H}(u), \qquad \Theta_1(u+2iK')=\mu\Theta_1(u),$$

$$\mathrm{H}_1(u+K+iK')=-i\lambda\Theta(u), \qquad \mathrm{H}_1(u+2iK')=\mu \mathrm{H}_1(u),$$

$$\Theta(u+K+iK')=\lambda \mathrm{H}_1(u), \qquad \Theta(u+2iK')=-\mu\Theta(u),$$

$$\mathrm{H}(u+K+iK')=\lambda\Theta_1(u), \qquad \mathrm{H}(u+2iK')=-\mu \mathrm{H}(u).$$

$$\mu=\mu(u)=e^{-\frac{\pi i}{K}(u+iK')}.$$

$$\Theta_1(u+2mK)=\Theta_1(u), \qquad \Theta_1(u+2miK')=A\Theta_1(u),$$

$$\mathrm{H}_1(u+2mK)=(-1)^m \mathrm{H}_1(u), \qquad \mathrm{H}_1(u+2miK')=A\mathrm{H}_1(u),$$

$$\Theta(u+2mK)=\Theta(u), \qquad \Theta(u+2miK')=(-1)^m A\Theta(u),$$

$$\mathrm{H}(u+2mK)=(-1)^m \mathrm{H}(u), \qquad \mathrm{H}(u+2miK')=(-1)^m A\mathrm{H}(u).$$

$$A=A(u)=e^{m^2\pi\frac{K'}{K}-\frac{m\pi i}{K}u}.$$

XXV.

$$\sqrt{e_1 - e_3}\,\sigma(u, \omega, \omega') = \frac{\mathrm{H}(2Kv)}{\mathrm{H}'(0)} e^{\frac{1}{2}\frac{\eta}{\omega}u^2}, \quad (u = 2\omega v), \quad \text{pp. } 378,\ 304.$$

$$\sigma_1 u = e^{-\eta u}\frac{\sigma(\omega + u)}{\sigma\omega} = e^{\eta u}\frac{\sigma(\omega - u)}{\sigma\omega} = \sigma_1(u, \omega, \omega') = \frac{\mathrm{H}_1(2Kv)}{\mathrm{H}_1(0)} e^{\frac{1}{2}\frac{\eta}{\omega}u^2} \quad \text{pp.} \begin{cases} 304, \\ 377. \end{cases}$$

$$\sigma_2 u = e^{-\eta'' u}\frac{\sigma(\omega'' + u)}{\sigma\omega''} = e^{\eta'' u}\frac{\sigma(\omega'' - u)}{\sigma\omega''} = \sigma_2(u, \omega, \omega') = \frac{\Theta_1(2Kv)}{\Theta_1(0)} e^{\frac{1}{2}\frac{\eta}{\omega}u^2}.$$

$$\sigma_3 u = e^{-\eta' u}\frac{\sigma(\omega' + u)}{\sigma\omega'} = e^{\eta' u}\frac{\sigma(\omega' - u)}{\sigma\omega'} = \sigma_3(u, \omega, \omega') = \frac{\Theta(2Kv)}{\Theta(0)} e^{\frac{1}{2}\frac{\eta}{\omega}u^2}.$$

XXVI. . . . . . . pp. 340, 380.

$$\begin{aligned}
\sigma(u + 2\omega) &= -\, e^{2\eta(u+\omega)}\,\sigma u, & \sigma(u + 2\omega'') &= -\, e^{2\eta''(u+\omega'')}\,\sigma u, \\
\sigma_1(u + 2\omega) &= -\, e^{2\eta(u+\omega)}\,\sigma_1 u, & \sigma_1(u + 2\omega'') &= e^{2\eta''(u+\omega'')}\,\sigma_1 u, \\
\sigma_2(u + 2\omega) &= e^{2\eta(u+\omega)}\,\sigma_2 u, & \sigma_2(u + 2\omega'') &= -\, e^{2\eta''(u+\omega'')}\,\sigma_2 u, \\
\sigma_3(u + 2\omega) &= e^{2\eta(u+\omega)}\,\sigma_3 u, & \sigma_3(u + 2\omega'') &= e^{2\eta''(u+\omega'')}\,\sigma_3 u.
\end{aligned}$$

$$\begin{aligned}
\sigma(u + 2\omega') &= -\, e^{2\eta'(u+\omega')}\,\sigma u, & \sigma(u + 2\tilde{\omega}) &= (-1)^{pr+p+r} e^{2\tilde{\eta}(u+\tilde{\omega})}\,\sigma u, \\
\sigma_1(u + 2\omega') &= e^{2\eta'(u+\omega')}\,\sigma_1 u, & \sigma_1(u + 2\tilde{\omega}) &= (-1)^{pr+p} e^{2\tilde{\eta}(u+\tilde{\omega})}\,\sigma_1 u, \\
\sigma_2(u + 2\omega') &= e^{2\eta'(u+\omega')}\,\sigma_2 u, & \sigma_2(u + 2\tilde{\omega}) &= (-1)^{pr} e^{2\tilde{\eta}(u+\tilde{\omega})}\,\sigma_2 u, \\
\sigma_3(u + 2\omega') &= -\, e^{2\eta'(u+\omega')}\,\sigma_3 u, & \sigma_3(u + 2\tilde{\omega}) &= (-1)^{pr+r} e^{2\tilde{\eta}(u+\tilde{\omega})}\,\sigma_3 u.
\end{aligned}$$

$$\begin{bmatrix} 2\tilde{\omega} = 2p\omega + 2r\omega', & p,\ r \textit{ any integers including} \\ 2\tilde{\eta} = 2p\eta + 2r\eta'. & \textit{zero} \end{bmatrix}.$$

XXVII. . . . . p. 224.

Function	Zeros
$H_1(u)$	$(2m+1)K + 2niK'$
$\Theta_1(u)$	$(2m+1)K + (2n+1)iK'$
$\Theta(u)$	$2mK + (2n+1)iK'$
$H(u)$	$2mK + 2niK'$

(m, n integers including zero.)

XXVIII.

$$sn\,u = \frac{1}{\sqrt{k}}\frac{H(u)}{\Theta(u)}, \quad \ldots \quad \text{p. 244.}$$

$$cn\,u = \sqrt{\frac{k'}{k}}\frac{H_1(u)}{\Theta(u)}, \qquad \sqrt{k} = \frac{H(K)}{\Theta(K)} = \frac{H_1(0)}{\Theta_1(0)},$$

$$dn\,u = \sqrt{k'}\frac{\Theta_1(u)}{\Theta(u)}, \qquad \sqrt{k'} = \frac{\Theta(0)}{\Theta_1(0)}.$$

XXIX.

$$\Theta(u) = \vartheta_0\left(\frac{u}{2K}\right), \qquad H_1(u) = \vartheta_2\left(\frac{u}{2K}\right), \quad \ldots \quad \text{p. 229.}$$

$$H(u) = \vartheta_1\left(\frac{u}{2K}\right), \qquad \Theta_1(u) = \vartheta_3\left(\frac{u}{2K}\right),$$

$$\vartheta_0(-u) = \vartheta_0(u), \quad \vartheta_1(-u) = -\vartheta_1(u), \quad \vartheta_2(-u) = \vartheta_2(u), \quad \vartheta_3(-u) = \vartheta_3(u).$$

XXX.

Function	Zeros
$\vartheta_0(u)$	$m + n\tau + \frac{\tau}{2}$
$\vartheta_1(u)$	$m + n\tau$
$\vartheta_2(u)$	$m + \frac{1}{2} + n\tau$
$\vartheta_3(u)$	$m + \frac{1}{2} + n\tau + \frac{\tau}{2}$

(m, n integers including zero.)

$$\tau = \frac{iK'}{K} = \frac{\omega'}{\omega} \text{ (p. 230).}$$

XXXI.

Function	Zeros
$\sigma_1 u$	$(2m+1)\omega + 2n\omega'$
$\sigma_2 u$	$(2m+1)\omega + (2n+1)\omega'$
$\sigma_3 u$	$2m\omega + (2n+1)\omega'$
σu	$2m\omega + 2n\omega'$

(m, n integers including zero.)

$$\sqrt{\wp u - e_1} = \frac{\sigma_1 u}{\sigma u}, \quad \sqrt{\wp u - e_2} = \frac{\sigma_2 u}{\sigma u}, \quad \sqrt{\wp u - e_3} = \frac{\sigma_3 u}{\sigma u}, \quad . \quad \text{p. } 373.$$

$$\wp' u = -2\frac{\sigma_1 u}{\sigma u}\frac{\sigma_2 u}{\sigma u}\frac{\sigma_3 u}{\sigma u}. \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{p. } 380.$$

XXXII. . . . . p. 384.

$$\sqrt{e_1 - e_2} = \frac{\sigma_2 \omega}{\sigma\omega} = \frac{e^{\eta''\omega}\sigma\omega'}{\sigma\omega\sigma\omega''}; \quad \sqrt{e_1 - e_3} = \frac{\sigma_3\omega}{\sigma\omega} = \frac{e^{-\eta'\omega}\sigma\omega''}{\sigma\omega\sigma\omega'};$$

$$\sqrt{e_2 - e_1} = \frac{\sigma_1\omega''}{\sigma\omega''} = -\frac{e^{\eta\omega''}\sigma\omega'}{\sigma\omega\sigma\omega''}; \quad \sqrt{e_2 - e_3} = \frac{\sigma_3\omega''}{\sigma\omega''} = -\frac{e^{\eta'\omega''}\sigma\omega}{\sigma\omega'\sigma\omega''};$$

$$\sqrt{e_3 - e_1} = \frac{\sigma_1\omega'}{\sigma\omega'} = \frac{e^{-\eta\omega'}\sigma\omega''}{\sigma\omega\sigma\omega'}; \quad \sqrt{e_3 - e_2} = \frac{\sigma_2\omega'}{\sigma\omega'} = \frac{e^{\eta''\omega'}\sigma\omega}{\sigma\omega'\sigma\omega''}.$$

$$\sqrt{e_3 - e_2} = -i\sqrt{e_2 - e_3}, \quad \sqrt{e_3 - e_1} = -i\sqrt{e_1 - e_3}, \quad \sqrt{e_2 - e_1} = -i\sqrt{e_1 - e_2},$$

where $\mathsf{R}\left(\frac{\omega'}{i\omega}\right) > 0$.

XXXIII . . . . p. [illegible]

$$\vartheta_0(u+\tfrac{1}{2})=\vartheta_3(u), \qquad \vartheta_0(u+1)=\vartheta_0(u),$$
$$\vartheta_1(u+\tfrac{1}{2})=\vartheta_2(u), \qquad \vartheta_1(u+1)=-\vartheta_1(u),$$
$$\vartheta_2(u+\tfrac{1}{2})=-\vartheta_1(u), \qquad \vartheta_2(u+1)=-\vartheta_2(u),$$
$$\vartheta_3(u+\tfrac{1}{2})=\vartheta_0(u), \qquad \vartheta_3(u+1)=\vartheta_3(u).$$

$$\vartheta_0\left(u+\frac{\tau}{2}\right)=iA\vartheta_1(u), \qquad \vartheta_0(u+\tau)=-B\vartheta_0(u),$$
$$\vartheta_1\left(u+\frac{\tau}{2}\right)=iA\vartheta_0(u), \qquad \vartheta_1(u+\tau)=-B\vartheta_1(u),$$
$$\vartheta_2\left(u+\frac{\tau}{2}\right)=A\vartheta_3(u), \qquad \vartheta_2(u+\tau)=B\vartheta_2(u),$$
$$\vartheta_3\left(u+\frac{\tau}{2}\right)=A\vartheta_2(u), \qquad \vartheta_3(u+\tau)=B\vartheta_3(u),$$
$$A=q^{-\frac{1}{4}}e^{-\pi i u}, \qquad B=q^{-1}e^{-2\pi i u}.$$

$$\vartheta_0\left(u+\frac{1+\tau}{2}\right)=A\vartheta_2(u), \qquad \vartheta_0(u+m+n\tau)=(-1)^n C\vartheta_0(u),$$
$$\vartheta_1\left(u+\frac{1+\tau}{2}\right)=A\vartheta_2(u), \qquad \vartheta_1(u+m+n\tau)=(-1)^{m+n} C\vartheta_1(u),$$
$$\vartheta_2\left(u+\frac{1+\tau}{2}\right)=-iA\vartheta_0(u), \qquad \vartheta_2(u+m+n\tau)=(-1)^m C\vartheta_2(u),$$
$$\vartheta_3\left(u+\frac{1+\tau}{2}\right)=iA\vartheta_1(u), \qquad \vartheta_3(u+m+n\tau)=C\vartheta_3(u),$$
$$C=q^{-n^2}e^{-2n\pi i u}.$$

XXXIV. . . . . p. 386.

$$\sigma(u\pm\omega)=\pm e^{\pm\eta u}\sigma\omega\sigma_1u=\pm\frac{1}{\sqrt[4]{e_1-e_2}\sqrt[4]{e_1-e_3}}e^{\pm\eta(u\pm\frac{1}{2}\omega)}\sigma_1u,$$

$$\sigma_1(u\pm\omega)=\mp\sqrt{e_1-e_2}\sqrt{e_1-e_3}e^{\pm\eta u}\sigma\omega\sigma u=\mp\sqrt[4]{e_1-e_2}\sqrt[4]{e_1-e_3}e^{\pm\eta(u\pm\frac{1}{2}\omega)}\sigma u,$$

$$\sigma_2(u\pm\omega)=\sqrt{e_1-e_2}e^{\pm\eta u}\sigma\omega\sigma_3u=\frac{\sqrt[4]{e_1-e_2}}{\sqrt[4]{e_1-e_3}}e^{\pm\eta(u\pm\frac{1}{2}\omega)}\sigma_3u,$$

$$\sigma_3(u\pm\omega)=\sqrt{e_1-e_3}e^{\pm\eta u}\sigma\omega\sigma_2u=\frac{\sqrt[4]{e_1-e_3}}{\sqrt[4]{e_1-e_2}}e^{\pm\eta(u\pm\frac{1}{2}\omega)}\sigma_2u,$$

$$\sigma(u\pm\omega'')=\pm e^{\pm\eta''u}\sigma\omega''\sigma_2u=\pm\frac{\sqrt{i}}{\sqrt[4]{e_1-e_2}\sqrt[4]{e_2-e_3}}e^{\pm\eta''(u\pm\frac{1}{2}\omega'')}\sigma_2u,$$

$$\sigma_1(u\pm\omega'')=\sqrt{e_2-e_1}e^{\pm\eta''u}\sigma\omega''\sigma_3u=\frac{1}{\sqrt{i}}\frac{\sqrt[4]{e_1-e_2}}{\sqrt[4]{e_2-e_3}}e^{\pm\eta''(u\pm\frac{1}{2}\omega'')}\sigma_3u,$$

$$\sigma_2(u\pm\omega'')=\pm\sqrt{e_2-e_1}\sqrt{e_2-e_3}e^{\pm\eta''x}\sigma\omega''\sigma u=\mp\frac{\sqrt[4]{e_1-e_2}\sqrt[4]{e_2-e_3}}{\sqrt{i}}e^{\pm\eta''(u\pm\frac{1}{2}\omega'')}\sigma u,$$

$$\sigma_3(u\pm\omega'')=\sqrt{e_2-e_3}e^{\pm\eta''u}\sigma\omega''\sigma_1u=\sqrt{i}\frac{\sqrt[4]{e_2-e_3}}{\sqrt[4]{e_1-e_2}}e^{\pm\eta''(u+\frac{1}{2}\omega'')}\sigma_1u,$$

$$\sigma(u\pm\omega')=\pm e^{\pm\eta'u}\sigma\omega'\sigma_3u=\pm\frac{i}{\sqrt[4]{e_1-e_3}\sqrt[4]{e_2-e_3}}e^{\pm\eta'(u\pm\frac{1}{2}\omega')}\sigma_3u,$$

$$\sigma_1(u\pm\omega')=\sqrt{e_3-e_1}e^{\pm\eta'u}\sigma\omega'\sigma_2u=\frac{\sqrt[4]{e_1-e_3}}{\sqrt[4]{e_2-e_3}}e^{\pm\eta'(u\pm\frac{1}{2}\omega')}\sigma_2u,$$

$$\sigma_2(u\pm\omega')=\sqrt{e_3-e_2}e^{\pm\eta'u}\sigma\omega'\sigma_1u=\frac{\sqrt[4]{e_2-e_3}}{\sqrt[4]{e_1-e_3}}e^{\pm\eta'(u\pm\frac{1}{2}\omega')}\sigma_1u,$$

$$\sigma_3(u\pm\omega')=\mp\sqrt{e_3-e_1}\sqrt{e_3-e_2}e^{\pm\eta'u}\sigma\omega'\sigma u=\pm i\sqrt[4]{e_1-e_3}\sqrt[4]{e_2-e_3}e^{\pm\eta'(u\pm\frac{1}{2}\omega')}\sigma u.$$

[Schwarz, *loc. cit.*, p. 26.]

XXXV. . . . pp. 220, 229, 378, 397.

$$\vartheta_0(u)=1+\sum_{m=1}^{m=\infty}(-1)^m 2q^{m^2}\cos 2m\pi u=1-2q\cos 2\pi u+2q^4\cos 4\pi u-\cdots,$$

$$\vartheta_1(u)=\sum_{m=0}^{m=\infty}(-1)^m 2q^{\frac{(2m+1)^2}{4}}\sin(2m+1)\pi u=2q^{\frac{1}{4}}\sin\pi u-2q^{\frac{9}{4}}\sin 3\pi u+\cdots,$$

$$\vartheta_2(u)=\sum_{m=0}^{m=\infty}2q^{\frac{(2m+1)^2}{4}}\cos(2m+1)\pi u=2q^{\frac{1}{4}}\cos\pi u+2q^{\frac{9}{4}}\cos 3\pi u+\cdots,$$

$$\vartheta_3(u)=1+\sum_{m=1}^{m=\infty}2q^{m^2}\cos 2m\pi u=1+2q\cos 2\pi u+2q^4\cos 4\pi u+\cdots.$$

XXXVI. . . . . p. 230.

$$\vartheta_0(u)=Q_0\prod_{m=1}^{m=\infty}(1-2q^{2m-1}\cos 2\pi u+q^{4m-2}),$$

$$\vartheta_1(u)=2Q_0q^{\frac{1}{4}}\sin\pi u\prod_{m=1}^{m=\infty}(1-2q^{2m}\cos 2\pi u+q^{4m}),$$

$$\vartheta_2(u)=2Q_0q^{\frac{1}{4}}\cos\pi u\prod_{m=1}^{m=\infty}(1+2q^{2m}\cos 2\pi u+q^{4m}),$$

$$\vartheta_3(u)=Q_0\prod_{m=1}^{m=\infty}(1+2q^{2m-1}\cos 2\pi u+q^{4m-2}).$$

XXXVII. . . . . p. 396.

$$Q_0=\prod_{m=1}^{m=\infty}(1-q^{2m}),\qquad Q_1=\prod_{m=1}^{m=\infty}(1+q^{2m}),$$

$$Q_2=\prod_{m=1}^{m=\infty}(1+q^{2m-1}),\qquad Q_3=\prod_{m=1}^{m=\infty}(1-q^{2m-1}).$$

$$Q_1Q_2Q_3=1,\qquad 16qQ_1^8=Q_2^8-Q_3^8.$$ . . . pp. 396, 409.

XXXIX.

If $$v = \frac{u}{2\omega}, \qquad z = e^{v\pi i}, \qquad \tau = \frac{\omega'}{\omega}, \qquad q = e^{\tau\pi i} = h,$$

$$\wp u = -\frac{\eta}{\omega} - \frac{\pi^2}{\omega^2}\left\{\frac{1}{(z - z^{-1})^2} + \sum_{m=1}^{m=\infty} \frac{h^{2m}z^{-2}}{(1 - h^{2m}z^{-2})^2} + \sum_{m=1}^{m=\infty} \frac{h^{2m}z^2}{(1 - h^{2m}z^2)^2}\right\},$$

p. 336.

$$\zeta u = \frac{\sigma'}{\sigma}(u) = \frac{\eta}{\omega}u + \frac{\pi i}{2\omega}\left\{\frac{z + z^{-1}}{z - z^{-1}} + \sum_{m=1}^{m=\infty}\frac{2h^{2m}z^{-2}}{1 - h^{2m}z^{-2}} - \sum_{m=1}^{m=\infty}\frac{2h^{2m}z^2}{1 - h^{2m}z^2}\right\},$$

p. 337.

$$\sigma u = \frac{2\omega}{\pi}\,\frac{z - z^{-1}}{2i}\,e^{2\eta\omega v^2}\prod_{m=1}^{m=\infty}\frac{1 - q^{2m}z^{-2}}{1 - q^{2m}}\prod_{m=1}^{m=\infty}\frac{1 - q^{2m}z^2}{1 - q^{2m}}.$$ p. 341.

$$= \frac{2\omega}{\pi}\sin \pi v\, e^{2\eta\omega v^2}\prod_{m=1}^{m=\infty}\frac{1 - 2q^{2m}\cos 2\pi v + q^{4m}}{(1 - q^{2m})^2}, .$$ p. 342.

$$\sigma_1(u) = \frac{z + z^{-1}}{2}\,e^{2\eta\omega v^2}\prod_{m=1}^{m=\infty}\frac{1 + q^{2m}z^{-2}}{1 + q^{2m}}\prod_{m=1}^{m=\infty}\frac{1 + q^{2m}z^2}{1 + q^{2m}}.$$ pp. $\begin{cases}342,\\379.\end{cases}$

$$= \cos \pi v\, e^{2\eta\omega v^2}\prod_{m=1}^{m=\infty}\frac{1 + 2q^{2m}\cos 2\pi v + q^{4m}}{(1 + q^{2m})^2},$$

$$\sigma_2 u = e^{2\eta\omega v^2}\prod_{m=1}^{m=\infty}\frac{1 + q^{2m-1}z^{-2}}{1 + q^{2m-1}}\prod_{m=1}^{m=\infty}\frac{1 + q^{2m-1}z^2}{1 + q^{2m-1}}$$

$$= e^{2\eta\omega v^2}\prod_{m=1}^{m=\infty}\frac{1 + 2q^{2m-1}\cos 2\pi v + q^{4m-2}}{(1 + q^{2m-1})^2},$$

$$\sigma_3 u = e^{2\eta\omega v^2}\prod_{m=1}^{m=\infty}\frac{1 - q^{2m-1}z^{-2}}{1 - q^{2m-1}}\prod_{m=1}^{m=\infty}\frac{1 - q^{2m-1}z^2}{1 - q^{2m-1}}$$

$$= e^{2\eta\omega v^2}\prod_{m=1}^{m=\infty}\frac{1 - 2q^{2m-1}\cos 2\pi v + q^{4m-2}}{(1 - q^{2m-1})^2}.$$

XL. . . . . pp. 397, 400.

$$\vartheta_1'(0) = 2\pi \sum_{m=0}^{m=\infty} (-1)^m (2m+1) q^{\frac{(2m+1)^2}{4}} = 2\pi(q^{\frac{1}{4}} - 3q^{\frac{9}{4}} + 5q^{\frac{25}{4}} - \cdots)$$

$$= 2K\sqrt{\frac{2Kkk'}{\pi}},$$

$$\vartheta_0(0) = 1 + 2\sum_{m=1}^{m=\infty} (-1)^m q^{m^2} = 1 - 2q + 2q^4 - 2q^9 + \cdots = \sqrt{\frac{2Kk'}{\pi}},$$

$$\vartheta_2(0) = 2\sum_{m=0}^{m=\infty} q^{\frac{(2m+1)^2}{4}} = 2q^{\frac{1}{4}} + 2q^{\frac{9}{4}} + 2q^{\frac{25}{4}} + \cdots = \sqrt{\frac{2Kk}{\pi}},$$

$$\vartheta_3(0) = 1 + 2\sum_{m=1}^{m=\infty} q^{m^2} = 1 + 2q + 2q^4 + 2q^9 + \cdots = \sqrt{\frac{2K}{\pi}}.$$

XLI. . . . . See p. 397.

$$\vartheta_0'(1) = 0 = \vartheta_2'(1) = \vartheta_3'(1), \qquad \vartheta_1'(1) = -\vartheta_1'(0).$$

$$\vartheta_0'(\tfrac{1}{2}) = 0 = \vartheta_1'(\tfrac{1}{2}) = \vartheta_3'(\tfrac{1}{2}), \qquad \vartheta_2'(\tfrac{1}{2}) = -\vartheta_1'(0).$$

XLII. . . . . See pp. 397, 411.

$$\vartheta_0'\left(\frac{\tau}{2}\right) = iq^{-\frac{1}{4}}\vartheta_1'(0), \qquad \vartheta_0'\left(\frac{1+\tau}{2}\right) = -i\pi q^{-\frac{1}{4}}\vartheta_2(0),$$

$$\vartheta_1'\left(\frac{\tau}{2}\right) = \pi q^{-\frac{1}{4}}\vartheta_0(0), \qquad \vartheta_1'\left(\frac{1+\tau}{2}\right) = -i\pi q^{-\frac{1}{4}}\vartheta_3(0),$$

$$\vartheta_2'\left(\frac{\tau}{2}\right) = -i\pi q^{-\frac{1}{4}}\vartheta_3(0), \qquad \vartheta_2'\left(\frac{1+\tau}{2}\right) = -\pi q^{-\frac{1}{4}}\vartheta_0(0),$$

$$\vartheta_3'\left(\frac{\tau}{2}\right) = -i\pi q^{-\frac{1}{4}}\vartheta_2(0), \qquad \vartheta_3'\left(\frac{1+\tau}{2}\right) = iq^{-\frac{1}{4}}\vartheta_1'(0).$$

$$\vartheta_0'(\tau) = 2i\pi q^{-1}\vartheta_0(0), \qquad \vartheta_0'(m+n\tau) = (-1)^{n+1} 2n\pi i\, q^{-n^2}\vartheta_0(0),$$

$$\vartheta_1'(\tau) = -q^{-1}\vartheta_1'(0), \qquad \vartheta_1'(m+n\tau) = (-1)^{m+n} q^{-n^2}\vartheta_1'(0),$$

$$\vartheta_2'(\tau) = -2i\pi q^{-1}\vartheta_2(0), \qquad \vartheta_2'(m+n\tau) = (-1)^{m+1} 2n\pi i q^{-n^2}\vartheta_2(0),$$

$$\vartheta_3'(\tau) = -2i\pi q^{-1}\vartheta_3(0), \qquad \vartheta_3'(m+n\tau) = -2n\pi i q^{-n^2}\vartheta_3(0),$$

$$\vartheta_1'(0) = 2\pi Q_0^3 q^{\frac{1}{4}}, \qquad \vartheta_0(0) = Q_0 Q_3^2,$$ . . pp. 397, 399.

$$\vartheta_2(0) = 2Q_0 Q_1^2 q^{\frac{1}{4}}, \qquad \vartheta_3(0) = Q_0 Q_2^2.$$

XLIII. . . . . pp. 385, 410.

$$\sigma\omega = \frac{e^{\frac{1}{2}\eta\omega}}{\sqrt[4]{e_1 - e_3}\,\sqrt[4]{e_1 - e_2}}; \qquad \sigma\omega'' = \frac{\sqrt{i}\,e^{\frac{1}{2}\eta''\omega''}}{\sqrt[4]{e_2 - e_3}\,\sqrt[4]{e_1 - e_2}};$$

$$\sigma\omega' = \frac{i e^{\frac{1}{2}\eta'\omega'}}{\sqrt[4]{e_2 - e_3}\,\sqrt[4]{e_1 - e_3}}.$$

XLIV. . . . . See p. 410.

	ω	ω''	ω'
σ	$\frac{2\omega}{\pi}\frac{Q_1^2}{Q_0^2}e^{\frac{\eta\omega}{2}}$	$\sqrt{i}\frac{2\omega}{\pi}\frac{Q_2^2}{2Q_0^2q^{\frac{1}{4}}}e^{\frac{\eta''\omega''}{2}}$	$i\frac{2\omega}{\pi}\frac{Q_3^2}{2Q_0^2q^{\frac{1}{4}}}e^{\frac{\eta'\omega'}{2}}$
σ_1	0	$-i\sqrt{i}\frac{Q_3^2}{2Q_1^2q^{\frac{1}{4}}}e^{\frac{\eta''\omega''}{2}}$	$\frac{Q_2^2}{2Q_1^2q^{\frac{1}{4}}}e^{\frac{\eta'\omega'}{2}}$
σ_2	$\frac{Q_3^2}{Q_2^2}e^{\frac{\eta\omega}{2}}$	0	$2\frac{Q_1^2q^{\frac{1}{4}}}{Q_2^2}e^{\frac{\eta'\omega'}{2}}$
σ_3	$\frac{Q_2^2}{Q_3^2}e^{\frac{\eta\omega}{2}}$	$2\sqrt{i}\frac{Q_1^2q^{\frac{1}{4}}}{Q_3^2}e^{\frac{\eta''\omega''}{2}}$	0

XLV.

$$\vartheta_1'(0) = \pi\,\vartheta_0(0)\,\vartheta_2(0)\,\vartheta_3(0), \quad \ldots\ldots\quad \text{p. 399.}$$

$$\vartheta_3^4(0) = \vartheta_2^4(0) + \vartheta_0^4(0),$$

$$\frac{\vartheta_1'''(0)}{\vartheta_1'(0)} = \frac{\vartheta_0''(0)}{\vartheta_0(0)} + \frac{\vartheta_2''(0)}{\vartheta_2(0)} + \frac{\vartheta_3''(0)}{\vartheta_3(0)}. \quad .\quad \text{pp. 398, 410.}$$

XLVI.

$$K = \frac{\pi}{2}\vartheta_3^2(0), \qquad q = e^{\tau\pi i}, \qquad \frac{iK'}{K} = \tau, \quad \ldots\ldots\quad \text{p. 400.}$$

$$\sqrt{k} = \frac{\vartheta_2(0)}{\vartheta_3(0)} = \frac{2q^{\frac{1}{4}} + 2q^{\frac{9}{4}} + 2q^{\frac{25}{4}} + \cdots}{1 + 2q + 2q^4 + 2q^9 + \cdots}, \quad \ldots\quad \text{p. 241.}$$

$$\sqrt{k'} = \frac{\vartheta_0(0)}{\vartheta_3(0)} = \frac{1 - 2q + 2q^4 - 2q^9 + \cdots}{1 + 2q + 2q^4 + 2q^9 + \cdots}, \quad \ldots\quad \text{p. 244.}$$

$$\sqrt{k} = 2q^{\frac{1}{4}}\frac{Q_1^2}{Q_2^2} = 2q^{\frac{1}{4}}\left[\frac{(1+q^2)(1+q^4)(1+q^6)\cdots}{(1+q)(1+q^3)(1+q^5)\cdots}\right]^2, \quad \text{p. 230.}$$

$$\sqrt{k'} = \frac{Q_3^2}{Q_2^2} = \left[\frac{(1-q)(1-q^3)(1-q^5)\cdots}{(1+q)(1+q^3)(1+q^5)\cdots}\right]^2, \quad \ldots\quad \text{p. 230.}$$

$$\sqrt[4]{k} = \sqrt{2}\,q^{\frac{1}{8}}\,\frac{\sum\limits_{m=-\infty}^{m=+\infty}(-1)^m q^{2m^2+m}}{\sum\limits_{m=-\infty}^{m=+\infty}(-1)^m q^{2m^2}}, \quad \ldots\ldots\quad \text{pp. 400, 403.}$$

$$\sqrt[4]{k'} = \frac{\sum\limits_{m=-\infty}^{m=+\infty}(-1)^m q^{2m^2+m}}{\sum\limits_{m=-\infty}^{m=+\infty} q^{2m^2+m}}.$$

XLVII.

$$G=(e_1-e_2)^2(e_2-e_3)^2(e_3-e_1)^2=\frac{g_2^3-27g_3^2}{16}=\frac{1}{16}\frac{\pi^{12}}{\omega^{12}}Q_0^{24}q^2, \quad . \quad \text{p. } 409.$$

$$2\omega\sqrt[8]{G}=\sqrt{\frac{\pi}{2\omega}}\,\vartheta_1'(0), \qquad \sqrt[4]{e_1-e_2}=\sqrt{\frac{\pi}{2\omega}}\vartheta_0(0)=\sqrt{\frac{\pi}{2\omega}}Q_0Q_3^2, \text{ pp. } \begin{cases}408,\\397.\end{cases}$$

$$\sqrt[4]{e_1-e_3}=\sqrt{\frac{\pi}{2\omega}}\vartheta_3(0)=\sqrt{\frac{\pi}{2\omega}}Q_0Q_2^2,$$

$$\sqrt[4]{e_2-e_3}=\sqrt{\frac{\pi}{2\omega}}\vartheta_2(0)=\sqrt{\frac{\pi}{2\omega}}2q^{\frac{1}{4}}Q_0Q_1^2.$$

$$e_1=\frac{\pi^2}{12\omega^2}Q_0^4(Q_2^8+Q_3^8), \qquad e_2=\frac{\pi^2}{12\omega^2}Q_0^4(16qQ_1^8-Q_3^8),$$

$$e_3=-\frac{\pi^2}{12\omega^2}Q_0^4(16qQ_1^8+Q_2^8).$$

$$e_1=\frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2[\vartheta_3^4(0)+\vartheta_0^4(0)], \quad . \quad . \quad . \quad . \quad . \quad \text{p. } 408.$$

$$e_2=\frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2[\vartheta_2^4(0)-\vartheta_0^4(0)],$$

$$e_3=\frac{1}{3}\left(\frac{\pi}{2\omega}\right)^2[-\vartheta_2^4(0)-\vartheta_3^4(0)].$$

$$g_2=\frac{2}{3}\left(\frac{\pi}{2\omega}\right)^4[\vartheta_0^8(0)+\vartheta_2^8(0)+\vartheta_3^8(0)], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{p. } 409.$$

$$g_3=\frac{4}{27}\left(\frac{\pi}{2\omega}\right)[\vartheta_2^4(0)+\vartheta_3^4(0)][\vartheta_0^4(0)+\vartheta_3^4(0)][\vartheta_0^4(0)-\vartheta_2^4(0)].$$

$$\sqrt{k}=\frac{\sqrt[4]{e_2-e_3}}{\sqrt[4]{e_1-e_3}}, \qquad \sqrt{k'}=\frac{\sqrt[4]{e_1-e_2}}{\sqrt[4]{e_1-e_3}}.$$

XLVIII. . . . . p. 409.

$$\frac{\pi}{\omega} Q_0^3 q^{\frac{1}{4}} \sigma u = e^{2\eta\omega v^2}\vartheta_1(v), \qquad \underline{u = 2\omega v.}$$

$$2\, Q_0 Q_1^2 q^{\frac{1}{4}} \sigma_1 u = e^{2\eta\omega v^2}\vartheta_2(v),$$

$$Q_0 Q_2^2 \sigma_2 u = e^{2\eta\omega v^2}\vartheta_3(v),$$

$$Q_0 Q_3^2 \sigma_3 u = e^{2\eta\omega v^2}\vartheta_0(v).$$

XLIX.

$$\sigma u = 2\omega \frac{\vartheta_1(v)}{\vartheta_1'(0)} e^{2\eta\omega v^2}, \quad . \; . \; . \; . \; . \quad \text{pp. 409, 304, 378.}$$

$$\sigma_\alpha u = \frac{\vartheta_{\alpha+1}(v)}{\vartheta_{\alpha+1}(0)} e^{2\eta\omega v^2}, \qquad [\alpha = 1, 2, 3;\ \vartheta_4 = \vartheta_0.]$$

$$\zeta u = \frac{\eta u}{\omega} + \frac{1}{2\omega}\frac{\vartheta_1'(v)}{\vartheta_1(v)}.$$

L. . . . . p. 409.

$$\sqrt{\frac{2\omega}{\pi}}\sqrt[8]{G}\,\sigma u = e^{2\eta\omega v^2}\vartheta_1(v),$$

$$\sqrt{\frac{2\omega}{\pi}}\sqrt[4]{e_2 - e_3}\,\sigma_1 u = e^{2\eta\omega v^2}\vartheta_2(v),$$

$$\sqrt{\frac{2\omega}{\pi}}\sqrt[4]{e_1 - e_3}\,\sigma_2 u = e^{2\eta\omega v^2}\vartheta_3(v),$$

$$\sqrt{\frac{2\omega}{\pi}}\sqrt[4]{e_1 - e_2}\,\sigma_3 u = e^{2\eta\omega v^2}\vartheta_0(v).$$

LI.

$\underline{q=h.}$

$$\sqrt{\frac{2\omega}{\pi}}=\frac{2q^{\frac{1}{4}}+2q^{\frac{9}{4}}+2q^{\frac{25}{4}}+\cdots}{\sqrt[4]{e_2-e_3}}=\frac{2}{\sqrt[4]{e_1-e_3}-\sqrt[4]{e_1-e_2}}(2q+2q^9+2q^{25}+\cdots),$$

p. 408.

$$\sqrt{\frac{2\omega}{\pi}}=\frac{1+2q+2q^4+2q^9+\cdots}{\sqrt[4]{e_1-e_3}}=\frac{2}{\sqrt[4]{e_1-e_3}+\sqrt[4]{e_1-e_2}}(1+2q^4+2q^{16}+\cdots),$$

p. 409.

$$2\eta\omega=\pi^2\left\{\frac{1}{6}-\sum_{m=1}^{m=\infty}\frac{4h^{2m}}{(1-h^{2m})^2}\right\}, \quad \text{p. 336.}$$

$$2\eta\omega=-2e_1\omega^2+\pi^2\left\{\frac{1}{2}+\sum_{m=1}^{m=\infty}\frac{4q^{2m}}{(1+q^{2m})^2}\right\}, \quad \text{p. 379.}$$

$$2\eta\omega=-2e_2\omega^2+\pi^2\sum_{m=1}^{m=\infty}\frac{4q^{2m-1}}{(1+q^{2m-1})^2},$$

$$2\eta\omega=-2e_3\omega^2-\pi^2\sum_{m=1}^{m=\infty}\frac{4q^{2m-1}}{(1-q^{2m-1})^2}.$$

$$2\eta\omega=-\frac{1}{6}\frac{\vartheta_1'''(0)}{\vartheta_1'(0)}=\frac{\pi^2}{6}\frac{1-3^3q^{1\cdot 2}+5^3q^{2\cdot 3}-\cdots}{1-3q^{1\cdot 2}+5q^{2\cdot 3}-\cdots}, \quad \text{pp. 410, 397.}$$

$$2\eta\omega=-2e_1\omega^2-\frac{1}{2}\frac{\vartheta_2''(0)}{\vartheta_2(0)}=-2e_1\omega^2+\frac{\pi^2}{2}\frac{1+3^2q^{1\cdot 2}+5^2q^{2\cdot 3}+\cdots}{1+q^{1\cdot 2}+q^{2\cdot 3}+\cdots},$$

$$2\eta\omega=-2e_2\omega^2-\frac{1}{2}\frac{\vartheta_3''(0)}{\vartheta_3(0)}=-2e_2\omega^2+4\pi^2\frac{q+4q^4+9q^9+\cdots}{1+2q+2q^4+2q^9+\cdots},$$

$$2\eta\omega=-2e_3\omega^2-\frac{1}{2}\frac{\vartheta_0''(0)}{\vartheta_0(0)}=-2e_3\omega^2-4\pi^2\frac{q-4q^4+9q^9-\cdots}{1-2q+2q^4-2q^9+\cdots}.$$

$$3\sum_{m=1}^{m=\infty}\frac{q^{2m}}{(1-q^{2m})^2}=\sum_{m=1}^{m=\infty}\frac{q^{2m-1}}{(1-q^{2m-1})^2}-\sum_{m=1}^{m=\infty}\frac{q^{2m-1}}{(1+q^{2m-1})^2}-\sum_{m=1}^{m=\infty}\frac{q^{2m}}{(1+q^{2m})^2},$$

p. 379.

LII.

$$\vartheta_0^2(u) = k\vartheta_1^2(u) + k'\vartheta_3^2(u),$$

$$\vartheta_2^2(u) = -k'\vartheta_1^2(u) + k\vartheta_3^2(u),$$

$$\vartheta_1^2(u) = k\vartheta_0^2(u) - k'\vartheta_2^2(u).$$

[Formulas D), p. 237.]

LIII.

$$\sigma_2^2 u - \sigma_3^2 u + (e_2 - e_3)\sigma^2 u = 0, \quad \ldots\ldots \quad \text{p. 381.}$$

$$\sigma_3^2 u - \sigma_1^2 u + (e_3 - e_1)\sigma^2 u = 0,$$

$$\sigma_1^2 u - \sigma_2^2 u + (e_1 - e_2)\sigma^2 u = 0,$$

$$(e_2 - e_3)\sigma_1^2 u + (e_3 - e_1)\sigma_2^2 u + (e_1 - e_2)\sigma_3^2 u = 0.$$

LIV. . . . . . . pp. 305, 381, 387.

$$\frac{\sigma u}{\sigma_3 u} = \frac{1}{\sqrt{e_1 - e_3}}\, sn(\sqrt{e_1 - e_3}\cdot u, k), \qquad \frac{\sigma_1 u}{\sigma u} = \sqrt{e_1 - e_3}\,\frac{cn(\sqrt{e_1 - e_3}\cdot u, k)}{sn(\sqrt{e_1 - e_3}\cdot u, k)},$$

$$\frac{\sigma_1 u}{\sigma_3 u} = cn(\sqrt{e_1 - e_3}\cdot u, k), \qquad \frac{\sigma_2 u}{\sigma u} = \sqrt{e_1 - e_3}\,\frac{dn(\sqrt{e_1 - e_3}\cdot u, k)}{sn(\sqrt{e_1 - e_3}\cdot u, k)},$$

$$\frac{\sigma_2 u}{\sigma_3 u} = dn(\sqrt{e_1 - e_3}\cdot u, k), \qquad \frac{\sigma_3 u}{\sigma u} = \sqrt{e_1 - e_3}\,\frac{1}{sn(\sqrt{e_1 - e_3}\cdot u, k)}.$$

$$\frac{\sigma_1 u}{\sigma_2 u} = \sin\operatorname{coam}(\sqrt{e_1 - e_3}\cdot u, k), \qquad \frac{\sigma u}{\sigma_1 u} = \frac{1}{\sqrt{e_1 - e_3}} \text{ multiplied by } \operatorname{tg}\operatorname{am}(\sqrt{e_1 - e_3}\cdot u, k),$$

$$\frac{\sigma(u)}{\sigma_2(u)} = \frac{1}{\sqrt{e_1 - e_2}} \text{ multiplied by } \cos\operatorname{coam}(\sqrt{e_1 - e_3}\cdot u, k), \qquad \frac{\sigma_2 u}{\sigma_1 u} = \frac{1}{\sin\operatorname{coam}(\sqrt{e_1 - e_3}\cdot u, k)},$$

$$\frac{\sigma_3(u)}{\sigma_2(u)} = \frac{\sqrt{e_1 - e_3}}{\sqrt{e_1 - e_2}} \text{ multiplied by } \Delta\operatorname{coam}(\sqrt{e_1 - e_3}\cdot u, k), \qquad \frac{\sigma_3 u}{\sigma_1 u} = \frac{1}{\cos\operatorname{am}(\sqrt{e_1 - e_3}\cdot u, k)}.$$

[Schwarz, *loc. cit.*, p. 30.]

LV.

Homogeneity. . . . . p. 343.

$$\sigma(\lambda u, \lambda\omega, \lambda\omega') = \lambda\sigma(u, \omega, \omega'),$$

$$\zeta(\lambda u, \lambda\omega, \lambda\omega') = \frac{1}{\lambda}\zeta(u, \omega, \omega'),$$

$$\wp(\lambda u, \lambda\omega, \lambda\omega') = \frac{1}{\lambda^2}\wp(u, \omega, \omega'),$$

$$\wp^{(n)}(\lambda u, \lambda\omega, \lambda\omega') = \frac{1}{\lambda^{n+2}}\wp^{(n)}(u, \omega, \omega'),$$

$$g_2(\lambda\omega, \lambda\omega') = \lambda^{-4}g_2(\omega, \omega'),$$

$$g_3(\lambda\omega, \lambda\omega') = \lambda^{-6}g_3(\omega, \omega'),$$

$$\sigma\left(\lambda u, \frac{g_2}{\lambda^4}, \frac{g_3}{\lambda^6}\right) = \lambda\sigma(u, g_2, g_3),$$

$$\zeta\left(\lambda u, \frac{g_2}{\lambda^4}, \frac{g_3}{\lambda^6}\right) = \frac{1}{\lambda}\zeta(u, g_2, g_3),$$

$$\wp\left(\lambda u, \frac{g_2}{\lambda^4}, \frac{g_3}{\lambda^6}\right) = \frac{1}{\lambda^2}\wp(u, g_2, g_3),$$

$$\wp^{(n)}\left(\lambda u, \frac{g_2}{\lambda^4}, \frac{g_3}{\lambda^6}\right) = \frac{1}{\lambda^{n+2}}\wp^{(n)}(u, g_2, g_3).$$

LVI. . . . . p. 252.

$$sn''u = -(1+k^2)sn\,u + 2k^2sn^3u,$$
$$cn''u = (2k^2-1)cn\,u - 2k^2cn^3u,$$
$$dn''u = (2-k^2)dn\,u - 2\,dn^3u;$$

$$sn^{(4)}u = (1+14k^2+k^4)sn\,u - 20k^2(1+k^2)sn^3u + 24k^4sn^5u,$$
$$cn^{(4)}u = (1-16k^2+16k^4)cn\,u + 20k^2(1-2k^2)cn^3u + 24k^4cn^5u,$$
$$dn^{(4)}u = (16-16k^2+k^4)dn\,u + 20(k^2-2)dn^3u + 24\,dn^5u.$$

(See also Formulas II.)

LVII. . . . . p. 252, *et seq.*

$$sn\,u = u - (1+k^2)\frac{u^3}{3!} + (1+14k^2+k^4)\frac{u^5}{5!} - \cdots,$$

$$sn'(0) = 1,\ -sn'''(0) = 1+k^2,\ sn^{(5)}(0) = 1+14k^2+k^4,$$
$$-sn^{(7)}(0) = 1+135k^2+135k^4+k^6,$$
$$sn^{(9)}(0) = 1+1228k^2+5478k^4+1228k^6+k^8,$$

. . . .

$$cn\,u = 1 - \frac{u^2}{2!} + (1+4k^2)\frac{u^4}{4!} - \cdots,$$

$$cn''(0) = -1,\ cn^{(4)}(0) = 1+4k^2,\ -cn^{(6)}(0) = 1+44k^2+16k^4,$$
$$cn^{(8)}(0) = 1+408k^2+912k^4+64k^6,$$
$$-cn^{(10)}(0) = 1+3688k^2+30768k^4+15808k^6+256k^8,$$

. . . .

$$dn\,u = 1 - \frac{k^2u^2}{2!} + k^2(k^2+4)\frac{u^4}{4!} - \cdots,$$

$$dn''(0) = -k^2,\ dn^{(4)}(0) = k^2(k^2+4),\ -dn^{(6)}(0) = k^2(k^4+44k^2+16),$$
$$dn^{(8)}(0) = k^2(k^6+408k^4+912k^2+64),$$
$$-dn^{(10)}(0) = k^2(k^8+3688k^6+30768k^4+15808k^2+256),$$

. . . .

[Gudermann, *Crelle*, Bd. XIX, p. 80.]

$$\frac{kK}{2\pi}sn\frac{2Ku}{\pi} = \frac{\sqrt{q}\sin u}{1-q} + \frac{\sqrt{q^3}\sin 3u}{1-q^3} + \frac{\sqrt{q^5}\sin 5u}{1-q^5} + \cdots, \quad \text{p. 256.}$$
$$\frac{kK}{2\pi}cn\frac{2Ku}{\pi} = \frac{\sqrt{q}\cos u}{1+q} + \frac{\sqrt{q^3}\cos 3u}{1+q^3} + \frac{\sqrt{q^5}\cos 5u}{1+q^5} + \cdots,$$
$$\frac{K}{2\pi}dn\frac{2Ku}{\pi} = \frac{1}{4} + \frac{q\cos 2u}{1+q^2} + \frac{q^2\cos 4u}{1+q^4} + \frac{q^3\cos 6u}{1+q^6} + \cdots,$$

LVIII.

$$\frac{1}{3!}\wp''(u)=\wp^2u-\frac{g_2}{2^2\cdot 3},\quad \frac{1}{5!}\wp^{(4)}(u)=\wp^3u-\frac{3\,g_2}{2^2\cdot 5}\wp u-\frac{g_3}{2\cdot 5},$$

$$\frac{1}{7!}\wp^{(6)}(u)=\wp^4u-\frac{g_2}{5}\wp^2u-\frac{g_3}{7}\wp u+\frac{g_2^{\,2}}{2^4\cdot 5\cdot 7},$$

$$\frac{1}{9!}\wp^{(8)}(u)=\wp^5u-\frac{g_2}{2^2}\wp^3u-\frac{5\,g_3}{2^2\cdot 7}\wp^2u+\frac{g_2^{\,2}}{2^3\cdot 3\cdot 5}\wp u+\frac{11\,g_2g_3}{2^4\cdot 3\cdot 5\cdot 7},$$

[See Art. 377.]

. . . .

LIX.

$$\wp u=\frac{1}{u^2}+c_2u^2+c_3u^4+c_4u^6+\cdots+c_nu^{2n-2}+\cdots\cdots \quad \text{pp. 326–8.}$$

$$\zeta u=\frac{1}{u}-\frac{c_2}{3}u^3-\frac{c_3}{5}u^5-\frac{c_4}{7}u^7-\cdots-\frac{c_n}{2n-1}u^{2n-1}-\cdots,$$

$$c_2=\frac{g_2}{2^2\cdot 5},\quad c_3=\frac{g_3}{2^2\cdot 7},\quad c_4=\frac{g_2^{\,2}}{2^4\cdot 3\cdot 5^2},$$

$$c_5=\frac{3\,g_2g_3}{2^4\cdot 5\cdot 7\cdot 11},\quad c_6=\frac{g_2^{\,3}}{2^5\cdot 3\cdot 5^3\cdot 13}+\frac{g_3^{\,2}}{2^4\cdot 7^2\cdot 13},\cdots.$$

$$(n-3)(2n+1)c_n=3[c_2c_{n-2}+c_3c_{n-3}+c_4c_{n-4}+\cdots+c_{n-2}c_2]\cdots \quad \text{p. 327.}$$

$[n>3]$

LX.

$$\sigma u=u-\frac{g_2}{2^4\cdot 3\cdot 5}u^5-\frac{g_3}{2^3\cdot 3\cdot 5\cdot 7}u^7-\cdots \quad \text{p. 328.}$$

$$\frac{\partial^2\sigma}{\partial u^2}=\frac{2}{3}g_2^{\,2}\frac{\partial\sigma}{\partial g_3}+12\,g_3\frac{\partial\sigma}{\partial g_2}-\frac{1}{12}g_2u^2\sigma.\quad\ldots \quad \text{p. 393.}$$

$$\sigma_\lambda u=1-\frac{1}{2}e_\lambda u^2-\frac{1}{48}(6\,e_\lambda^{\,2}-g_2)u^4-\cdots \quad \text{p. 394.}$$

$(\lambda=1,2,3.)$

LXI. . . . . pp. 236, 246.

$$\vartheta^2{}_{\alpha+1}\vartheta_1(u+v)\vartheta_1(u-v)=\vartheta_1{}^2(u)\vartheta^2{}_{\alpha+1}(v)-\vartheta_{\alpha+1}(u)\vartheta_1{}^2(v),$$
$$(\alpha=1,2,3;\ \vartheta_4=\vartheta_0).$$

$$\vartheta_3{}^2\vartheta_0(u+v)\vartheta_0(u-v)=\vartheta_1{}^2(u)\vartheta_2{}^2(v)+\vartheta_3{}^2(u)\vartheta_0{}^2(v),$$

$$\vartheta_3{}^2\vartheta_2(u+v)\vartheta_2(u-v)=\vartheta_3{}^2(u)\vartheta_2{}^2(v)-\vartheta_1{}^2(u)\vartheta_0{}^2(v),$$

$$\vartheta_3{}^2\vartheta_3(u+v)\vartheta_3(u-v)=\vartheta_1{}^2(u)\vartheta_1{}^2(v)+\vartheta_3{}^2(u)\vartheta_3{}^2(v).$$

$$\vartheta_2\vartheta_3\vartheta_1(u+v)\vartheta_0(u-v)=\vartheta_1(u)\vartheta_0(u)\vartheta_2(v)\vartheta_3(v)+\vartheta_2(u)\vartheta_3(u)\vartheta_0(v)\vartheta_1(v),$$

$$\vartheta_2\vartheta_3\vartheta_2(u+v)\vartheta_3(u-v)=\vartheta_2(u)\vartheta_3(u)\vartheta_2(v)\vartheta_3(v)-\vartheta_0(u)\vartheta_1(u)\vartheta_0(v)\vartheta_1(v).$$

$$\vartheta_2\vartheta_0\vartheta_2(u+v)\vartheta_0(u-v)=\vartheta_0(u)\vartheta_2(u)\vartheta_0(v)\vartheta_2(v)-\vartheta_1(u)\vartheta_3(u)\vartheta_1(v)\vartheta_3(v),$$

$$\vartheta_2\vartheta_0\vartheta_1(u+v)\vartheta_3(u-v)=\vartheta_1(u)\vartheta_3(u)\vartheta_0(v)\vartheta_2(v)+\vartheta_0(u)\vartheta_2(u)\vartheta_1(v)\vartheta_3(v).$$

$$\vartheta_3\vartheta_0\vartheta_3(u+v)\vartheta_0(u-v)=\vartheta_0(u)\vartheta_3(u)\vartheta_0(v)\vartheta_3(v)-\vartheta_1(u)\vartheta_2(u)\vartheta_1(v)\vartheta_2(v),$$

$$\vartheta_3\vartheta_0\vartheta_1(u+v)\vartheta_2(u-v)=\vartheta_1(u)\vartheta_2(u)\vartheta_0(v)\vartheta_3(v)+\vartheta_0(u)\vartheta_3(u)\vartheta_1(v)\vartheta_2(v).$$

$$\vartheta_0\vartheta_0\vartheta_0(u+v)\vartheta_0(u-v)=\vartheta_0(u)\vartheta_0(u)\vartheta_0(v)\vartheta_0(v)-\vartheta_1(u)\vartheta_1(u)\vartheta_1(v)\vartheta_1(v).$$

$$\vartheta_3{}^3\vartheta_3(2u)=\vartheta_3{}^4(u)+\vartheta_1{}^4(u), \quad . \ . \ . \ . \ . \ . \ . \ . \ . \quad \text{p. 237.}$$

$$\vartheta_2{}^3\vartheta_2(2u)=\vartheta_3{}^4(u)-\vartheta_0{}^4(u),$$

$$\vartheta_3{}^2\vartheta_0\vartheta_0(2u)=\vartheta_3{}^2(u)\vartheta_0{}^2(u)+\vartheta_2{}^2(u)\vartheta_1{}^2(u).$$

LXII.

$$\sigma(u+u_1)\sigma(u-u_1)\sigma(u_2+u_3)\sigma(u_2-u_3),$$
$$+\sigma(u+u_2)\sigma(u-u_2)\sigma(u_3+u_1)\sigma(u_3-u_1),$$
$$+\sigma(u+u_3)\sigma(u-u_3)\sigma(u_1+u_2)\sigma(u_1-u_2)=0. \quad . \quad . \quad \text{p. } 390.$$

$$\sigma(u+v)\sigma(u-v)=\sigma^2u\,\sigma_\lambda^2v-\sigma_\lambda^2u\,\sigma^2v, \quad . \quad . \quad . \quad . \quad . \quad \text{p. } 391.$$

$$(e_\nu-e_\mu)\sigma(u+v)\sigma(u-v)=\sigma_\mu^2u\,\sigma_\nu^2v-\sigma_\nu^2u\,\sigma_\mu^2v,$$

$$\sigma(u+v)\sigma_\lambda(u-v)=\sigma_\lambda^2u\,\sigma_\lambda^2v-(e_\lambda-e_\mu)(e_\lambda-e_\nu)\sigma^2u\,\sigma^2v,$$

$$(e_\nu-e_\mu)\sigma_\lambda(u+v)\sigma_\lambda(u-v)=(e_\lambda-e_\mu)\sigma_\nu^2u\,\sigma_\nu^2v-(e_\lambda-e_\nu)\sigma_\mu^2u\,\sigma_\mu^2v,$$

$$\sigma_\lambda(u+v)\sigma_\lambda(u-v)=\sigma_\mu^2u\,\sigma_\lambda^2v-(e_\lambda-e_\mu)\sigma^2u\,\sigma_\nu^2v,$$

$$\sigma_\lambda(u+v)\sigma_\lambda(u-v)=\sigma_\lambda^2u\,\sigma_\mu^2v-(e_\lambda-e_\mu)\sigma_\nu^2u\,\sigma^2v,$$

$$\sigma_\lambda(u+v)\sigma(u-v)=\sigma_\lambda u\,\sigma u\,\sigma_\mu v\,\sigma_\nu v-\sigma_\mu u\,\sigma_\nu u\,\sigma_\lambda v\,\sigma v,$$

$$\sigma(u+v)\sigma_\lambda(u-v)=\sigma_\lambda u\,\sigma u\,\sigma_\mu v\,\sigma_\nu v+\sigma_\mu u\,\sigma_\nu u\,\sigma_\lambda v\,\sigma v,$$

$$\sigma_\mu(u+v)\sigma_\lambda(u-v)=\sigma_\lambda u\,\sigma_\mu u\,\sigma_\lambda v\,\sigma_\mu v-(e_\mu-e_\lambda)\sigma u\,\sigma_\nu u\,\sigma v\,\sigma_\nu v.$$

$\left[\underline{\lambda,\mu,\nu=1,2,3.}\right]$ [Schwarz, *loc. cit.*, p. 51.]

LXIII. . . . (See pp. 273, 349, 364.)

$$sn(u \pm v) = (sn\,u\,cn\,v\,dn\,v \pm sn\,v\,cn\,u\,dn\,u) \div D,$$

$$\underline{\text{where } D = 1 - k^2 sn^2 u\, sn^2 v.}$$

$$cn(u \pm v) = (cn\,u\,cn\,v \mp sn\,u\,sn\,v\,dn\,u\,dn\,v) \div D,$$

$$dn(u \pm v) = (dn\,u\,dn\,v \mp k^2 sn\,u\,sn\,v\,cn\,u\,cn\,v) \div D,$$

$$sn(u + v) + sn(u - v) = 2\,sn\,u\,cn\,v\,dn\,v \div D,$$

$$sn(u + v) - sn(u - v) = 2\,sn\,v\,cn\,u\,dn\,u \div D,$$

$$sn(u + v)\,sn(u - v) = (sn^2 u - sn^2 v) \div D,$$

$$cn(u + v) + cn(u - v) = 2\,cn\,u\,cn\,v \div D,$$

$$cn(u - v) - cn(u + v) = 2\,sn\,u\,dn\,u\,sn\,v\,dn\,v \div D,$$

$$dn(u + v) + dn(u - v) = 2\,dn\,u\,dn\,v \div D,$$

$$dn(u - v) - dn(u + v) = 2\,k^2\,sn\,u\,cn\,u\,sn\,v\,cn\,v \div D,$$

$$1 + k^2 sn(u + v)\,sn(u - v) = (dn^2 v + k^2 sn^2 u\,cn^2 v) \div D,$$

$$1 + sn(u + v)\,sn(u - v) = (cn^2 v + sn^2 u\,dn^2 v) \div D,$$

$$1 + cn(u + v)\,cn(u - v) = (cn^2 u + cn^2 v) \div D,$$

$$1 + dn(u + v)\,dn(u - v) = (dn^2 u + dn^2 v) \div D,$$

$$1 - k^2 sn(u + v)\,sn(u - v) = (dn^2 u + k^2 sn^2 v\,cn^2 u) \div D,$$

$$1 - sn(u + v)\,sn(u - v) = (cn^2 u + sn^2 v\,dn^2 u) \div D,$$

$$1 - cn(u + v)\,cn(u - v) = sn^2 u\,dn^2 v + sn^2 v\,dn^2 u \div D,$$

$$1 - dn(u + v)\,dn(u - v) = k^2(sn^2 u\,cn^2 v + sn^2 v\,cn^2 u) \div D.$$

LXIV.

$$\wp u - \wp v = -\frac{\sigma(u+v)\,\sigma(u-v)}{\sigma^2 u\,\sigma^2 v}, \quad \ldots\ldots \quad \text{p. 352.}$$

$$\zeta(u+v) + \zeta(u-v) - 2\,\zeta u = \frac{\wp' u}{\wp u - \wp v}, \quad \ldots \quad \text{p. 352.}$$

$$\zeta(u+v) - \zeta(u-v) - 2\,\zeta v = \frac{-\wp' v}{\wp u - \wp v},$$

$$\zeta(u \pm v) = \zeta u \pm \zeta v + \frac{1}{2}\,\frac{\wp' u \mp \wp' v}{\wp u - \wp v},$$

℘([illegible]

$$- \frac{1}{2}\,\frac{\partial}{\partial u}\left[\frac{\wp' u \mp \wp' v}{\wp u - \wp v}\right] \quad \ldots\ldots\ldots\ldots \quad \text{p. 352.}$$

$$\mp \frac{1}{2}\,\frac{\partial}{\partial v}\left[\frac{\wp' u \mp \wp' v}{\wp u - \wp v}\right]$$

$$+ \frac{(6\,\wp^2 u - \frac{1}{2} g_2)(\wp v - \wp u) + 4\,\wp^3 u - g_2 \wp u - g_3 \mp \wp' u\,\wp' v}{2(\wp u - \wp v)^2}$$

$$+ \frac{(6\,\wp^2 v - \frac{1}{2} g_2)(\wp u - \wp v) + 4\,\wp^3 v - g_2 \wp v - g_3 \mp \wp' u\,\wp' v}{2(\wp u - \wp v)^2},$$

$$\wp(u \pm v) = \frac{2(\wp u\,\wp v - \frac{1}{4} g_2)(\wp u + \wp v) - g_3 \mp \wp' u\,\wp' v}{2(\wp u - \wp v)^2} \quad \ldots\ldots \quad \text{p. 353.}$$

$$= \frac{1}{4}\left[\frac{\wp' u \mp \wp' v}{\wp u - \wp v}\right]^2 - \wp u - \wp v, \quad \ldots\ldots\ldots \quad \text{pp. 366, 367.}$$

1 / [illegible] v)

$$= \frac{2(\wp u\,\wp v - \frac{1}{4} g_2)(\wp u + \wp v) - g_3 \pm \wp' u\,\wp' v}{2(\wp u\,\wp v + \frac{1}{4} g_2)^2 + 2\,g_3(\wp u + \wp v)}. \quad \ldots\ldots \quad \text{p. 355.}$$

LXIII (*Continued*).

$\{1 \pm \quad sn(u+v)\}\ \{1 \pm \quad sn(u-v)\} = (cn\,v \pm \quad sn\,u\,dn\,v)^2 \div D,$

$\{1 \pm \quad sn(u+v)\}\ \{1 \mp \quad sn(u-v)\} = (cn\,u \pm \quad sn\,v\,dn\,u)^2 \div D.$

$\{1 \pm k\,sn(u+v)\}\ \{1 \pm k\,sn(u-v)\} = (dn$

$\{1 \pm k\,sn(u+v)\}\ \{1 \mp k\,sn(u-v)\} = (dn$

$+v)\}\ \{1 \pm \quad cn(u-v)\} = (cn$

$+v)\}\ \{1 \mp \quad cn(u-v)\} = (sn$

$+v)\}\ \{1 \pm \quad dn(u-v)\} = (dn$

$+v)\}\ \{1 \mp \quad dn(u-v)\} = k^2($ $D.$

$cn(u-v) = (sn\,u\,cn\,u\,dn\,v + sn\,v$

$cn(u+v) = (sn\,u\,cn\,u\,dn\,v - sn\,v$

$dn(u-v) = (sn\,u\,dn\,u\,cn\,v + sn\,v$

$dn(u+v) = (sn\,u\,dn\,u\,cn\,v - sn\,v$

$dn(u-v) = (cn\,u\,dn\,u\,cn\,v\,dn\,v -$

$dn(u+v) = (cn\,u\,dn\,u\,cn\,v\,dn\,v + k'^2\,sn\,u\,sn\,v) \div D.$

$(u+v) + \mathrm{am}(u-v)\} = 2\,sn\,u\,cn\,u\,dn\,v \div D,$

$(u+v) - \mathrm{am}(u-v)\} = 2\,sn\,v\,cn\,v\,dn\,u \div D,$

$(u+v) + \mathrm{am}(u-v)\} = (cn^2u - sn^2u\,dn^2v) \div D,$

$(u+v) - \mathrm{am}(u-v)\} = (cn^2v - sn^2v\,dn^2u)$

(Jacobi, Werke, I,

LXV.

$$\wp(u+v)+\wp(u-v)=\frac{2(\wp u\wp v-\tfrac{1}{4}g_2)(\wp u+\wp v)-g_3}{(\wp u-\wp v)^2} \quad . \quad . \quad \text{p. 353.}$$

$$=2\wp u-\frac{\partial^2}{\partial u^2}\log(\wp u-\wp v),$$

$$=2\wp v-\frac{\partial^2}{\partial v^2}\log(\wp u-\wp v).$$

$$\wp(u+v)-\wp(u-v)=-\frac{\wp' u\wp' v}{(\wp u-\wp v)^2}=-\frac{\partial^2}{\partial u\partial v}\log(\wp u-\wp v),$$

$$\wp(u+v)\wp(u-v)=\frac{(\wp u\wp v+\tfrac{1}{4}g_2)^2+g_3(\wp u+\wp v)}{(\wp u-\wp v)^2},$$

$$4[\wp u+\wp v+\wp(u+v)]=\left(\frac{\wp' u-\wp' v}{\wp u-\wp v}\right)^2=\left(\frac{\wp'(u+v)+\wp' v}{\wp(u+v)-\wp v}\right)^2. \quad \text{p. 354.}$$

$$\begin{vmatrix} 1, & \wp(u+v), & -\wp'(u+v) \\ 1, & \wp u, & \wp' u \\ 1, & \wp v, & \wp' v \end{vmatrix}=0. \quad . \quad . \quad . \quad . \quad \text{p. 354.}$$

$$\wp(2u)=\frac{(\wp^2 u+\tfrac{1}{4}g_2)^2+2g_3\wp u}{4\wp^3 u-g_2\wp u-g_3}=\wp u-\frac{1}{4}\frac{d^2}{du^2}\log\wp' u, \quad . \quad . \quad \text{p. 355.}$$

$$\zeta(2u)=2\zeta u+\frac{1}{2}\frac{\wp'' u}{\wp' u},$$

$$\sigma(2u)=\sigma^4 u\frac{d^3\log\sigma u}{du^3}=2\sigma u(\sigma' u)^3-3\sigma^2 u\sigma' u\sigma'' u+\sigma^3 u\sigma''' u, \text{ p. 356.}$$

$$\sigma(2u)=2\sigma u\,\sigma_1 u\,\sigma_2 u\,\sigma_3 u. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{p. 380.}$$

(Schwarz, *loc. cit.*, p. 14.)

LXVI.

$$E(u) = \int_0^u dn^2 u\, du = \int_0^\phi \sqrt{1 - k^2 \sin^2\phi}\, d\phi = E(\phi, k), \quad \ldots \ldots \quad \text{p. 285.}$$

$$E(k, z) = \int_0^z \sqrt{\frac{1 - k^2 z^2}{1 - z^2}}\, dz; \; E = \int_0^1 \sqrt{\frac{1 - k^2 z^2}{1 - z^2}}\, dz; \; E' = \int_0^1 \sqrt{\frac{1 - k'^2 z^2}{1 - z^2}}\, dz.$$

$$E = \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2\phi}\, d\phi, \quad E' = \int_0^{\frac{\pi}{2}} \sqrt{1 - k'^2 \sin^2\phi}\, d\phi.$$

$$KE' + K'E - KK' = \frac{\pi}{2}, \quad \ldots \ldots \ldots \quad \text{p. 291.}$$

$$J = K - E, \quad J' = E'; \quad J'K - K'J = \frac{\pi}{2}.$$

$$\Theta(u) = \Theta(0) e^{\int_0^u Z(u) du}. \quad \ldots \ldots \ldots \ldots \quad \text{p. 292.}$$

$$Z(u) = \left(1 - \frac{E}{K}\right) u - \int_0^u k^2 sn^2 u\, du = \int_0^u \left(dn^2 u - \frac{E}{K}\right) du = E(u) - u\frac{E}{K}.$$

$$dn^2 u = \frac{E}{K} + Z'(u), \qquad Z'(0) = 1 - \frac{E}{K},$$

$$k^2 sn^2 u = Z'(0) - Z'(u), \qquad k^2 cn^2 u = k^2 - Z'(0) + Z'(u),$$

$$Z'(K) = Z'(0) - k^2,$$

$$Z(u) = \frac{\Theta'(u)}{\Theta(u)},$$

$$Z(-u) = -Z(u), \qquad Z(0) = 0,$$

$$Z(u + 2K) = Z(u), \qquad Z(u + 2iK') = Z(u) - \frac{\pi i}{K}, \quad \ldots \quad \text{p. 294.}$$

$$Z(K) = 0, \quad \ldots \quad \text{p. 292.} \qquad Z(iK') = \infty \begin{cases} \textit{Simple pole,} \\ \textit{residue} = 1. \end{cases}$$

$$iZ(iu, k) = -\tan \operatorname{am}(u, k') dn(u, k') + \frac{\pi u}{2KK'} + Z(u, k'). \quad \text{p. 293.}$$

LXVII. . . . . pp. 302–303.

$$\eta = \sqrt{e_1 - e_3}\left\{E - \frac{e_1}{e_1 - e_3}K\right\}, \quad \eta' = -i\sqrt{e_1 - e_3}\left\{E' + \frac{e_3}{e_1 - e_3}K'\right\},$$

$$E = \frac{1}{\sqrt{e_1 - e_3}}\{\eta + e_1\omega\}, \quad E' = \frac{i}{\sqrt{e_1 - e_3}}\{\eta' + e_3\omega'\}.$$

Formulas for ζu are found under Nos. XXII and XLIV.

$$\eta = -\left[e_3 + (e_1 - e_3)\frac{J}{K}\right]\omega, \quad \eta' = -\left[e_3 + (e_1 - e_3)\frac{J}{K}\right]\omega' - \frac{\pi i}{2\omega}.$$

$$\eta'' = -\left[e_3 + (e_1 - e_3)\frac{J}{K}\right]\omega'' - \frac{\pi i}{2\omega}.$$

$$\eta\omega' - \eta'\omega = \frac{\pi i}{2}.$$

$$E(\sqrt{e_1 - e_3}\cdot u) = \frac{1}{\sqrt{e_1 - e_3}}\left(\frac{\sigma_3' u}{\sigma_3 u} + e_1 u\right), \quad . \quad . \quad . \quad . \quad \text{p. 307.}$$

$$Z(\sqrt{e_1 - e_3}\cdot u) = \frac{1}{\sqrt{e_1 - e_3}}\left[\zeta(u + \omega') - \frac{\eta}{\omega}u - \eta'\right]. \quad \text{p. 308.}$$

$$Z(u) = \frac{2\pi}{K}\sum_{m=1}^{m=\infty}\frac{q^m \sin\dfrac{m\pi u}{K}}{1 - q^{2m}} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{p. 295.}$$

$$Z(u + v) = Z(u) + Z(v) - k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn}(u + v), \quad . \quad . \quad . \quad \text{p. 350.}$$

$$E(u + v) = E(u) + E(v) - k^2 \operatorname{sn} u \operatorname{sn} v \operatorname{sn}(u + v).$$

LXVIII.

$$\Pi[\mathsf{z},\sqrt{\mathsf{Z}(\mathsf{z})};\ \alpha_1, \sqrt{\mathsf{Z}(\alpha_1)};\ \alpha_2, \sqrt{\mathsf{Z}(\alpha_2)}] = \Pi(\mathsf{z};\ \alpha_1;\ \alpha_2)$$

$$= \frac{1}{2}\int^{\mathsf{z},\sqrt{\mathsf{Z}(\mathsf{z})}}\left[\frac{\sqrt{\mathsf{Z}(\alpha_2)}+\sqrt{\mathsf{Z}(\mathsf{z})}}{\mathsf{z}-\alpha_2} - \frac{\sqrt{\mathsf{Z}(\alpha_1)}+\sqrt{\mathsf{Z}(\mathsf{z})}}{\mathsf{z}-\alpha_1}\right]\frac{d\mathsf{z}}{\sqrt{\mathsf{Z}(\mathsf{z})}}, \quad \text{p. 414.}$$

$$\mathsf{Z}(\mathsf{z}) = (1-\mathsf{z}^2)(1-k^2\mathsf{z}^2).$$

$$\Pi(n, k, \phi) = \int\frac{d\phi}{(1+n\sin^2\phi)\Delta\phi}, \quad \dots\dots \quad \text{p. 420.}$$

$$\Pi(u, a) = \int_0^u \frac{k^2 sn a\, cn a\, dn a\, sn^2 u}{1-k^2 sn^2 a\, sn^2 u}du = \frac{1}{2}\log\frac{\Theta(u-a)}{\Theta(u+a)} + u\frac{\Theta'(a)}{\Theta(a)}. \quad \text{p. 420.}$$

$$\Pi(u, a) - \Pi(a, u) = uE(a) - aE(u), \quad \dots\dots \quad \text{p. 421.}$$

$$\Pi(u, a) = -\Pi(-u, a), \quad \Pi(0, a) = 0 = \Pi(u, K),$$

$$\Pi(u, iK') = \infty, \quad \Pi(K, a) = KE(a) - aE = KZ(a),$$

$$\Pi(u+2K, a) = \Pi(u, a) + 2KZ(a),$$

$$\Pi(u+2iK', a) = \Pi(u, a) + 2iK'Z(a) + \frac{\pi i a}{K}, \quad \dots\dots \quad \text{p. 421.}$$

$$\Pi(iu, ia+K) = \Pi(u, a+K', k'), \quad \dots\dots \quad \text{p. 422.}$$

$$\Pi\left(\frac{2Ku}{\pi}, \frac{2Ka}{\pi}\right) = \frac{2Ku}{\pi}\frac{\Theta'\left(\frac{2Ka}{\pi}\right)}{\Theta\left(\frac{2Ka}{\pi}\right)} - 2\sum_{m=1}^{m=\infty}\frac{q^m\sin 2ma\sin 2mu}{m(1-q^{2m})}. \quad \text{p. 423.}$$

Addition-theorems are found on p. 426.

LXIX.

$$\Pi(t, \sqrt{S(t)};\ \alpha, \sqrt{S(\alpha)};\ \infty) = \Pi(t; \alpha; \infty)$$

$$= \frac{1}{2}\int^{t, \sqrt{S(t)}} \frac{\sqrt{S(t)} + \sqrt{S(\alpha)}}{t - \alpha} \frac{dt}{\sqrt{S(t)}}, \quad S(t) = 4t^3 - g_2 t - g_3, \ . \quad \text{p. } 419.$$

$$t = \wp u, \qquad \sqrt{S(t)} = -\wp' u, \qquad \alpha = \wp u_0, \qquad \sqrt{S(\alpha)} = -\wp' u_0,$$

$$\Pi(t; \alpha; \infty) = \frac{1}{2}\int^{u} \frac{\wp' u + \wp' u_0}{\wp u - \wp u_0} du$$

$$= \log \frac{\sigma(u_0 - u)}{\sigma u\, \sigma u_0} + u \frac{\sigma' u_0}{\sigma u_0},$$

$$\Pi(t;\ \alpha;\ \infty) - \Pi(\alpha;\ t;\ \infty) = u \frac{\sigma' u_0}{\sigma u_0} - u_0 \frac{\sigma' u}{\sigma u} + (2n + 1)\pi i. \ . \quad \text{p. } 420.$$

Addition-theorem on p. 429.

LXX.

$$\int_0^u E(u) du = \log \Omega(u), \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad \text{p. } 423.$$

$$\Omega(iu) = e^{-\frac{u^2}{2}} cn(u, k')\, \Omega(u. k'), \ . \ . \ . \ . \quad \text{p. } 424.$$

$$(\sqrt{e_1 - e_3} \cdot u) = e^{\frac{1}{2} e_1 u^2} \sigma_3 u, \ . \ . \ . \ . \ . \ . \ . \ . \ . \quad \text{p. } 425.$$

$$\Pi(u, a) = uE(a) + \frac{1}{2} \log \frac{\Omega(u - a)}{\Omega(u + a)}, \quad . \ . \quad \text{p. } 424.$$

$$\Pi(\sqrt{e_1 - e_3} \cdot u, \sqrt{e_1 - e_3} \cdot a) = \frac{1}{2} \log \frac{\sigma_3(u - a)}{\sigma_3(u + a)} + u \frac{\sigma_3' a}{\sigma_3 a}. \quad \text{p. } 425.$$

LXXI.

If $f(u)$ is a rational function of u, we may write

(1) $$f(u) = A + \sum_i \left[A_{1i} v_i - \frac{A_{2i}}{1!} v_i' + \frac{A_{3i}}{2!} v_i'' - \cdots \pm \frac{A_{n_i i}}{(n_i - 1)!} v_i^{(n_i - 1)} \right],$$

where $v_i = \dfrac{1}{u - a_i}$. . . . . p. 9.

(2) $$f(u) = C \frac{(u - c_1)(u - c_2) \ldots (u - c_m)}{(u - b_1)(u - b_2) \ldots (u - b_n)}. \quad \ldots \quad \text{p. 9.}$$

If $\phi(u)$ is a rational function of $\sin u$ and $\cos u$, we may write

(1) $$\phi(u) = P(e^{iu}) + \Phi(u), \quad \ldots \quad \text{p. 22.}$$

where

$$\Phi(u) = B + \sum_i \left[B_{1i} \cot \frac{u - a_i}{2} - \frac{B_{2i}}{1!} \frac{d}{du} \cot \frac{u - a_i}{2} \right.$$

$$\left. + \frac{B_{3i}}{2!} \frac{d^2}{du^2} \cot \frac{u - a_i}{2} - \cdots \pm \frac{B_{n_i i}}{(n_i - 1)!} \frac{d^{n_i - 1}}{du^{n_i - 1}} \cot \frac{u - a_i}{2} \right].$$

(2) $$\phi(u) = C_1 e^{\mu u i} \frac{\sin (u - c_1) \sin (u - c_2) \ldots \sin (u - c_m)}{\sin (u - b_1) \sin (u - b_2) \ldots \sin (u - b_n)}. \quad \ldots \quad \text{p. 25.}$$

If $F(u)$ is a doubly periodic function, we may write

(1) $$F(u) = D + \sum_i \left[D_{1i} Z_0(u - u_i) - \frac{D_{2i}}{1!} Z_0'(u - u_i) + \cdots \right.$$

$$\left. \pm \frac{D_{n_i i}}{(n_i - 1)!} Z_0^{(n_i - 1)}(u - u_i) \right], \quad \ldots \quad \text{pp. 120 and 433.}$$

where the transcendental function $Z_0(u)$ becomes infinite of the first order for $u = 0$, the residue being *unity*.

(2) $$F(u) = C_2 \frac{\sigma(u - u_1^0) \sigma(u - u_2^0) \ldots \sigma(u - u_r^0)}{\sigma(u - u_1) \sigma(u - u_2) \ldots \sigma(u - u_r)}, \quad \ldots \quad \text{p. 439.}$$

where $u_1^0 + u_2^0 + \cdots + u_r^0 = u_1 + u_2 + \cdots + u_r$.

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